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An Application of a Measure of Noncompactness in the Study of Asymptotic Stability

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Abstract—Using the technique of fixed-point theorem of Darbo type associated with measures of noncompactness, we obtain an existence result for some functional-integral equation. Moreover, the choice of suitable measure of noncompactness allows us to characterize solutions of the considered equation in terms of asymptotic stability. The method applied here also creates a generalization of the classical Banach fixed- point principle.

Keywords—Measure of noncompactness, Modulus of continuity, Functional-integral equation, Asymptotic stability.

1. INTRODUCTION

Fixed-point theorems used in nonlinear functional analysis allow us, in general, to obtain existence theorems concerning investigated functional-operator equations. It is rather difficult to obtain characterizations of solutions of considered equations with help of those theorems. In this paper, we are going to show how the technique associated with certain measure of noncompactness can be used in order to obtain both existence results concerning some functional-integral equations and simultaneously to characterize asymptotic stability of solutions of those equations.

In this paper, we will use axiomatically defined measures of noncompactness as presented in the book [1]. The basic tool used in our considerations is a fixed-point theorem of Darbo type (cf. [1,2]). Axiomatically defined measures of noncompactness were used in a lot of papers (cf. [1,3,4] and references therein). Nevertheless, the application of measures of noncompactness in the study of asymptotic stability of solutions of functional-integral equation, which will be presented in this paper, seems to be new and original.

Let us also mention that the method used in the paper creates some generalization of the classical Banach fixed-point principle.

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2. MEASURES OF NONCOMPACTNESS AND A FIXED-POINT THEOREM

In this section, we collect all auxiliary facts which will be used in the sequel. Assume that $(E, \|.\|)$ is an infinite-dimensional Banach space with the zero element θ. By the symbol $B(x,r)$, we denote the closed ball centered at x and with radius r. If X is a subset of E, then $\bar{X}$, $\operatorname{Conv} X$ stand for the closure and convex closure of X, respectively. The family of all nonempty and bounded subsets of E will be denoted by M_E while N_E denotes its subfamily consisting of all relatively compact sets.

DEFINITION. *(See [1].) A mapping $\mu : M_E \to R_+ = [0,\infty)$ will be called a measure of noncompactness in the space E provided it satisfies the following conditions:*

1^0 *the family* $\ker \mu = \{X \in M_E : \mu(X) = 0\}$ *is nonempty and* $\ker \mu \subset N_E$;
2^0 $X \subset Y \Longrightarrow \mu(X) \le \mu(Y)$;
3^0 $\mu(\bar{X}) = \mu(X)$;
4^0 $\mu(\operatorname{Conv} X) = \mu(X)$;
5^0 $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ *for* $\lambda \in [0,1]$;
6^0 *if (X_n) is a sequence of sets from M_E such that $X_{n+1} \subset X_n$, $\bar{X}_n = X_n$ $(n = 1,2,\dots)$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the intersection $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty and $X_\infty \in \ker \mu$.*

The family $\ker \mu$ described in Axiom 1^0 is called the kernel of the measure of noncompactness μ. A measure μ is said to be *sublinear* if it satisfies the following two conditions:

7^0 $\mu(\lambda X) = |\lambda|\,\mu(X)$ for $\lambda \in \mathbb{R}$;
8^0 $\mu(X + Y) \le \mu(X) + \mu(Y)$.

For further facts concerning measures of noncompactness and its properties we refer to [1]. We will only need the following fixed-point theorem of Darbo [2]. Let us mention that a generalization of this theorem was given by Sadovskii [5].

THEOREM 1. *Let Q be a nonempty bounded closed convex subset of the space E and let $F : Q \to Q$ be a continuous operator such that $\mu(FX) \le k\mu(X)$ for all nonempty subsets X of Q, where $k \in [0,1)$ is a constant. Then F has a fixed point in the set Q.*

It can be shown that in the situation described in the above theorem the set $\operatorname{fix} F$ of all fixed points of F belonging to Q is a member of $\ker \mu$. This fact permits us to characterize solutions of investigated operator equation.

In what follows, we will work in the Banach space $BC(\mathbb{R}_+)$ contained of all real functions defined, bounded and continuous on $\mathbb{R}_+$. The norm in $BC(\mathbb{R}_+)$ is defined as the standard supreme norm, i.e.,

$$\|x\| = \sup\{|x(t)| : t \ge 0\}.$$

We will use a measure of noncompactness in the space $BC(\mathbb{R}_+)$ which was constructed in the paper [6]. In order to define this measure let us fix a nonempty bounded subset X of $BC(\mathbb{R}_+)$ and a positive number $T > 0$. For $x \in X$ and $\varepsilon \ge 0$ denote by $\omega^T(x,\varepsilon)$, the modulus of continuity of x on the interval $[0,T]$, i.e.,

$$\omega^T(x,\varepsilon) = \sup\left[|x(t) - x(s)| : t,s \in [0,T],\ \ |t-s| \le \varepsilon\right].$$

Moreover, let us put

$$\omega^T(X,\varepsilon) = \sup\left[\omega^T(x,\varepsilon) : x \in X\right],$$
$$\omega_o^T(X) = \lim_{\varepsilon\to 0} \omega^T(X,\varepsilon),$$
$$\omega_o(X) = \lim_{T\to\infty} \omega_o^T(X).$$

If t is a fixed number from $\mathbb{R}_+$, let us denote

$$X(t) = \{x(t) : x \in X\}$$

and

$$\operatorname{diam} X(t) = \sup\{|x(t) - y(t)| : x, y \in X\}.$$

Finally, consider the function μ defined on $M_{BC(\mathbb{R}_+)}$ by the formula

$$\mu(X) = \omega_o(X) + \limsup_{t\to\infty} \operatorname{diam} X(t).$$

It can be shown [6] that the function $\mu(X)$ defines sublinear measure of noncompactness in the space $BC(\mathbb{R}_+)$ which majored the ball measure of noncompactness in the sense of the above accepted definition. The kernel $\ker \mu$ of this measure contains nonempty and bounded sets X such that functions belonging to X are locally equicontinuous on $\mathbb{R}_+$ and "the thickness of the bundle" formed by functions from X tends to zero at infinity.

3. MAIN RESULT AND REMARKS

We start with a general case. Let F be an operator transforming the space $BC(\mathbb{R}_+)$ into itself and such that

$$|(Fx)(t) - (Fy)(t)| \leq k\,|x(t) - y(t)| + a(t) \tag{1}$$

for all functions $x, y \in BC(\mathbb{R}_+)$ and for any $t \in \mathbb{R}_+$, where k is a constant from the interval $[0, 1)$ and $a : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that $\lim_{t\to\infty} a(t) = 0$. Further, assume that $x = x(t)$ ($x \in BC(\mathbb{R}_+)$) is a solution of the operator equation

$$x = Fx. \tag{2}$$

Then we have the following simple result.

THEOREM 2. *Under the above assumptions, the function x is asymptotically stable solution of equation (2) that means that for any $\varepsilon > 0$ there exists $T > 0$ such that for every $t \geq T$ and for every other solution y of equation (2) the following inequality holds to be true:*

$$|x(t) - y(t)| \leq \varepsilon.$$

PROOF. Suppose contrary. This implies that there exists a number $\varepsilon_o > 0$ such that for any $T > 0$ there are $t \geq T$ and a solution y of equation (2) with the property

$$|x(t) - y(t)| \geq \varepsilon_o.$$

Hence, without loss of generality choose a nondecreasing sequence (t_n) such that $t_n \to \infty$ and a sequence (y_n) of solutions of equation (2) such that

$$|x(t_n) - y_n(t_n)| \geq \varepsilon_o. \tag{3}$$

On the other hand, keeping in mind (1), we obtain

$$|x(t_n) - y_n(t_n)| \leq k\,|x(t_n) - y_n(t_n)| + a(t_n). \tag{4}$$

Now, linking (3) and (4), we arrive at the following estimate:

$$(1-k)\varepsilon_o \leq (1-k)\,|x(t_n) - y_n(t_n)| \leq a(t_n).$$

Thus, we obtain a contradiction with the fact that $a(t_n) \to 0$ when $n \to \infty$. This completes the proof. ∎

In the following, we study the functional-integral equation of the form

$$x(t) = f(t, x(t)) + \int_o^t u(t, s, x(s))\, ds, \tag{5}$$

where $t \geq 0$. Assume that the functions involved in equation (5) satisfy the following conditions:

(i) $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and the function $t \to f(t,0)$ is a member of the space $BC(\mathbb{R}_+)$;

(ii) there exists a constant $k \in [0,1)$ such that

$$|f(t,x) - f(t,y)| \leq k\,|x-y|,$$

for any $t \geq 0$ and for all $x, y \in \mathbb{R}$;

(iii) $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exist continuous functions $a, b : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\lim_{t\to\infty} a(t) \int_o^t b(s)\,ds = 0,$$

and

$$|u(t,s,x)| \leq a(t)b(s)$$

for all $t, s \in \mathbb{R}_+$ $(s \leq t)$ and for any $x \in \mathbb{R}$.

Now we may formulate our main result.

THEOREM 3. *Under Assumptions (i)–(iii), equation (5) has at least one solution $x = x(t)$ belonging to the space $BC(\mathbb{R}_+)$ and being asymptotically stable on the interval $\mathbb{R}_+$.*

PROOF. First, let us define the function $v = v(t)$ by putting

$$v(t) = a(t) \int_o^t b(s)\,ds.$$

In view of our assumptions, we have that $v = v(t)$ is continuous on $\mathbb{R}_+$ and $v(t) \to 0$ as $t \to \infty$.

Next, let us fix a function $x \in BC(\mathbb{R}_+)$ and put

$$(Fx)(t) = f(t,x(t)) + \int_o^t u(t,s,x(s))\,ds.$$

Then, in view of the assumptions, we infer that Fx is continuous on $\mathbb{R}_+$. On the other hand, we get

$$\begin{aligned} |(Fx)(t)| &\leq |f(t,x(t)) - f(t,0)| + |f(t,0)| + \int_o^t |u(t,s,x(s)|\,ds \\ &\leq k\,|x(t)| + |f(t,0)| + a(t)\int_o^t b(s)\,ds = k\,|x(t)| + |f(t,0)| + v(t). \end{aligned} \tag{6}$$

Hence, we deduce that Fx is bounded on $\mathbb{R}_+$. This allows us to infer that $Fx \in BC(\mathbb{R}_+)$ which means that the operator F transforms the space $BC(\mathbb{R}_+)$ into itself.

Now, let us observe that from estimate (6), we obtain

$$\|Fx\| \leq k\,\|x\| + Q,$$

where

$$Q = \sup\{|f(t,0)| + v(t) : t \geq 0\} < \infty.$$

This yields that the operator F transforms the ball $B_r = B(\theta, r)$ into itself, where $r = Q/(1-k)$.

In what follows, we show that the operator F is continuous on the ball B_r. To do this, let us fix arbitrarily a number $\varepsilon > 0$ and take two functions $x, y \in B_r$ such that $\|x - y\| \leq \varepsilon$. Then, keeping in mind our assumptions, we get

$$\begin{aligned} |(Fx)(t) - (Fy)(t)| &\leq k\varepsilon + \int_o^t |u(t,s,x(s)) - u(t,s,y(s))|\,ds \\ &\leq k\varepsilon + 2a(t)\int_o^t b(s)\,ds = k\varepsilon + 2v(t). \end{aligned} \tag{7}$$

Now, denote by T a real positive number such that

$$2a(t)\int_{o}^{t} b(s)\,ds \le \varepsilon$$

for $t \ge T$.

Consider two cases.

1^0 $t \ge T$.

Then, from estimate (7), we obtain

$$|(Fx)(t) - (Fy)(t)| \le k\varepsilon + \varepsilon = (k+1)\varepsilon.$$

2^0 $t < T$.

Then, let $\omega = \omega(\varepsilon)$ be the function defined by the formula

$$\omega(\varepsilon) = \sup\left\{|u(t,s,x) - u(t,s,y)| : t,s \in [0,T],\ x,y \in [-r,r],\ |x-y| \le \varepsilon\right\}.$$

Taking into account that the function $u = u(t,s,x)$ is uniformly continuous on the set $[0,T] \times [0,T] \times [-r,r]$, we obtain that $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then, in virtue of (7), we have

$$|(Fx)(t) - (Fy)(t)| \le k\varepsilon + \int_{o}^{t} \omega(\varepsilon)\,ds \le k\varepsilon + T\omega(\varepsilon).$$

Now, linking Cases 1^0 and 2^0, we can deduce that F is continuous on B_r.

In the sequel, let us take a set $X \subset B_r$, $X \neq \emptyset$. Further, fix numbers $T > 0$, $\varepsilon > 0$ and a function $x \in X$. Then, choosing $t,s \in [0,T]$ such that $|t-s| \le \varepsilon$ and taking into account our assumptions, we get

$$\begin{aligned}
|(Fx)(t) - (Fx)(s)| &\le |f(t,x(t)) - f(s,x(s))| + \left|\int_{o}^{t} u(t,\tau,x(\tau))\,d\tau - \int_{o}^{s} u(s,\tau,x(\tau))\,d\tau\right| \\
&\le |f(t,x(t)) - f(t,x(s))| + |f(t,x(s)) - f(s,x(s))| + \left|\int_{o}^{t} u(t,\tau,x(\tau))\,d\tau - \int_{o}^{s} u(s,\tau,x(\tau))\,d\tau\right| \\
&\le k\,|x(t) - x(s)| + |f(t,x(s)) - f(s,x(s))| + \left|\int_{s}^{t} u(t,\tau,x(\tau))\,d\tau\right| \\
&\quad + \left|\int_{o}^{s} [u(t,\tau,x(\tau)) - u(s,\tau,x(\tau))]\,d\tau\right| \\
&\le k\omega^T(x,\varepsilon) + \omega_r^T(f,\varepsilon) + \varepsilon\ \sup\left[a(t)b(\tau) : 0 \le t \le T,\ 0 \le \tau \le T\right] + T\bar{\omega}_r^T(u,\varepsilon),
\end{aligned}$$

where

$$\begin{aligned}
\omega_r^T(f,\varepsilon) &= \sup\left\{|f(t,x) - f(s,x)| : t,s \in [0,T],\ |t-s| \le \varepsilon,\ |x| \le r\right\}, \\
\bar{\omega}_r^T(u,\varepsilon) &= \sup\{|u(t,\tau,x) - u(s,\tau,x)| : t,s \in [0,T],\ |t-s| \le \varepsilon,\ \tau \in [0,T],\ |x| \le r\}.
\end{aligned}$$

Applying the accepted assumptions, we infer easily that the function $f = f(t,x)$ is uniformly continuous on the set $[0,T] \times [-r,r]$, while the function $u = u(t,\tau,x)$ is uniformly continuous on the set $[0,T] \times [O,T] \times [-r,r]$. Hence, we deduce that $\omega_r^T(f,\varepsilon) \to 0$ and $\bar{\omega}_r^T(u,\varepsilon) \to 0$ as $\varepsilon \to 0$.

Now, from the above-obtained estimate, we get

$$\omega^T(Fx,\varepsilon) \le k\omega^T(x,\varepsilon) + \omega_r^T(f,\varepsilon) + T\bar{\omega}_r^T(u,\varepsilon) + \varepsilon \sup\left\{a(t)b(\tau) : t,\tau \in [0,T]\right\}.$$

This yields

$$\omega_o^T(FX) \le k\omega_o^T(X),$$

and consequently,

$$\omega_o(FX) \le k\omega_o(X). \tag{8}$$

Further, for an arbitrary fixed $t \in \mathbb{R}_+$ and for $x, y \in X$ we derive the following estimate:

$$\begin{aligned} |(Fx)(t) - (Fy)(t)| &\le |f(t, x(t)) - f(t, y(t))| + \int_o^t |u(t, s, x(s))|\, ds + \int_o^t |u(t, s, y(s))|\, ds \\ &\le k\,|x(t) - y(t)| + 2\int_o^t a(t)b(s)\, ds \\ &= k\,|x(t) - y(t)| + 2v(t). \end{aligned} \tag{9}$$

Hence, we obtain

$$\operatorname{diam}(FX)(t) \le k \operatorname{diam} X(t) + 2v(t).$$

This yields

$$\limsup_{t\to\infty} \operatorname{diam}(FX)(t) \le k \limsup_{t\to\infty} \operatorname{diam} X(t). \tag{10}$$

Now, linking (8) and (10), we arrive at the following inequality:

$$\mu(FX) \le k\mu(X).$$

The above inequality in conjunction with Theorem 1 allows us to deduce that there exists a solution $x = x(t)$ of equation (5) in the space $BC(\mathbb{R}_+)$. Moreover, in view of (9) and Theorem 2, we infer that $x(t)$ is asymptotically stable on the interval $\mathbb{R}_+$. This completes the proof. ∎

REMARKS.

(1) Observe that the information about the asymptotic stability of the solution $x = x(t)$ of equation (5) can be also deduced from the fact that the set of all solutions of equation (5) belongs to $\ker \mu$ (cf. Theorem 1). Keeping in mind the description of the kernel of the measure of noncompactness μ (cf. Section 2), we obtain that every solution $x = x(t)$ of equation (5) is asymptotically stable.

(2) Following are examples of functional-integral equations satisfying the assumptions of Theorem 3

$$x(t) = \frac{t}{1+t^2}x(t) + \int_o^t e^{-t}\frac{sx(s)}{1+|x(s)|}\, ds,$$

$$x(t) = \frac{\ln(1+t)}{1+t}\sin x(t) + \int_o^t \frac{s^2 \operatorname{arctg} x(s)}{1+t^4}\, ds.$$

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