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A semilinear elliptic system involving nonlinear boundary condition and sign-changing weight function

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Abstract

In this paper, we study the combined effect of concave and convex nonlinearities on the number of solutions for a semilinear elliptic system $(E_{\lambda,\mu})$ involving nonlinear boundary condition and sign-changing weight function. With the help of the Nehari manifold, we prove that the system has at least two nontrivial nonnegative solutions when the pair of the parameters (λ,μ) belongs to a certain subset of $\mathbb{R}^2$.

Keywords: Semilinear elliptic systems; Nonlinear boundary conditions; Nehari manifold

1. Introduction

In this paper, we consider the multiplicity results of nontrivial nonnegative solutions of the following semilinear elliptic system:

$$\begin{cases} -\Delta u + u = \frac{\alpha}{\alpha+\beta} f(x)|u|^{\alpha-2}u|v|^{\beta} & \text{in } \Omega, \\ -\Delta v + v = \frac{\beta}{\alpha+\beta} f(x)|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = \lambda g(x)|u|^{q-2}u, \quad \frac{\partial v}{\partial n} = \mu h(x)|v|^{q-2}v & \text{on } \partial\Omega, \end{cases} \tag{$E_{\lambda,\mu}$}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $\alpha > 1, \beta > 1$ satisfying $2 < \alpha + \beta < 2^*$ ($2^* = \frac{2N}{N-2}$ if $N \geqslant 3$, $2^* = \infty$ if $N = 2$), $1 < q < 2$, the pair of parameters $(\lambda, \mu) \in \mathbb{R}^2 \setminus \{(0,0)\}$ and the weight functions f, g, h are satisfying the following conditions:

(A) $f \in C(\overline{\Omega})$ with $\|f\|_\infty = 1$ and $f^+ = \max\{f, 0\} \not\equiv 0$;
(B) $g, h \in C(\partial\Omega)$ with $\|g\|_\infty = \|h\|_\infty = 1$, $g^\pm = \max\{\pm g, 0\} \not\equiv 0$ and $h^\pm = \max\{\pm h, 0\} \not\equiv 0$.

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Semilinear elliptic problems with nonlinear boundary condition are widely studied; we refer the reader to Garcia-Azorero, Peral and Rossi [5] and Wu [6,7]. Recently, in [6] the author considered a semilinear elliptic equation involving sign-changing weight function, and showed multiplicity results with respect to the parameter via the extraction of Palais-Smale sequences in the Nehari manifold.

Because the two sublinear boundary conditions in problem $(E_{\lambda,\mu})$ are homogeneous of the same degree $q-1$ and so the problem $(E_{\lambda,\mu})$ is similar to the Ambrosetti, Brezis and Cerami problem [1] (a semilinear elliptic equation involving concave and convex nonlinearities). Thus, the existence of more than one nontrivial solution for problem $(E_{\lambda,\mu})$ is expected. In this paper, we give a very simple variational method which is similar to the "fibering method" of Pohozaev's (see [3] or [4]) to prove the existence of at least two nontrivial nonnegative solutions of problem $(E_{\lambda,\mu})$. In particular we do this without the extraction of Palais-Smale sequences in the Nehari manifold. Throughout this section, we let S and $\overline{S}$ be the best Sobolev and the best Sobolev trace constants for the embedding of $H_0^1(\Omega)$ in $L^{\alpha+\beta}(\Omega)$ and $H_0^1(\Omega)$ in $L^q(\partial\Omega)$, respectively. And let $C_0=(\frac{q}{2})^{2/(2-q)}C(\alpha,\beta,q,S,\overline{S})$ be a positive number where $C(\alpha,\beta,q,S,\overline{S})=(\frac{\alpha+\beta-q}{2-q}S^{\alpha+\beta})^{2/(2-\alpha-\beta)}(\frac{\alpha+\beta-2}{\alpha+\beta-q}\overline{S}^{-q})^{\frac{2}{2-q}}$. Then we have the following result.

Theorem 1.1. *If the parameters* λ,μ *satisfy*

$$0<|\lambda|^{\frac{2}{2-q}}+|\mu|^{\frac{2}{2-q}}<C_0,$$

then problem $(E_{\lambda,\mu})$ *has at least two solutions* (u_0^+,v_0^+) *and* (u_0^-,v_0^-) *such that* $u_0^\pm\geqslant 0$, $v_0^\pm\geqslant 0$ *in* Ω *and* $u_0^\pm\neq 0$, $v_0^\pm\neq 0$. *Furthermore, if* $f\geqslant 0$, *then* $u_0^\pm>0$, $v_0^\pm>0$ *in* Ω.

This paper is organized as follows. In Section 2, we give some the properties of the Nehari manifold. In Section 3, we prove Theorem 1.1.

2. Nehari manifold

Problem $(E_{\lambda,\mu})$ is posed in the framework of the Sobolev space $H=H^1(\Omega)\times H^1(\Omega)$ with the standard norm

$$\|(u,v)\|_H=\left(\int_\Omega\left(|\nabla u|^2+u^2\right)dx+\int_\Omega\left(|\nabla v|^2+v^2\right)dx\right)^{\frac{1}{2}}.$$

Moreover, a pair of functions $(u,v)\in H$ is said to be a weak solution of problem $(E_{\lambda,\mu})$ if

$$\int_\Omega(\nabla u\nabla\varphi_1+u\varphi_1)\,dx+\int_\Omega(\nabla v\nabla\varphi_2+v\varphi_2)\,dx-\frac{\alpha}{\alpha+\beta}\int_\Omega f|u|^{\alpha-2}u|v|^\beta\varphi_1\,dx-\frac{\beta}{\alpha+\beta}\int_\Omega f|u|^\alpha|v|^{\beta-2}v\varphi_2\,dx$$
$$-\lambda\int_{\partial\Omega}g|u|^{q-2}u\varphi_1\,ds-\mu\int_{\partial\Omega}h|v|^{q-2}v\varphi_2\,ds=0$$

for all $(\varphi_1,\varphi_2)\in H$. Thus, the corresponding energy functional of problem $(E_{\lambda,\mu})$ is defined by

$$J_{\lambda,\mu}(u,v)=\frac{1}{2}\|(u,v)\|_H^2-\frac{1}{\alpha+\beta}\int_\Omega f|u|^\alpha|v|^\beta\,dx-\frac{1}{q}K_{\lambda,\mu}(u,v)$$

for $(u,v)\in H$, where $K_{\lambda,\mu}(u,v)=\lambda\int_{\partial\Omega}g|u|^q\,ds+\mu\int_{\partial\Omega}h|v|^q\,ds$.

As the energy functional $J_{\lambda,\mu}$ is not bounded below on H, it is useful to consider the functional on the Nehari manifold

$$\mathbf{N}_{\lambda,\mu}=\left\{(u,v)\in H\setminus\{(0,0)\}\;\middle|\;\left\langle J'_{\lambda,\mu}(u,v),(u,v)\right\rangle=0\right\}.$$

Thus, $(u,v)\in\mathbf{N}_{\lambda,\mu}$ if and only if

$$\left\langle J'_{\lambda,\mu}(u,v),(u,v)\right\rangle=\|(u,v)\|_H^2-\int_\Omega f|u|^\alpha|v|^\beta\,dx-K_{\lambda,\mu}(u,v)=0. \tag{2.1}$$

Note that $\mathbf{N}_{\lambda,\mu}$ contains every nonzero solution of problem $(E_{\lambda,\mu})$. Moreover, we have the following results.

Lemma 2.1. *The energy functional $J_{\lambda,\mu}$ is coercive and bounded below on $\mathbf{N}_{\lambda,\mu}$.*

Proof. If $(u,v)\in\mathbf{N}_{\lambda,\mu}$, then by the Sobolev imbedding theorem

$$J_{\lambda,\mu}(u,v)=\frac{\alpha+\beta-2}{2(\alpha+\beta)}\big\|(u,v)\big\|_H^2-\left(\frac{\alpha+\beta-q}{q(\alpha+\beta)}\right)K_{\lambda,\mu}(u,v)$$
$$\geqslant\frac{\alpha+\beta-2}{2(\alpha+\beta)}\big\|(u,v)\big\|_H^2-\overline{S}^q\left(\frac{\alpha+\beta-q}{q(\alpha+\beta)}\right)\big(|\lambda|^{\frac{2}{2-q}}+|\mu|^{\frac{2}{2-q}}\big)^{\frac{2-q}{2}}\big\|(u,v)\big\|_H^q. \tag{2.2}$$

Thus, J_λ is coercive and bounded below on $\mathbf{N}_{\lambda,\mu}$. □

Define

$$\Phi_{\lambda,\mu}(u,v)=\big\langle J'_{\lambda,\mu}(u,v),(u,v)\big\rangle.$$

Then for $(u,v)\in\mathbf{N}_{\lambda,\mu}$,

$$\big\langle\Phi'_{\lambda,\mu}(u,v),(u,v)\big\rangle=2\big\|(u,v)\big\|_H^2-(\alpha+\beta)\int_\Omega f|u|^\alpha|v|^\beta\,dx-qK_{\lambda,\mu}(u,v) \tag{2.3}$$
$$=(2-\alpha-\beta)\int_\Omega f|u|^\alpha|v|^\beta\,dx-(q-2)K_{\lambda,\mu}(u,v). \tag{2.4}$$

Now, we split $\mathbf{N}_{\lambda,\mu}$ into three parts:

$$\mathbf{N}^+_{\lambda,\mu}=\big\{(u,v)\in\mathbf{N}_{\lambda,\mu}\;\big|\;\big\langle\Phi'_{\lambda,\mu}(u,v),(u,v)\big\rangle>0\big\};$$
$$\mathbf{N}^0_{\lambda,\mu}=\big\{(u,v)\in\mathbf{N}_{\lambda,\mu}\;\big|\;\big\langle\Phi'_{\lambda,\mu}(u,v),(u,v)\big\rangle=0\big\};$$
$$\mathbf{N}^-_{\lambda,\mu}=\big\{(u,v)\in\mathbf{N}_{\lambda,\mu}\;\big|\;\big\langle\Phi'_{\lambda,\mu}(u,v),(u,v)\big\rangle<0\big\}.$$

Then, we have the following results.

Lemma 2.2. *Suppose that (u_0,v_0) is a local minimizer for $J_{\lambda,\mu}$ on $\mathbf{N}_{\lambda,\mu}$ and that $(u_0,v_0)\notin\mathbf{N}^0_{\lambda,\mu}$. Then $J'_{\lambda,\mu}(u_0,v_0)=0$ in H^{-1} (the dual space of the Sobolev space H).*

Proof. Our proof is almost the same as that in Brown and Zhang [3, Theorem 2.3] (or see Binding, Drabek, and Huang [2]). □

Lemma 2.3. *We have*

(i) *if $(u,v)\in\mathbf{N}^+_{\lambda,\mu}$, then $K_{\lambda,\mu}(u,v)>0$;*
(ii) *if $(u,v)\in\mathbf{N}^0_{\lambda,\mu}$, then $K_{\lambda,\mu}(u,v)>0$ and $\int_\Omega f|u|^\alpha|v|^\beta\,dx>0$;*
(iii) *if $(u,v)\in\mathbf{N}^-_{\lambda,\mu}$, then $\int_\Omega f|u|^\alpha|v|^\beta\,dx>0$.*

Proof. The proof is immediate from (2.1) and (2.4). □

Moreover, we have the following result.

Lemma 2.4. *If*

$$0<|\lambda|^{\frac{2}{2-q}}+|\mu|^{\frac{2}{2-q}}<C(\alpha,\beta,q,S,\overline{S}),$$

then $\mathbf{N}^0_{\lambda,\mu}=\emptyset$.

Proof. Suppose otherwise, that is there exists $(\lambda, \mu) \in \mathbb{R}^2 \setminus \{(0,0)\}$ with

$$0 < |\lambda|^{\frac{2}{2-q}} + |\mu|^{\frac{2}{2-q}} < C(\alpha, \beta, q, S, \overline{S})$$

such that $\mathbf{N}^0_{\lambda,\mu} \neq \emptyset$. Then for $(u, v) \in \mathbf{N}^0_{\lambda,\mu}$ we have

$$0 = \langle \Phi'_{\lambda,\mu}(u,v), (u,v) \rangle = (2-q)\|(u,v)\|_H^2 - (\alpha+\beta-q)\int_\Omega f|u|^\alpha |v|^\beta\,dx$$

$$= (2-\alpha-\beta)\|(u,v)\|_H^2 - (q-\alpha-\beta)K_{\lambda,\mu}(u,v).$$

By the Hölder inequality and the Sobolev imbedding theorem,

$$\|(u,v)\|_H \geqslant \left(\frac{\alpha+\beta-q}{2-q} S^{\alpha+\beta}\right)^{\frac{1}{2-\alpha-\beta}}$$

and

$$\|(u,v)\|_H \leqslant \left(\frac{\alpha+\beta-q}{\alpha+\beta-2}\right)^{\frac{1}{2-q}} \overline{S}^{\frac{q}{2-q}} \left(|\lambda|^{\frac{2}{2-q}} + |\mu|^{\frac{2}{2-q}}\right)^{\frac{1}{2}}.$$

This implies

$$|\lambda|^{\frac{2}{2-q}} + |\mu|^{\frac{2}{2-q}} \geqslant C(\alpha, \beta, q, S, \overline{S})$$

which is a contradiction. Thus, we can conclude that if

$$0 < |\lambda|^{\frac{2}{2-q}} + |\mu|^{\frac{2}{2-q}} < C(\alpha, \beta, q, S, \overline{S}),$$

we have $\mathbf{N}^0_{\lambda,\mu} = \emptyset$. □

By Lemma 2.4, we write $\mathbf{N}_{\lambda,\mu} = \mathbf{N}^+_{\lambda,\mu} \cup \mathbf{N}^-_{\lambda,\mu}$ and define

$$\theta^+_{\lambda,\mu} = \inf_{(u,v)\in\mathbf{N}^+_{\lambda,\mu}} J_{\lambda,\mu}(u,v); \qquad \theta^-_{\lambda,\mu} = \inf_{(u,v)\in\mathbf{N}^-_{\lambda,\mu}} J_{\lambda,\mu}(u,v).$$

Then we have the following result.

Theorem 2.5. *If* $0 < |\lambda|^{\frac{2}{2-q}} + |\mu|^{\frac{2}{2-q}} < C_0$, *then we have*

(i) $\theta^+_{\lambda,\mu} < 0$;
(ii) $\theta^-_{\lambda,\mu} > d_0$ *for some* $d_0 = d_0(\alpha, \beta, q, \overline{S}, S, \lambda, \mu) > 0$.

Proof. (i) Let $(u,v) \in \mathbf{N}^+_{\lambda,\mu}$. By (2.3)

$$\frac{2-q}{\alpha+\beta-q}\|(u,v)\|_H^2 > \int_\Omega f|u|^\alpha |v|^\beta\,dx$$

and so

$$J_{\lambda,\mu}(u,v) = \left(\frac{1}{2}-\frac{1}{q}\right)\|(u,v)\|_H^2 + \left(\frac{1}{q}-\frac{1}{\alpha+\beta}\right)\int_\Omega f|u|^\alpha |v|^\beta\,dx$$

$$< \left[\left(\frac{1}{2}-\frac{1}{q}\right) + \left(\frac{1}{q}-\frac{1}{\alpha+\beta}\right)\frac{2-q}{\alpha+\beta-q}\right]\|(u,v)\|_H^2$$

$$= -\frac{(2-q)(\alpha+\beta-2)}{2q(\alpha+\beta)}\|(u,v)\|_H^2 < 0.$$

Thus, $\theta^+_{\lambda,\mu} < 0$.

(ii) Let $(u,v) \in \mathbf{N}^-_{\lambda,\mu}$. By (2.3)

$$\frac{2-q}{\alpha+\beta-q}\left\|(u,v)\right\|_H^2 < \int_\Omega f|u|^\alpha|v|^\beta\,dx.$$

Moreover, by the Sobolev imbedding theorem

$$\int_\Omega f|u|^\alpha|v|^\beta\,dx \leqslant S^{\alpha+\beta}\left\|(u,v)\right\|_H^{\alpha+\beta}.$$

This implies

$$\left\|(u,v)\right\|_H > \left(\frac{2-q}{(\alpha+\beta-q)S^{\alpha+\beta}}\right)^{\frac{1}{\alpha+\beta-2}} \quad \text{for all } (u,v) \in \mathbf{N}^-_{\lambda,\mu}. \tag{2.5}$$

By (2.2) in the proof of Lemma 2.1

$$\begin{aligned} J_{\lambda,\mu}(u,v) &\geqslant \left\|(u,v)\right\|_H^q\left[\frac{\alpha+\beta-2}{2(\alpha+\beta)}\left\|(u,v)\right\|_H^{2-q} - \overline{S}^q\left(\frac{\alpha+\beta-q}{q(\alpha+\beta)}\right)\left(|\lambda|^{\frac{2}{2-q}}+|\mu|^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}}\right] \\ &> \left(\frac{2-q}{(\alpha+\beta-q)S^{\alpha+\beta}}\right)^{\frac{q}{\alpha+\beta-2}} \\ &\quad \times\left[\frac{\alpha+\beta-2}{2(\alpha+\beta)}\left(\frac{2-q}{(\alpha+\beta-q)S^{\alpha+\beta}}\right)^{\frac{2-q}{\alpha+\beta-2}} - \overline{S}^q\left(\frac{\alpha+\beta-q}{q(\alpha+\beta)}\right)\left(|\lambda|^{\frac{2}{2-q}}+|\mu|^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}}\right]. \end{aligned}$$

Thus, if

$$0 < |\lambda|^{\frac{2}{2-q}} + |\mu|^{\frac{2}{2-q}} < C_0,$$

then

$$J_{\lambda,\mu}(u,v) > d_0 \quad \text{for all } (u,v) \in \mathbf{N}^-_{\lambda,\mu},$$

for some $d_0 = d_0(\alpha,\beta,q,\overline{S},S,\lambda,\mu) > 0$. This completes the proof. □

For each $(u,v) \in H$ with $\int_\Omega f|u|^\alpha|v|^\beta\,dx > 0$, we write

$$t_{\max} = \left(\frac{(2-q)\|(u,v)\|_H^2}{(\alpha+\beta-q)\int_\Omega f|u|^\alpha|v|^\beta\,dx}\right)^{\frac{1}{\alpha+\beta-2}} > 0.$$

Then the following lemma hold.

Lemma 2.6. *For each $(u,v) \in H$ with $\int_\Omega f|u|^\alpha|v|^\beta\,dx > 0$, we have*

(i) *if $K_{\lambda,\mu}(u,v) \leqslant 0$, then there is unique $t^- > t_{\max}$ such that $(t^-u, t^-v) \in \mathbf{N}^-_{\lambda,\mu}$ and*

$$J_{\lambda,\mu}\left(t^-u,t^-v\right) = \sup_{t\geqslant 0} J_{\lambda,\mu}(tu,tv);$$

(ii) *if $K_{\lambda,\mu}(u,v) > 0$, then there are unique $0 < t^+ < t_{\max} < t^-$ such that $(t^+u,t^+v) \in \mathbf{N}^+_{\lambda,\mu}$, $(t^-u,t^-v) \in \mathbf{N}^-_{\lambda,\mu}$ and*

$$J_{\lambda,\mu}\left(t^+u,t^+v\right) = \inf_{0\leqslant t\leqslant t_{\max}} J_{\lambda,\mu}(tu,tv); \qquad J_{\lambda,\mu}\left(t^-u,t^-v\right) = \sup_{t\geqslant 0} J_{\lambda,\mu}(tu,tv).$$

Proof. Fix $(u,v) \in H$ with $\int_\Omega f|u|^\alpha|v|^\beta\,dx > 0$. Let

$$m(t) = t^{2-q}\left\|(u,v)\right\|_H^2 - t^{\alpha+\beta-q}\int_\Omega f|u|^\alpha|v|^\beta\,dx \quad \text{for } t \geqslant 0. \tag{2.6}$$

Clearly, $m(0)=0$, $m(t)\to-\infty$ as $t\to\infty$. Since

$$m'(t)=(2-q)t^{1-q}\left\|(u,v)\right\|_H^2-(\alpha+\beta-q)t^{\alpha+\beta-q-1}\int\limits_\Omega f|u|^\alpha|v|^\beta\,dx$$

we have $m'(t)=0$ at $t=t_{\max}$, $m'(t)>0$ for $t\in[0,t_{\max})$ and $m'(t)<0$ for $t\in(t_{\max},\infty)$. Then $m(t)$ achieves its maximum at $t_{\max}$, is increasing for $t\in[0,t_{\max})$ and decreasing for $t\in(t_{\max},\infty)$. Moreover,

$$\begin{aligned} m(t_{\max})&=\left\|(u,v)\right\|_H^q\left[\left(\frac{2-q}{\alpha+\beta-q}\right)^{\frac{2-q}{\alpha+\beta-2}}-\left(\frac{2-q}{\alpha+\beta-q}\right)^{\frac{\alpha+\beta-q}{\alpha+\beta-2}}\right]\left(\frac{\|(u,v)\|_H^{\alpha+\beta}}{\int_\Omega f|u|^\alpha|v|^\beta\,dx}\right)^{\frac{2-q}{\alpha+\beta-2}}\\ &\geqslant\left\|(u,v)\right\|_H^q\left(\frac{\alpha+\beta-2}{\alpha+\beta-q}\right)\left(\frac{\alpha+\beta-q}{2-q}S^{\alpha+\beta}\right)^{\frac{2-q}{2-\alpha-\beta}}. \end{aligned}\tag{2.7}$$

(i) $K_{\lambda,\mu}(u,v)\leqslant0$. There is a unique $t^->t_{\max}$ such that $m(t^-)=K_{\lambda,\mu}(u,v)$ and $m'(t^-)<0$. Now,

$$(2-q)\left(t^-\right)^2\left\|(u,v)\right\|_H^2-(\alpha+\beta-q)\left(t^-\right)^{\alpha+\beta}\int\limits_\Omega f|u|^\alpha|v|^\beta\,dx=\left(t^-\right)^{1+q}m'\left(t^-\right)<0,$$

and

$$\left\langle J'_{\lambda,\mu}\left(t^-u,t^-v\right),\left(t^-u,t^-v\right)\right\rangle=\left(t^-\right)^q\left[m\left(t^-\right)-K_{\lambda,\mu}(u,v)\right]=0.$$

Thus, $(t^-u,t^-v)\in\mathbf{N}^-_{\lambda,\mu}$. Since for $t>t_{\max}$, we have

$$(2-q)\left\|(tu,tv)\right\|_H^2-(\alpha+\beta-q)\int\limits_\Omega f|tu|^\alpha|tv|^\beta\,dx<0,\qquad \frac{d^2}{dt^2}J_{\lambda,\mu}(tu,tv)<0$$

and

$$\frac{d}{dt}J_{\lambda,\mu}(tu,tv)=t\left\|(u,v)\right\|_H^2-t^qK_{\lambda,\mu}(u,v)-t^{\alpha+\beta}\int\limits_\Omega f|u|^\alpha|v|^\beta\,dx=0\quad\text{for } t=t^-.$$

Thus, $J_{\lambda,\mu}(t^-u,t^-v)=\sup_{t\geqslant0}J_{\lambda,\mu}(tu,tv)$.

(ii) $K_{\lambda,\mu}(u,v)>0$. By (2.7) and

$$\begin{aligned} m(0)&=0\\ &<K_{\lambda,\mu}(u,v)\\ &\leqslant\overline{S}^q\left(|\lambda|^{\frac{2}{2-q}}+|\mu|^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}}\left\|(u,v)\right\|_H^q\\ &<\left\|(u,v)\right\|_H^q\left(\frac{\alpha+\beta-2}{\alpha+\beta-q}\right)\left(\frac{\alpha+\beta-q}{2-q}S^{\alpha+\beta}\right)^{\frac{2-q}{2-\alpha-\beta}}\\ &\leqslant m(t_{\max}), \end{aligned}$$

for $0<|\lambda|^{\frac{2}{2-q}}+|\mu|^{\frac{2}{2-q}}<C(\alpha,\beta,q,S,\overline{S})$, there are unique t^+ and t^- such that $0<t^+<t_{\max}<t^-$,

$$m\left(t^+\right)=K_{\lambda,\mu}(u,v)=m\left(t^-\right)$$

and

$$m'\left(t^+\right)>0>m'\left(t^-\right).$$

We have $(t^+u,t^+v)\in\mathbf{N}^+_{\lambda,\mu}$, $(t^-u,t^-v)\in\mathbf{N}^-_{\lambda,\mu}$, and $J_{\lambda,\mu}(t^-u,t^-v)\geqslant J_{\lambda,\mu}(tu,tv)\geqslant J_{\lambda,\mu}(t^+u,t^+v)$ for each $t\in[t^+,t^-]$ and $J_{\lambda,\mu}(t^+u,t^+v)\leqslant J_{\lambda,\mu}(tu,tv)$ for each $t\in[0,t^+]$. Thus,

$$J_{\lambda,\mu}\left(t^+u,t^+v\right)=\inf_{0\leqslant t\leqslant t_{\max}}J_{\lambda,\mu}(tu,tv);\qquad J_{\lambda,\mu}\left(t^-u,t^-v\right)=\sup_{t\geqslant0}J_{\lambda,\mu}(tu,tv).$$

This completes the proof. □

For each $(u,v) \in H$ with $K_{\lambda,\mu}(u,v) > 0$, we write

$$\bar{t}_{\max} = \left(\frac{(\alpha+\beta-q) K_{\lambda,\mu}(u,v)}{(\alpha+\beta-2)\|(u,v)\|_H^2} \right)^{\frac{1}{2-q}} > 0. \tag{2.8}$$

Then we have the following lemma.

Lemma 2.7. *For each $(u,v) \in H$ with $K_{\lambda,\mu}(u,v) > 0$, we have*

(i) *if $\int_\Omega f|u|^\alpha |v|^\beta\,dx \leqslant 0$, then there is unique $0 < t^+ < \bar{t}_{\max}$ such that $(t^+u, t^+v) \in \mathbf{N}^+_{\lambda,\mu}$ and*
$$J_{\lambda,\mu}(t^+u, t^+v) = \inf_{t \geqslant 0} J_{\lambda,\mu}(tu, tv);$$

(ii) *if $\int_\Omega f|u|^\alpha |v|^\beta\,dx > 0$, then there are unique $0 < t^+ < \bar{t}_{\max} < t^-$ such that $(t^+u, t^+v) \in \mathbf{N}^+_{\lambda,\mu}$, $(t^-u, t^-v) \in \mathbf{N}^-_{\lambda,\mu}$ and*
$$J_{\lambda,\mu}(t^+u, t^+v) = \inf_{0 \leqslant t \leqslant \bar{t}_{\max}} J_{\lambda,\mu}(tu, tv); \qquad J_{\lambda,\mu}(t^-u, t^-v) = \sup_{t \geqslant 0} J_{\lambda,\mu}(tu, tv).$$

Proof. Fix $(u,v) \in H$ with $K_{\lambda,\mu}(u,v) > 0$. Let

$$\bar{m}(t) = t^{2-\alpha-\beta} \|(u,v)\|_H^2 - t^{q-\alpha-\beta} K_{\lambda,\mu}(u,v) \quad \text{for } t > 0. \tag{2.9}$$

Clearly, $\bar{m}(t) \to -\infty$ as $t \to 0^+$, $\bar{m}(t) \to 0$ as $t \to \infty$. Since

$$\bar{m}'(t) = (2-\alpha-\beta) t^{1-\alpha-\beta} \|(u,v)\|_H^2 - (q-\alpha-\beta) t^{q-\alpha-\beta-1} K_{\lambda,\mu}(u,v)$$

we have $\bar{m}'(t) = 0$ at $t = \bar{t}_{\max}$, $\bar{m}'(t) > 0$ for $t \in [0, \bar{t}_{\max})$ and $\bar{m}'(t) < 0$ for $t \in (\bar{t}_{\max}, \infty)$. Then $\bar{m}(t)$ achieves its maximum at $\bar{t}_{\max}$, is increasing for $t \in (0, \bar{t}_{\max})$ and decreasing for $t \in (\bar{t}_{\max}, \infty)$. Similar to the argument in Lemma 2.6, we can obtain the results of Lemma 2.7. □

3. Proof of Theorem 1.1

First, we establish the existence of a local minimum for $J_{\lambda,\mu}$ on $\mathbf{N}^+_{\lambda,\mu}$.

Theorem 3.1. *If $0 < |\lambda|^{\frac{2}{2-q}} + |\mu|^{\frac{2}{2-q}} < C_0$, then $J_{\lambda,\mu}$ has a minimizer (u_0^+, v_0^+) in $\mathbf{N}^+_{\lambda,\mu}$ and it satisfies*

(i) $J_{\lambda,\mu}(u_0^+, v_0^+) = \theta^+_{\lambda,\mu}$;
(ii) *(u_0^+, v_0^+) is a solution of problem $(E_{\lambda,\mu})$, such that $u_0^+ \geqslant 0$, $v_0^+ \geqslant 0$ in Ω and $u_0^+ \neq 0$, $v_0^+ \neq 0$.*

Proof. Let $\{(u_n, v_n)\}$ be a minimizing sequence for $J_{\lambda,\mu}$ on $\mathbf{N}^+_{\lambda,\mu}$. Then by Lemma 2.1 and the compact imbedding theorem, there exist a subsequence $\{(u_n, v_n)\}$ and $(u_0^+, v_0^+) \in H$ such that (u_0^+, v_0^+) is a solution of problem $(E_{\lambda,\mu})$ and

$$\begin{aligned}
&u_n \rightharpoonup u_0^+ \quad \text{weakly in } H_0^1(\Omega),\\
&u_n \to u_0^+ \quad \text{strongly in } L^q(\partial\Omega) \text{ and in } L^{\alpha+\beta}(\Omega),\\
&v_n \rightharpoonup v_0^+ \quad \text{weakly in } H_0^1(\Omega),\\
&v_n \to v_0^+ \quad \text{strongly in } L^q(\partial\Omega) \text{ and in } L^{\alpha+\beta}(\Omega).
\end{aligned}$$

This implies

$$\begin{aligned}
&K_{\lambda,\mu}(u_n, v_n) \to K_{\lambda,\mu}(u_0^+, v_0^+) \quad \text{as } n \to \infty,\\
&\int_\Omega f|u_n|^\alpha |v_n|^\beta \to \int_\Omega f|u_0^+|^\alpha |v_0^+|^\beta \quad \text{as } n \to \infty.
\end{aligned}$$

Since

$$J_{\lambda,\mu}(u_n, v_n) = \frac{\alpha+\beta-2}{2(\alpha+\beta)}\left\|(u_n, v_n)\right\|_H^2 - \frac{\alpha+\beta-q}{q(\alpha+\beta)}K_{\lambda,\mu}(u_n, v_n)$$

and by Theorem 2.5(i)

$$J_{\lambda,\mu}(u_n, v_n) \to \theta^+_{\lambda,\mu} < 0 \quad \text{as } n \to \infty.$$

Letting $n \to \infty$, we see that $K_{\lambda,\mu}(u_0^+, v_0^+) > 0$. Now we prove that

$$u_n \to u_0^+ \quad \text{strongly in } H^1(\Omega),$$
$$v_n \to v_0^+ \quad \text{strongly in } H^1(\Omega).$$

Supposing the contrary, then either

$$\left\|u_0^+\right\|_{H^1} < \liminf_{n\to\infty} \|u_n\|_{H^1} \quad \text{or} \quad \left\|v_0^+\right\|_{H^1} < \liminf_{n\to\infty} \|v_n\|_{H^1}. \tag{3.1}$$

Fix $(u, v) \in H$ with $K_{\lambda,\mu}(u, v) > 0$. Let

$$\phi_{(u,v)}(t) = \bar{m}(t) - \int_\Omega f|u|^\alpha|v|^\beta\,dx$$

where $\bar{m}(t)$ is as in (2.9). Clearly, $\phi_{(u,v)}(t) \to -\infty$ as $t \to 0^+$ and

$$\phi_{(u,v)}(t) \to -\int_\Omega f|u|^\alpha|v|^\beta\,dx \quad \text{as } t \to \infty.$$

Since $\phi'_{(u,v)}(t) = \bar{m}'(t)$, similar argument as in the proof of Lemma 2.7 we have $\phi_{(u,v)}(t)$ achieves its maximum at $\bar{t}_{\max}(u, v)$, is increasing for $t \in (0, \bar{t}_{\max}(u, v))$ and decreasing for $t \in (\bar{t}_{\max}(u, v), \infty)$, where

$$\bar{t}_{\max}(u, v) = \left(\frac{(\alpha+\beta-q)K_{\lambda,\mu}(u, v)}{(\alpha+\beta-2)\|(u, v)\|_H^2}\right)^{\frac{1}{2-q}}$$

is as in (2.8). Since $K_{\lambda,\mu}(u_0^+, v_0^+) > 0$, by Lemma 2.7, there is a unique $0 < t_0^+ < \bar{t}_{\max}(u_0^+, v_0^+)$ such that $(t_0^+u_0^+, t_0^+v_0^+) \in \mathbf{N}^+_{\lambda,\mu}$ and

$$J_{\lambda,\mu}\left(t_0^+u_0^+, t_0^+v_0^+\right) = \inf_{0\leqslant t\leqslant \bar{t}_{\max}(u_0^+, v_0^+)} J_{\lambda,\mu}\left(tu_0^+, tv_0^+\right).$$

Then

$$\phi_{(u_0^+,v_0^+)}\left(t_0^+\right) = \left(t_0^+\right)^{-(\alpha+\beta)}\left(\left\|\left(t_0^+u_0^+, t_0^+v_0^+\right)\right\|_H^2 - K_{\lambda,\mu}\left(t_0^+u_0^+, t_0^+v_0^+\right) - \int_\Omega f\left|t_0^+u_0^+\right|^\alpha\left|t_0^+v_0^+\right|^\beta dx\right) = 0. \tag{3.2}$$

By (3.1) and (3.2) we obtain

$$\phi_{(u_n,v_n)}\left(t_0^+\right) > 0 \quad \text{for } n \text{ sufficiently large.} \tag{3.3}$$

Since $(u_n, v_n) \in \mathbf{N}^+_{\lambda,\mu}$, we have $\bar{t}_{\max}(u_n, v_n) > 1$. Moreover,

$$\phi_{(u_n,v_n)}(1) = \left\|(u_n, v_n)\right\|_H^2 - K_{\lambda,\mu}(u_n, v_n) - \int_\Omega f|u_n|^\alpha|v_n|^\beta\,dx = 0$$

and $\phi_{(u_n,v_n)}(t)$ is increasing for $t \in (0, \bar{t}_{\max}(u_n, v_n))$. This implies $\phi_{(u_n,v_n)}(t) \leqslant 0$ for all $t \in (0, 1]$ and n sufficiently large. We obtain $1 < t_0^+ \leqslant \bar{t}_{\max}(u_0^+, v_0^+)$. But $(t_0^+u_0^+, t_0^+v_0^+) \in \mathbf{N}^+_{\lambda,\mu}$ and

$$J_{\lambda,\mu}\left(t_0^+u_0^+, t_0^+v_0^+\right) = \inf_{0\leqslant t\leqslant \bar{t}_{\max}(u_0^+, v_0^+)} J_{\lambda,\mu}\left(tu_0^+, tv_0^+\right).$$

This implies

$$J_{\lambda,\mu}\big(t_0^+u_0^+, t_0^+v_0^+\big) < J_{\lambda,\mu}\big(u_0^+, v_0^+\big) < \lim_{n\to\infty} J_{\lambda,\mu}(u_n, v_n) = \theta_{\lambda,\mu}^+,$$

which is a contradiction. Hence

$$u_n \to u_0^+ \quad \text{strongly in } H^1(\Omega),$$
$$v_n \to v_0^+ \quad \text{strongly in } H^1(\Omega).$$

This implies

$$J_{\lambda,\mu}(u_n, v_n) \to J_{\lambda,\mu}\big(u_0^+, v_0^+\big) = \theta_{\lambda,\mu}^+ \quad \text{as } n \to \infty.$$

Thus, (u_0^+, v_0^+) is a minimizer for $J_{\lambda,\mu}$ on $\mathbf{N}_{\lambda,\mu}^+$. Since $J_{\lambda,\mu}(u_0^+, v_0^+) = J_{\lambda,\mu}(|u_0^+|, |v_0^+|)$ and $(|u_0^+|, |v_0^+|) \in \mathbf{N}_{\lambda,\mu}^+$, by Lemma 2.2 we may assume that (u_0^+, v_0^+) is a nonnegative solution of problem $(E_{\lambda,\mu})$. Finally, we prove that $u_0^+ \neq 0$, $v_0^+ \neq 0$. We assume that, without loss of generality, $v_0^+ \equiv 0$. Then as u_0^+ is a nonzero solution of

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = \lambda g(x)|u|^{q-2}u & \text{on } \partial\Omega, \end{cases}$$

we have

$$\big\|u_0^+\big\|_{H^1}^2 = \lambda \int_{\partial\Omega} g\big|u_0^+\big|^q \, ds > 0.$$

Moreover, by the conditions (A), (B) we may choose $w \in H^1(\Omega) \setminus \{0\}$ such that

$$\|w\|_{H^1}^2 = \mu \int_{\partial\Omega} h|w|^q \, ds > 0$$

and

$$\int_{\Omega} f\big|u_0^+\big|^\alpha |w|^\beta \, dx \geqslant 0.$$

Now

$$K_{\lambda,\mu}\big(u_0^+, w\big) = \lambda \int_{\partial\Omega} g\big|u_0^+\big|^q \, ds + \mu \int_{\partial\Omega} h|w|^q \, ds > 0$$

and so by Lemma 2.7 there is a unique $0 < t^+ < \bar{t}_{\max}$ such that $(t^+u_0^+, t^+w) \in \mathbf{N}_{\lambda,\mu}^+$. Moreover,

$$\bar{t}_{\max} = \left(\frac{(\alpha+\beta-q)K_{\lambda,\mu}(u_0^+, w)}{(\alpha+\beta-2)\|(u_0^+, w)\|_H^2}\right)^{\frac{1}{2-q}} = \left(\frac{\alpha+\beta-q}{\alpha+\beta-2}\right)^{\frac{1}{2-q}} > 1$$

and

$$J_{\lambda,\mu}\big(t^+u_0^+, t^+w\big) = \inf_{0\leqslant t\leqslant \bar{t}_{\max}} J_{\lambda,\mu}\big(tu_0^+, tw\big).$$

This implies

$$J_{\lambda,\mu}\big(t^+u_0^+, t^+w\big) \leqslant J_{\lambda,\mu}\big(u_0^+, w\big) < J_{\lambda,\mu}\big(u_0^+, 0\big) = \theta_{\lambda,\mu}^+$$

which is a contradiction. □

Next, we establish the existence of a local minimum for $J_{\lambda,\mu}$ on $\mathbf{N}_{\lambda,\mu}^-$.

Theorem 3.2. *If $0 < |\lambda|^{\frac{2}{2-q}} + |\mu|^{\frac{2}{2-q}} < C_0$, then $J_{\lambda,\mu}$ has a minimizer (u_0^-, v_0^-) in $\mathbf{N}_{\lambda,\mu}^-$ and it satisfies*

(i) $J_{\lambda,\mu}(u_0^-, v_0^-) = \theta_{\lambda,\mu}^-$;
(ii) *(u_0^-, v_0^-) is a solution of problem $(E_{\lambda,\mu})$, such that $u_0^- \geqslant 0$, $v_0^- \geqslant 0$ in Ω and $u_0^- \neq 0$, $v_0^- \neq 0$.*

Proof. Let $\{(u_n, v_n)\}$ be a minimizing sequence for $J_{\lambda,\mu}$ on $\mathbf{N}^-_{\lambda,\mu}$. Then by Lemma 2.1 and the compact imbedding theorem there exist a subsequence $\{(u_n, v_n)\}$ and $(u_0^-, v_0^-) \in H$ such that

$$\begin{aligned} &u_n \rightharpoonup u_0^- \quad \text{weakly in } H_0^1(\Omega),\\ &u_n \to u_0^- \quad \text{strongly in } L^q(\partial\Omega) \text{ and in } L^{\alpha+\beta}(\Omega),\\ &v_n \rightharpoonup v_0^- \quad \text{weakly in } H_0^1(\Omega),\\ &v_n \to v_0^- \quad \text{strongly in } L^q(\partial\Omega) \text{ and in } L^{\alpha+\beta}(\Omega). \end{aligned}$$

This implies

$$\begin{aligned} &K_{\lambda,\mu}(u_n, v_n) \to K_{\lambda,\mu}\left(u_0^-, v_0^-\right) \quad \text{as } n \to \infty,\\ &\int_\Omega f|u_n|^\alpha|v_n|^\beta \to \int_\Omega f\left|u_0^-\right|^\alpha\left|v_0^-\right|^\beta \quad \text{as } n \to \infty. \end{aligned}$$

Moreover, by (2.3) we obtain

$$\int_\Omega f|u_n|^\alpha|v_n|^\beta\,dx > \frac{2-q}{\alpha+\beta-q}\left\|(u_n, v_n)\right\|_H^2. \tag{3.4}$$

By (2.5) and (3.4) there exists a positive number $\overline{C}$ such that

$$\int_\Omega f|u_n|^\alpha|v_n|^\beta\,dx > \overline{C}.$$

This implies

$$\int_\Omega f\left|u_0^-\right|^\alpha\left|v_0^-\right|^\beta\,dx \geqslant \overline{C}. \tag{3.5}$$

Now we prove that

$$\begin{aligned} &u_n \to u_0^- \quad \text{strongly in } H_0^1(\Omega),\\ &v_n \to v_0^- \quad \text{strongly in } H_0^1(\Omega). \end{aligned}$$

Suppose otherwise, then either $\|u_0^-\|_{H^1} < \liminf_{n\to\infty}\|u_n\|_{H^1}$ or $\|v_0^-\|_{H^1} < \liminf_{n\to\infty}\|v_n\|_{H^1}$. By Lemma 2.6, there is a unique t_0^- such that $(t_0^- u_0^-, t_0^- v_0^-) \in \mathbf{N}^-_{\lambda,\mu}$. Since $(u_n, v_n) \in \mathbf{N}^-_{\lambda,\mu}$, $J_{\lambda,\mu}(u_n, v_n) \geqslant J_{\lambda,\mu}(tu_n, tv_n)$ for all $t \geqslant 0$, we have

$$J_{\lambda,\mu}\left(t_0^- u_0^-, t_0^- v_0^-\right) < \lim_{n\to\infty} J_{\lambda,\mu}\left(t_0^- u_n, t_0^- v_n\right) \leqslant \lim_{n\to\infty} J_{\lambda,\mu}(u_n, v_n) = \theta^-_{\lambda,\mu}$$

and this is contradiction. Hence

$$\begin{aligned} &u_n \to u_0^- \quad \text{strongly in } H_0^1(\Omega),\\ &v_n \to v_0^- \quad \text{strongly in } H_0^1(\Omega). \end{aligned}$$

This implies

$$J_{\lambda,\mu}(u_n, v_n) \to J_{\lambda,\mu}\left(u_0^-, v_0^-\right) = \theta^-_{\lambda,\mu} \quad \text{as } n \to \infty.$$

Since $J_{\lambda,\mu}(u_0^-, v_0^-) = J_{\lambda,\mu}(|u_0^-|, |v_0^-|)$ and $(|u_0^-|, |v_0^-|) \in \mathbf{N}^-_{\lambda,\mu}$, by Lemma 2.2 and (3.5) we may assume that (u_0^-, v_0^-) is a solution of problem $(E_{\lambda,\mu})$, such that $u_0^- \geqslant 0$, $v_0^- \geqslant 0$ in Ω and $u_0^- \neq 0$, $v_0^- \neq 0$. □

Now, we complete the proof of Theorem 1.1: By Theorems 3.1, 3.2 problem $(E_{\lambda,\mu})$ has two solutions $(u_0^+, v_0^+) \in \mathbf{N}^+_{\lambda,\mu}$ and $(u_0^-, v_0^-) \in \mathbf{N}^-_{\lambda,\mu}$ such that $u_0^\pm \geqslant 0$, $v_0^\pm \geqslant 0$ in Ω and $u_0^\pm \neq 0$, $v_0^\pm \neq 0$. Since $\mathbf{N}^+_{\lambda,\mu} \cap \mathbf{N}^-_{\lambda,\mu} = \emptyset$, this implies that (u_0^+, v_0^+) and (u_0^-, v_0^-) are distinct. Moreover, if $f \geqslant 0$, then by the maximum principle we obtain $u_0^\pm > 0$, $v_0^\pm > 0$ in Ω.

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