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Convergence of Best Approximations from Unbounded Sets

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Given a metric space whose bounded sets are relatively compact (i.e., have compact closures), we show that a nearest point selection from a sequence of Kuratowski converging sets converges to the nearest point in the limit set whenever the latter point is unique. The result is extended to Kuratowski limits of linear varieties in infinite dimensional Hilbert spaces where this nearest point (relative to the origin) is necessarily unique. Finally, we show that the Kuratowski limit of hyperplanes must itself be a hyperplane and that a necessary and sufficient condition for the associated nearest points to the origin to converge as above is that the canonial points parametrizing the hyperplanes converge. © 1992 Academic Press, Inc.

1. INTRODUCTION

Suppose X is a metric space, S is a non-empty subset of X, and $\{S_n\}$ is a sequence of non-empty subsets of X. Assume that $\lim S_n = S$ in the sense of Kuratowski (Section 2). The general problem we consider is that of finding a *constructive* way of selecting a sequence of points $\{s_n\}$, one from each S_n, which converges to a point s in S. Our particular approach is to use best approximations, i.e., from each S_n (resp. S), we select a point s_n (resp. s) which is nearest to some reference point p in X. The problem is then to find conditions on X, S, and the S_n under which $\{s_n\}$ converges to s.

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This problem was considered by the authors in [18] for the case where X is compact and the selections $\{s_n\}$ are more general, i.e., not necessarily nearest-point selections. The main advantage of the compactness assumption is that (for closed subsets) $\lim S_n = S$ is equivalent to convergence of $\{S_n\}$ to S in the underlying Hausdorff metric. However, a significant disadvantage is that ordinary Euclidean space is excluded. In [18], among other things, we gave sufficient conditions for $\{s_n\}$ to converge to s in the presence of compactness for X. Our main objective here is to conclude such convergence without the compactness assumption on X. Unfortunately, as we shall see, some weaker forms of compactness still appear to be necessary.

In Section 2, we establish the ncessary mathematical preliminaries. In Section 3, we consider the case where X has the property that bounded subsets are relatively compact (i.e., have compact closure, e.g., Euclidean space) and the S_n are arbitrary. If p in X has unique best approximation in S and $\lim S_n = S$, then $\lim s_n = s$, where s_n is any best approximation to p in S_n, each n, and where s is the *unique* best approximation to$p$ in S. If every p in X admits a unique best approximation s in S (depending on p), then $\lim S_n = S$ if and only if $\lim s_n = s$, for *any* nearest-point selection $\{s_n\}$ from the S_n relative to any p in X.

In Section 4, we assume S_n and S are closed linear varieties in Hilbert space, i.e.,

$$S_n = \{x \in X : A_n x = y_n\}, \qquad n = 1, 2, \ldots,$$
$$S = \{x \in X : Ax = y\},$$

where X, Y are Hilbert spaces, A_n, A are bounded linear operators from X into Y, and y_n, y are elements of Y, $n = 1, 2, \ldots$. Assuming AA^* is invertible, if $A_n \to A$ *uniformly* and $y_n \to y$, then $\lim S_n = S$. (Later on, we show that this is not true if the A_n converge strongly to A.) Moreover, we exhibit a selection (possibly not nearest-point) from the S_n which converges to a point in S. If the A_n and A have closed ranges, then this seection will be a nearest-point selection relative to the origin in X.

In Section 5, we restrict our attention to the case where Y is the real numbers and each S_n is a closed hyperplane not passing through the origin. Recall that each such hyperplane is uniquely determined (as in Section 2) by a non-zero element a_n of X. We first show that the limit of hyperplanes is again a hyperplane. Specifically, for S non-empty and not containing the origin, if $\lim S_n = S$, then there exists non-zero a in X such that S is the closed hyperplane defined by a and the a_n converge weakly to a. In the presence of this weak convergence, we give necessary and sufficient conditions for nearest-point selection convergence, i.e., $\lim s_n = s$, where this time, the nearest-points are taken relative to the origin in X. We also give

a condition on the a_n relative to a under which $\lim s_n = s$ is equivalent to $\lim S_n = S$.

Finally, in Section 6, we apply our selection convergence results to the following problems:

(i) approximating a solution to an infinite system of inequalities in unbounded variables;

(ii) approximating a best approximation (to an arbitrary point) in a convex body in Euclidean space.

2. Preliminaries

We begin by defining Kuratowski convergence [12] for a sequence of sets in a metric space. For the moment, suppose (X, d) is an arbitrary metric space with $S_n \subseteq X$, $n = 1, 2, \ldots$. Define:

(i) $\liminf S_n =$ set of points x in X for which there exists $x_n \in S_n$, for n sufficiently large, such that $d(x_n, x) \to 0$, as $n \to \infty$.

(ii) $\limsup S_n =$ set of points x in X for which there exists a subsequence $\{S_{n_j}\}$ of $\{S_n\}$ and a corresponding sequence $\{x_j\}$ such that $x_j \in S_{n_j}$, all j, and $d(x_j, x) \to 0$, as $j \to \infty$,

In general, $\liminf S_n \subseteq \limsup S_n$. If $S \subseteq X$ is such that $S \subseteq \liminf S_n$ and $\limsup S_n \subseteq S$, i.e., $\liminf S_n = \limsup S_n = S$, then we write $\lim S_n = S$ and say that $\{S_n\}$ *Kuratowski converges* to S. If $\{S_n\}$ converges to S in the underlying Hausdorff metric, then $\lim S_n = S$ [9, p. 171]. However, the converse is false in general. If X happens to be locally compact, then the closed subsets of X may be topologized so that $\lim S_n = S$ is equivalent to convergence relative to this topology [17].

Let $p \in X$ and $S \subseteq X$. Then there may or may not exist s in S such that $d(p, s) = d(p, S)$, where

$$d(p, S) = \begin{cases} \inf\{d(p, y) : y \in S\}, & S \neq \phi, \\ \infty, & S = \phi. \end{cases}$$

If such s in S exists and is unique, i.e., p has unique best approximation in S, then we say that p is a *uniqueness point* for S in X. We denote the set of such p in X by $U(S)$. In general, $S \subseteq U(S) \subseteq X$. Note that $U(S) = X$ if S is a singleton. For convenience, define $U(\phi) = X$.

In Section 3, it will be important that bounded subsets of X have compact closures. Such X must necessarily be a locally compact metric space. Of course, all compact metric spaces and all closed subsets of finite dimensional normed linear spaces have this property. Moreover, all finite

products of such spaces also have it. Thus, there exist non-compact, non-Euclidean metric spaces having the property that bounded subsets have compact closures.

The following results will be useful in later sections. Let $\theta \in X$ denote a fixed, arbitrary point. If X is a normed linear space, then θ will be assumed to be the zero element. For $x \in X$ and $r > 0$, we denote by $B_x(r)$ the closed ball of radius r centered at x. A subset B of X is *bounded* if there exists $r > 0$ for which $B \subseteq B_\theta(r)$.

LEMMA 2.1. *Suppose bounded subsets of X have compact closures.*

(i) *If S is a closed, non-empty subset of X, then $d(x, S)$ is attained for each x in X.*

(ii) *If $S_n \subseteq X$, all n, then* $\lim S_n = \phi$ *if and only if for each bounded (resp. compact) subset B of X, $B \cap S_n = \phi$ eventually.*

Proof. (i) Let $x \in X$. Since S is non-empty, we have that $d(x, S) < \infty$. For each positive integer n, let $y_n \in S$ be such that

$$d(x, S) \leqslant d(x, y_n) \leqslant d(x, S) + \frac{1}{n}.$$

Then $d(x, y_n) \to d(x, S)$, as $n \to \infty$. Hence, the sequence $\{y_n\}$ is contained in S and is bounded. Let $r > 0$ be sufficiently large such that $\{y_n\} \subseteq B_\theta(r)$. Since $B_\theta(r)$ is compact by hypothesis, passing to a subsequence if necessary, we may assume that there exists $y \in B_\theta(r)$ such that $y_n \to y$, as $n \to \infty$. Since S is closed, $y \in S$ necessarily. Moreover, $d(x, y_n) \to d(x, y)$, as $n \to \infty$, so that $d(x, y) = d(x, S)$.

(ii) This is proved as in [16], where it was assumed that $X = \mathbf{R}^n$. ∎

In Section 5, we will be interested in convergence of closed hyperplanes in Hilbert space. Thus, for the moment, suppose X is a real Hilbert space. It is well known [8, 11] that closed hyperplanes in X which do not pass through the origin θ are in one-to-one correspondence with the non-zero elements of X, where the correspondence $a \to H(a)$ is given by

$$H(a) = \{x \in X : \langle x, a \rangle = 1\},$$

for $a \in X$, $a \neq \theta$ ($H(a) = \phi$, for $a = \theta$). Since $H(a)$ is closed and convex, there exists [1, 8] a unique best approximation s in $H(a)$ to θ, where $s \neq \theta$. Necessarily, s is the orthogonal projection of θ in $H(a)$ and must satisfy [8]

$$\langle x, s \rangle = \|s\|^2, \qquad x \in H(a).$$

Consequently, a must be equal to $(1/\|s\|^2)s$, i.e.,

$$H(a) = \{x \in X: \langle x, s \rangle = \|s\|^2\}$$

for such a. Moreover, $\|a\| = 1/\|s\|$ and $s = (1/\|a\|^2)a$.

3. Convergence of Best Approximations

Suppose once again that X is a metric space. Before we can establish our main result in this section, we require the following.

LEMMA 3.1. *Suppose bounded subsets of X have compact closures. Let $p \in X$ and $S_n \subseteq X$, all n. Suppose $\{s_n\}$ is a nearest-point selection from the S_n relative to p, i.e., $s_n \in S_n$ and $d(p, s_n) = d(p, S_n)$, all n. If* $\limsup S_n \neq \phi$, *then $\{s_n\}$ has a convergent subsequence in X.*

Proof. Suppose not. Let $r > 0$ be arbitrary and consider the closed ball $B_p(r)$ which is compact by hypothesis. If $\{s_n\}$ is not eventually outside $B_p(r)$, then there exists a subsequence inside $B_p(r)$ which must in turn have a convergent subsequence. Contradiction. Thus, $\{s_n\}$ must be outside $B_p(r)$ eventually. Hence, for each $r > 0$, there exists n_r such that $d(p, s_n) > r$, for each $n \geqslant n_r$.

Now let $x \in \limsup S_n$, which is non-empty by hypothesis. Let $\delta > 0$ and set $r = d(p, x) + \delta$, so that $r > d(p, x) \geqslant 0$. Let n_r be as above. By the choice of x, there exists a subsequence $\{S_{n_j}\}$ of $\{S_n\}$ and a corresponding sequence $\{y_j\}$ such that $y_j \in S_{n_j}$, all j, and $y_j \to x$, as $j \to \infty$. In particular, let j_r be sufficiently large such that $n_j \geqslant n_r$, for $j \geqslant j_r$. Then $j \geqslant j_r$ implies that $d(p, s_{n_j}) > r$. Also, $d(p, s_{n_j}) \leqslant d(p, y_j)$, all j, since s_{n_j} is a point in S_{n_j} nearest to p. Moreover, $d(p, y_j) \to d(p, x)$, as $j \to \infty$. Hence for $j \geqslant j_r$, we have

$$d(p, x) + \delta < d(p, s_{n_j}) \leqslant d(p, y_j).$$

Contradiction. Therefore, $\{s_n\}$ has a convergent subsequence in X. ∎

Our main result in this section is the following.

THEOREM 3.2. *Suppose bounded subsets of X have compact closures. Let S be a non-empty subset of X and $\{S_n\}$ a sequence of non-empty subsets of X. If* $\lim S_n = S$, *then for each p in $U(S)$ and each nearest-point selection $\{s_n\}$ from the S_n relative to p, the sequence $\{s_n\}$ converges to the unique point s in S nearest to p.*

Proof. If $\{s_n\}$ does not converge to s, then there exists a subsequence

$\{s_{n_j}\}$ of $\{s_n\}$ which is bounded away from s; i.e., there exists $r > 0$ such that $d(s_{n_j}, s) > r$, $j = 1, 2, \ldots$. We also have that

$$\phi \neq S = \lim_n S_n = \lim_j S_{n_j}$$

[3, p. 121], so that $\limsup_j S_{n_j} \neq \phi$. Hence, by Lemma 3.1, $\{s_{n_j}\}$ has a convergent subsequence. Passing to a subsequence if necessary, we may assume there exists x in X such that $s_{n_j} \to x$, as $j \to \infty$. Thus, $x \in \limsup S_n = S$ by hypothesis. Moreover,

$$r < d(s_{n_j}, s) \leqslant d(s_{n_j}, x) + d(x, s),$$

i.e.,

$$d(x, s) > r - d(s_{n_j}, x), \qquad \text{all } j.$$

Since $d(s_{n_j}, x) \to 0$, as $j \to \infty$, we have that $d(x, s) \geqslant r > 0$. Consequently, $x \neq s$ and necessarily $d(p, s) < d(p, x)$, since $p \in U(S)$. Define

$$\rho = \tfrac{1}{2}[d(p, x) - d(p, s)],$$

from which it follows that

$$d(p, s) + \rho = d(p, x) - \rho > 0$$

and

$$d(p, s) < d(p, x) - \rho.$$

Thus s is in the interior of the ball $B_p(\sigma)$, where $\sigma = d(p, x) - \rho$. Let $0 < \delta < \rho$ be such that $B_s(\delta) \subseteq B_p(\sigma)$. If $y \in B_s(\delta)$, then $d(s, y) < \delta$ and $d(p, y) < d(p, x) - \rho$.

Now $s \in S$ implies that $s \in \lim_n S_n = \lim_j S_{n_j}$. Thus, there exists a sequence in the S_{n_j} which converges to s. Consequently, the S_{n_j} must eventually intersect $B_s(\delta)$, i.e., there exists j_δ sufficiently large such that

$$S_{n_j} \cap B_s(\delta) \neq \phi, \qquad j \geqslant j_\delta.$$

For such j, let $y_j \in S_{n_j} \cap B_s(\delta)$. Then $d(s, y_j) < \delta$. Moreover,

$$\begin{aligned} d(p, s_{n_j}) &\leqslant d(p, y_j) \\ &\leqslant d(p, s) + d(s, y_j) \\ &< d(p, s) + \delta \\ &< d(p, s) + \rho, \end{aligned}$$

i.e., $d(p, s_{n_j}) < d(p, x) - \rho$, for $j \geqslant j_\delta$. Since $d(p, s_{n_j}) \to d(p, x)$, as $j \to \infty$, it follows that $d(p, x) \leqslant d(p, x) - \rho$, where $\rho > 0$. Contradiction. Therefore, $s_n \to s$, as $n \to \infty$. ∎

Remarks. 1. If we assume in addition that S and the S_n are closed, then alternate proofs of Theorem 3.2 may be obtained by (1) combining Theorem 3.1 of [3] with Lemma 1.0 of [4] or (2) by suitably adapting the second part of the proof of Theorem 1.0 of [15].

2. Theorem 3.2 requires that we know $U(S)$ in order to apply it. In general, this is difficult to determine. In some applications, it is possible to verify directly that a *particular* point p is in $U(S)$ (see [14], for example). However, it is desirable that $U(S) = X$, so that p may be chosen arbitrarily in X. This will be the case, for example, if X is a Hilbert space and S is closed and convex [1, p. 15].

THEOREM 3.3 (Selection Convergence). *Suppose bounded subsets of X have compact closures. Let S be a non-empty, closed subset of X and $\{S_n\}$ a sequence of non-empty, closed subsets of X. Assume that $U(S) = X$. Then the following are equivalent:*

(i) $\lim S_n = S$.

(ii) $\lim d(x, S_n) = d(x, S)$, *for all x in X.*

(iii) *For each p in X, and each nearest-point selection $\{s_n\}$ from the S_n relative to p, the sequence $\{s_n\}$ converges to the unique point s in S nearest to p.*

Proof. Statements (i) and (ii) are well known to be equivalent; see [10, Theorem 5] or [4, Theorem 2.3], for example. Thus, it suffices to show (iii) implies (ii) and (i) implies (iii).

(iii) implies (ii). Let $x \in X$. By Lemma 2.1, there exists $s_n \in S_n$, all n, and $s \in S$ such that $d(x, s_n) = d(x, S_n)$, all n, and $d(x, s) = d(x, S)$. By hypothesis, $d(x, s_n) \to d(x, s)$, so that $d(x, S_n) \to d(x, S)$, as $n \to \infty$.

(i) implies (iii). This follows immediately from Theorem 3.2. ∎

Remark. Notable versions of Theorem 3.3 have been obtained by Beer [5] and Tsukada [20] in the context of closed, convex subsets of a reflexive Banach space (with additional structure) relative to set convergence in the sense of Mosco.

Recall that X is locally compact if bounded subsets have compact closures. Let X_∞ denote the one-point compactification of X, where ∞ is the point at infinity and $X_\infty = X \cup \{\infty\}$. The following is our selection convergence result for $S = \phi$. Recall that $U(\phi) = X$.

PROPOSITION 3.4. *Suppose bounded subsets of X have compact closures. Let $\{S_n\}$ be a sequence of non-empty, closed subsets of X. Then the following are equivalent:*

(i) $\lim S_n = \phi$, *i.e.*, $\lim S_n = \{\infty\}$ *in* X_∞.

(ii) $\lim d(x, S_n) = \infty$, *for all* $x \in X$.

(iii) *For each p in X, and each nearest-point selection $\{s_n\}$ from the S_n relative to p, the sequence $\{s_n\}$ is eventually outside every bounded (resp. compact) subset of X, i.e., $s_n \to \infty$ in X_∞.*

Proof. Part (i) implies (iii) by (ii) of Lemma 2.1.

(iii) implies (ii). Let $x \in X$. Then by Lemma 2.1(i), for each n, there exists $s_n \in S_n$ such that $d(x, s_n) = d(x, S_n)$. Since $s_n \to \infty$ by hypothesis, for each $r > 0$, there exists n_r such that for $n \geqslant n_r$, $s_n \notin B_x(r)$, i.e., $d(x, s_n) > r$. Consequently, $d(x, S_n) \to \infty$.

(ii) implies (i). Let $x \in \limsup S_n$. Then there exists a subsequence $\{S_{n_m}\}$ of $\{S_n\}$ and a corresponding sequence $\{x_m\}$ such that $x_m \in S_{n_m}$, all m, and $x_m \to x$, i.e., $d(x_m, x) \to 0$. Let $p \in X$. Then by the hypothesis, $d(p, S_{n_m}) \to \infty$, so that $d(p, x_m) \to \infty$. Since $d(p, x) \geqslant d(p, x_m) - d(x, x_m)$, it follows that $d(p, x) = \infty$. Contradiction. Thus, $\limsup S_n = \phi$ and hence, $\lim S_n = \phi$. ∎

Remark. Part (iii) of Proposition 3.4 is consistent with (iii) of Theorem 3.3 since the point ∞ at infinity may be interpreted as "the unique point in ϕ nearest p."

4. CONVERGENCE OF BEST APPROXIMATIONS FROM LINEAR VARIETIES

Let X and Y be Hilbert spaces, with $x \in X$, $y \in Y$, and $x_n \in X$, $y_n \in Y_n$, all n. Let A, A_n be bounded linear operators from X into Y, all n. Consider the corresponding linear varieties $S = \{x \in X: Ax = y\}$, and $S_n = \{x \in X: A_n x = y_n\}$, $n = 1, 2, \ldots$. In this section, we study convergence of linear varieties, as well as convergence of nearest-point selections (relative to the origin), in the presence of converging defining parameters.

LEMMA 4.1. *Suppose $A_n \to A$ uniformly and AA^* is invertible. Then $A_n A_n^*$ is eventually invertible. In this event, $S \neq \phi$ and $S_n \neq \phi$, eventually.*

Proof. Let G denote the set of bounded linear operators on Y whose inverses exist and are bounded. Then G is an open set in the space of all bounded linear operators on Y equipped with the uniform topology, i.e., operator norm topology, and the mapping $T \to T^{-1}$ is a homeomorphism of G [7, p. 584; 19, p. 193]. By hypothesis, $A_n A_n^* \to AA^*$ uniformly and

$AA^* \in G$. Hence, $A_n A_n^* \in G$, i.e., $A_n A_n^*$ has a bounded inverse, and $(A_n A_n^*)^{-1} \to (AA^*)^{-1}$ uniformly, for large n. Moreover, $A^*(AA^*)^{-1}y$ is an element of S and $A_n^*(A_n A_n^*)^{-1}y_n$ is an element of S_n, for large n. ∎

THEOREM 4.2. *Suppose AA^* is invertible, $A_n \to A$ uniformly, and $y_n \to y$. Then*:

(i) $\lim S_n = S$.

(ii) *The selection $s_n = A_n^*(A_n A_n^*)^{-1}y_n$ in S_n, for large n, converges to the selection $s = A^*(AA^*)^{-1}y$ in S.*

(iii) *If A has closed range and A_n has closed range eventually, then the points s_n, s are points from the S_n, S respectively, which are nearest to θ, and $s_n \to s$.*

Proof. (i) Let $x \in \limsup S_n$. Then there exists a subsequence $\{S_{n_k}\}$ of $\{S_n\}$ and a corresponding sequence $\{x_k\}$ such that $x_k \in S_{n_k}$, i.e., $A_{n_k} x_k = y_{n_k}$, all k, and $x_k \to x$. To show $x \in S$, we have

$$\begin{aligned}\|Ax - y\| &\leqslant \|Ax - A_{n_k}x\| + \|A_{n_k}x - y_{n_k}\| + \|y_{n_k} - y\| \\ &\leqslant \|A - A_{n_k}\|\,\|x\| + \|A_{n_k}x - A_{n_k}x_k\| + \|y_{n_k} - y\| \\ &\leqslant \|A - A_{n_k}\|\,\|x\| + \|A_{n_k}\|\,\|x - x_k\| + \|y_{n_k} - y\|,\end{aligned}$$

where the right hand side goes to zero, as $k \to \infty$. Thus, $\limsup S_n \subseteq S$.

Next suppose that $x \in S$, so that $Ax = y$. To show $x \in \liminf S_n$, by the proof of Lemma 4.1, we may let

$$x_n = A_n^*(A_n A_n^*)^{-1}(y_n - A_n x) + x,$$

for large n. Clearly, $x_n \in S_n$, for such n, and $x_n \to x$ because

$$\begin{aligned}\|x_n - x\| &\leqslant \|A_n^*\|\,\|(A_n A_n^*)^{-1}\|\,\|y_n - A_n x\| \\ &\leqslant \|A_n^*\|\,\|(A_n A_n^*)^{-1}\|\,[\|y_n - y\| + \|Ax - A_n x\|],\end{aligned}$$

where the right side goes to zero, as $n \to \infty$. Hence, $S \subseteq \liminf S_n$.

(ii) For convenience, let $s_n = A_n^*(A_n A_n^*)^{-1}y_n$, for large n, and $s = A^*(AA^*)^{-1}y$. Clearly, $s_n \in S_n$, for such n, and $s \in S$. Also,

$$\begin{aligned}\|s_n - s\| &\leqslant \|A_n^*(A_n A_n^*)^{-1}y_n - A^*(AA^*)^{-1}y_n\| \\ &\quad + \|A^*(AA^*)^{-1}y_n - A^*(AA^*)^{-1}y\| \\ &\leqslant \|A_n^*(A_n A_n^*)^{-1} - A^*(AA^*)^{-1}\|\,\|y_n\| \\ &\quad + \|A^*(AA^*)^{-1}\|\,\|y_n - y\|,\end{aligned}$$

where the right side goes to zero, as $n \to \infty$, i.e., $s_n \to s$.

(iii) If A and A_n have closed ranges, then s and s_n are the points in S and S_n, respectively, which are closest to θ [13, pp. 161–162]. ∎

Remarks. 1. Under the hypotheses of this theorem, parts (i) and (ii) guarantee existence of a convergent selection. However, s and s_n need not be of minimum norm.

2. If the A_n are required only to converge *strongly* to A, then the theorem is not true. We give a counter-example in Section 5 (Example 2).

3. In the first part of the proof of (i) above, it suffices to assume A_n converges strongly to A in order to conclude that $\limsup S_n \subseteq S$.

5. Convergence of Best Approximations from Hyperplanes

In this section we consider the convergence of closed hyperplanes in a Hilbert space versus the convergence of the corresponding sequence of points nearest the origin θ. It is likely that the results of this section can be obtained in Banach spaces which, along with their duals, are reflexive, strictly convex, and have Kadec norms (property (H) in [20]). We leave this pursuit to the interested reader.

Recall the results and notation of Section 2.

Lemma 5.1. *Let $\{H(a_n)\}$ be a sequence of closed hyperplanes in the Hilbert space X not passing through the origin, where $\{a_n\}$ is the corresponding sequence of non-zero defining elements of X. If $\theta \notin \liminf H(a_n)$, then $\{a_n\}$ has a weakly convergent subsequence in X.*

Proof. Let $r > 0$. Suppose $\{a_n\}$ is not eventually outside $B_\theta(r)$. Then there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that $a_{n_j} \in B_\theta(r)$, all j. The Alaoglu Theorem [7] then guarantees a weakly convergent subsequence.

Now suppose $\{a_n\}$ is eventually outside $B_\theta(r)$ for each r. Then $\|a_n\| \to \infty$ necessarily. Let s_n be the unique point in $H(a_n)$ nearest to θ. Of course, $s_n \neq \theta$, all n. Moreover, we have that $a_n = (1/\|s_n\|^2)s_n$ and $\|s_n\| = \|a_n\|^{-1}$, all n, where $0 < \|a_n\| < \infty$. Hence, $\|s_n\| \to 0$, i.e., $s_n \to \theta$ in X. But then $\theta \in \liminf H(a_n)$, since $s_n \in H(a_n)$, all n. Contradiction. Thus, $\{a_n\}$ must have a weakly convergent subsequence. ∎

The following theorem shows that the limit of hyperplanes is itself a hyperplane and moreover, the defining elements of the sequence converge weakly to the defining element of the limit.

Theorem 5.2. *Suppose X is a Hilbert space and $\{H(a_n)\}$ is a sequence of closed hyperplanes in X not passing through the origin. Suppose $\lim H(a_n) = S$, where S is a non-empty subset of X not containing θ. Then*

there exists $a \neq \theta$ in X such that $S = \{x \in X: \langle x, a \rangle = 1\}$, i.e., S is a closed hyperplane $H(a)$ in X, and $\{a_n\}$ converges weakly to a.

Proof. Since $\theta \notin S$, necessarily $\theta \notin \liminf H(a_n)$. Hence, by Lemma 5.1, there exists a in X and a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that $\{a_{n_j}\}$ converges weakly to a. Define $H = \{x \in X: \langle x, a \rangle = 1\}$. Note that if $a = \theta$, then $H = \phi$. We next show that $S = H$.

Let $x \in S$. Then $x \in \liminf H(a_{n_j})$, so that there exists $x_j \in H(a_{n_j})$, all j, such that $x_j \to x$. Then $\langle x_j, a_{n_j} \rangle| = 1$, all j, and

$$\begin{aligned} |1 - \langle x, a \rangle| &= |\langle x_j, a_{n_j} \rangle - \langle x, a \rangle| \\ &\leqslant |\langle x_j, a_{n_j} \rangle - \langle x, a_{n_j} \rangle| + |\langle x, a_{n_j} \rangle - \langle x, a \rangle| \\ &= |\langle x_j - x, a_{n_j} \rangle| + |\langle x, a_{n_j} \rangle - \langle x, a \rangle| \\ &\leqslant \|x_j - x\| \, \|a_{n_j}\| + |\langle x, a_{n_j} \rangle - \langle x, a \rangle|, \qquad \text{all } j, \end{aligned}$$

where $\langle x, a_{n_j} \rangle \to \langle x, a \rangle$ and $x_j \to x$, as $j \to \infty$. Also, $\|a_{n_j}\|$ is bounded [11, p. 200]. Thus, $\langle x, a \rangle = 1$, i.e., $x \in H$. Hence, $S \subseteq H$.

Conversely, since $H \neq \phi$, we have that $a \neq \theta$. Let $x \in H$, so that $\langle x, a \rangle = 1$. Define

$$\alpha_j = 1/\|a_{n_j}\|^2 (1 - \langle x, a_{n_j} \rangle)$$

and

$$x_j = x + \alpha_j a_{n_j}, \qquad \text{all } j.$$

Then $x_j \in H(a_{n_j})$, since $\langle x_j, a_{n_j} \rangle = 1$, all j. Next we show that $\{x_j\}$ converges to x in X. We have

$$\begin{aligned} \|x_j - x\| &= |\alpha_j| \, \|a_{n_j}\| \\ &= |1 - \langle x, a_{n_j} \rangle| / \|a_{n_j}\|, \qquad \text{all } j. \end{aligned}$$

If $\{\|a_{n_j}\|\}$ is not bounded away from 0, then there exists a subsequence (which we may assume is $\{\|a_{n_j}\|\}$) such that $\|a_{n_j}\| \to 0$, i.e., $a_{n_j} \to \theta$, as $j \to \infty$. Contradiction, since $a \neq \theta$. Thus $\{\|a_{n_j}\|\}$ is bounded away from 0, i.e., there exists $\varepsilon > 0$ such that $\|a_{n_j}\| \geqslant \varepsilon$, all j, which implies that

$$\|x_j - x\| \leqslant (1/\varepsilon) \, |1 - \langle x, a_{n_j} \rangle|, \qquad \text{all } j,$$

which converges to 0, since $x \in H$. Consequently,

$$x \in \liminf H(a_{n_j}) = \lim H(a_n) = S,$$

i.e., $H \subseteq S$.

Thus, $H = S$, where $H = H(a)$, for $a \neq \theta$. Hence S is a closed hyperplane $H(a)$ in X not passing through θ.

Finally, we show that $\{a_n\}$ converges weakly to a. This follows from (1) $\Rightarrow$ (2) of Theorem 4.1 of [2]. We give an alternate proof in this context.

Let $x \in X$ and consider the sequence $\{\langle x, a_n \rangle\}$. Let $\{\langle x, a_{n_j} \rangle\}$ be any subsequence of $\{\langle x, a_n \rangle\}$, so that $\{a_{n_j}\}$ is a subsequence of $\{a_n\}$. Necessarily, $S = \lim H(a_{n_j})$. Applying the previous part of this proof to $\{H(a_{n_j})\}$, we see that there exists $b \in X$, $b \neq \theta$, and a subsequence $\{a_{n_{j_k}}\}$ of $\{a_{n_j}\}$ such that $\{a_{n_{j_k}}\}$ converges weakly to b. Also,

$$S = \{x \in X : \langle x, b \rangle = 1\}.$$

But S is uniquely determined by a, i.e., $a = b$, so that $\{a_{n_{j_k}}\}$ converges weakly to a. In particular, $\{\langle x, a_{n_{j_k}} \rangle\}$ is a subsequence of $\{\langle x, a_{n_j} \rangle\}$ which converges to $\langle x, a \rangle$. Consequently, $\{\langle x, a_n \rangle\}$ converges to $\langle x, a \rangle$ [6, p. 88]. Since x is arbitrary, $\{a_n\}$ converges weakly to a. ∎

We have the following converse to Theorem 5.2.

PROPOSITION 5.3. *Let X be a Hilbert space. Suppose $\{a_n\}$ is a sequence in X and $a \in X$. If $\{a_n\}$ converges weakly to a, then* $\lim H(a_n) = H(a)$, *where $H(a) = \phi$ if $a = \theta$.*

Proof. Let $x \in H(a)$, so that $\langle x, a \rangle = 1$. Let $\alpha_n = \langle x, a_n \rangle$, all n, so that $\alpha_n \to 1$, as $n \to \infty$. In particular, $\alpha_n \neq 0$, eventually. Define $x_n = \alpha_n^{-1} x$, for large n, so that $x_n \in H(a_n)$, all such n. Necessarily, $x_n \to x$, as $n \to \infty$. Hence, $x \in \liminf H(a_n)$, i.e., $H(a) \subseteq \liminf H(a_n)$.

Now, let $x \in \limsup H(a_n)$. Then there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ and a corresponding sequence $\{x_j\}$ such that $x_j \in H(a_{n_j})$, i.e., $\langle x_j, a_{n_j} \rangle = 1$, all j, and $x_j \to x$. Arguing as in the proof of Theorem 5.2, this may be shown to imply that $\langle x, a \rangle = 1$, so that $x \in H$. Thus, $\limsup H(a_n) \subseteq H(a)$. ∎

COROLLARY 5.4. *Let X be a Hilbert space, $\{a_n\} \subseteq X$, $a \in X$, $a \neq \theta$. Then* $\lim H(a_n) = H(a)$ *if and only if $\{a_n\}$ converges weakly to a.*

Proof. This follows from Theorem 5.2 and Proposition 5.3. It also follows from Corollary 4.2 of [2]. ∎

Remark. In Corollary 5.4, $a_n \to a$ if and only if $\lim H(a_n) = H(a)$ in the sense of Mosco. See Theorem 5.1 and Corollary 5.5 of [2].

It is natural to ask for conditions under which $\{a_n\}$ converges to a (strongly) in Theorem 5.2. The next result shows that this is equivalent to convergence of the best approximations (to the origin).

PROPOSITION 5.5. *Let X be a Hilbert space. Suppose $\{a_n\}$ and a are non-zero elements of X with s_n (resp. s) the nearest-point in $H(a_n)$ (resp. $H(a)$) to θ, all n. Assume $\{a_n\}$ converges weakly to a. Then the following are equivalent:*

(i) *$\{a_n\}$ converges to a.*

(ii) *$\{s_n\}$ converges to s.*

(iii) *$\{s_n\}$ converges weakly to s.*

(iv) *$\{\|a_n\|\}$ converges to $\|a\|$.*

(v) *$\{\|s_n\|\}$ converges to $\|s\|$.*

Proof. From Section 2, we see that (i) implies (ii) and (iv) is equivalent to (v), while (ii) implies (iii) in general. Thus, it suffices to show that (iii) implies (iv) and (iv) implies (i).

(iii) implies (iv). Suppose $\{s_n\}$ converges weakly to s. Let $x \in X$ be such that $\langle x, s\rangle \neq 0$ (for example, $x = s$). Then $\langle x, s_n\rangle \to \langle x, s\rangle$ and hence, $\langle x, s_n\rangle \neq 0$, eventually. By the results of Section 2, $\langle x, s_n\rangle = (1/\|a_n\|^2)\langle x, a_n\rangle$ and $\langle x, s\rangle = (1/\|a\|^2)\langle x, a\rangle$, so that

$$\|a_n\| = \left[\frac{\langle x, a_n\rangle}{\langle x, s_n\rangle}\right]^{1/2}$$

and

$$\|a\| = \left[\frac{\langle x, a\rangle}{\langle x, s\rangle}\right]^{1/2}.$$

Hence, by our hypotheses, $\{\|a_n\|\}$ converges to $\|a\|$.

(iv) implies (i). Suppose $\{\|a_n\|\}$ converges to $\|a\|$. Then $\limsup \|a_n\| \leqslant \|a\|$. Since $\{a_n\}$ converges to a weakly, it follows that $\{a_n\}$ converges to a by [11, p. 206]. ∎

EXAMPLE 1. Let $\{H(a_n)\}$ be a sequence of (closed) hyperplanes in $\mathbf{R}^m$ not passing through the origin, i.e., $a_n \in \mathbf{R}^m$, $a_n \neq \theta$, all n. Let $s_n \neq \theta$ denote the unique point in $H(a_n)$ nearest to $H(a_n)$, all n. Suppose S is a non-empty subset of $\mathbf{R}^m$ not containing the origin such that $\lim H(a_n) = S$. Then S is also a hyperplane not passing through the origin (Theorem 5.2). Moreover, if s is the unique point in S nearest to θ, then $s_n \to s$, as $n \to \infty$ (Proposition 5.5).

EXAMPLE 2. Suppose X is a Hilbert space, $\{a_n\}$ is a sequence in X which converges weakly, but *not* strongly, to a in X, $a \neq \theta$. Necessarily, $a_n \neq \theta$ eventually. Then, by Proposition 5.3, $\lim H(a_n) = H(a)$. But, the nearest-

point selection $\{s_n\}$ from the $H(a_n)$ relative to θ does not converge to the point s in $H(a)$ nearest θ, by Proposition 5.5. Thus, it is easy to find examples where $\lim H(a_n) = H(a)$ but $s_n \not\to s$.

Alternately, the sequence of bounded linear operators $A_n(x) = \langle x, a_n \rangle$, $x \in X$, converges strongly but not uniformly to the bounded linear operator $A(x) = \langle x, a \rangle$, $x \in X$. (Recall Theorem 4.2.) Note that AA^* is invertible and, for each n, $y_n = 1$ and A_n has closed range; similarly for y and A. Thus, strong convergence of A_n to A is not sufficient in Theorem 4.2.

The next result gives a sufficient condition for strong convergence, as well as nearest-point convergence, to hold in Theorem 5.2.

PROPOSITION 5.6. *Let the notation be as in Proposition* 5.5. *Suppose* $\limsup \|a_n\| \leqslant \|a\|$. *Then the following are equivalent*:

(i) $\lim H(a_n) = H(a)$.

(ii) $\{a_n\}$ *converges weakly to* a.

(iii) $\{a_n\}$ *converges to* a.

(iv) $\{s_n\}$ *converges to* s.

Proof. Part (iii) is equivalent to (iv) by the results of Section 2, and (ii) is equivalent to (iii) by [11, p. 206]. Part (i) is equivalent to (ii) by Corollary 5.4. ∎

6. APPLICATIONS

In this section, we give two applications of our convergence results. The first is an extension to the case of unbounded variables of an application given in [18].

Solving Systems of Inequalities

Consider an infinite system of inequalities

$$g_i(x_1, ..., x_m) \leqslant b_i, \qquad i = 1, 2, ...,$$

where $x = (x_1, ..., x_m) \in \mathbf{R}^m$; i.e., each x_j is an unbounded real variable. We assume the g_i are convex and continuous, all i. Define

$$S_n = \{x \in \mathbf{R}^m : g_i(x) \leqslant b_i, i = 1, ..., n\}, \qquad n = 1, 2, ...$$

Then each S_n is the closed, convex subset of $\mathbf{R}^m$ satisfying the first n inequalities. If we define

$$S = \{s \in \mathbf{R}^m : g_i(x) \leqslant b_i \text{ all } i\},$$

then S is the closed, convex subset of $\mathbf{R}^m$ satisfying all the inequalities. We assume $S \neq \phi$. Our objective is to approximate a point in S. Observe that $S = \bigcap_{n=1}^{\infty} S_n$, $S_{n+1} \subseteq S_n$ so that $S_n \neq \phi$, all n. Moreover, $\lim S_n = S$ [12, p. 339] and $U(S) = \mathbf{R}^m = U(S_n)$, all n.

THEOREM 6.1. *Suppose $S \neq \phi$. Let p be any point in $\mathbf{R}^m$. Let s_n (resp. s) be the unique point in S_n (resp. S) nearest to p, $n = 1, 2, \ldots$. Then the sequence $\{s_n\}$ converges to s.*

Proof. Apply Theorem 3.2. ∎

Approximating Best Approximations

Suppose S is a non-empty, convex body of $\mathbf{R}^m$, that is, a convex subset of $\mathbf{R}^m$ which is the closure of its interio S^0, so that S^0 is also non-empty. Let p be any point in $\mathbf{R}^m$ and consider the problem of approximating the unique point in S which is nearest to p. To do this, we employ the following grid approximation technique.

For each n, let

$$Z_n = \{j/n : j \text{ is an integer}\},$$

$$G_n = \prod_{k=1}^{\infty} Z_k,$$

and define

$$S_n = S \cap G_n \cap B_\theta(n).$$

Then each S_n is a finite, discrete subset of S. In particular, for each $k = 1, 2, \ldots$, we have $S_{2^{k+1}} \subseteq S_{2^k}$. The S_n are not nested however.

LEMMA 6.2. *If S is a non-empty convex body, then* $\lim S_n = S$.

Proof. Since $S_n \subseteq S$, all n, it follows that $\limsup S_n \subseteq S$. Conversely, let $x \in S^0$, which is open. Let V be a bounded open, neighborhood of x such that $V \subseteq S^0$. Suppose n_1 is sufficiently large such that $V \subseteq B_\theta(n)$, for $n \geqslant n_1$. Choose $\varepsilon > 0$ sufficiently small so that the m-dimensional open cube W around x of side ε satisfies $W \subseteq V$. Let $n_2 \geqslant n_1$ be such that $1/n < \varepsilon/2$, for $n \geqslant n_2$. It is not difficult to see that for each $j = 1, \ldots, m$ and $n = 1, 2, \ldots$, there exists an integer k_j^n such that $k_j^n/n \leqslant x_j \leqslant (k_j^n + 1)/n$. In addition, for $n \geqslant n_2$, we have

$$|k_j^n/n - x_j| < \varepsilon/2, \qquad j = 1, \ldots, m.$$

Hence, for $n \geqslant n_2$, the point x^n defined by

$$x^n = (k_1^n/n, k_2^n/n, \ldots, k_m^n/n)$$

belongs to W. Consequently,

$$x^n \in V \cap G_n \cap B_\theta(n) \subseteq V \cap S_n, \qquad \text{for} \quad n \geqslant n_2,$$

i.e., the S_n eventually intersect an arbitrary neighborhood of x. This implies $x \in \liminf S_n$, so that $S^0 \subseteq \liminf S_n$ in general. But $\liminf S_n$ is closed [9, 12], so that $S \subseteq \liminf S_n$, which completes the proof. ∎

Remarks. The S_n are eventually non-empty. This follows from the proof of Lemma 6.2. Since they are also finite, there exists a point in S_n which is nearest to p, for n sufficiently large.

THEOREM 6.3. *Suppose S is a non-empty convex body, $p \in \mathbf{R}^m$, and the S_n are as above. Let s_n be any point in S_n nearest to p and s the unique point in S nearest to p. Then the sequence $\{s_n\}$ converges to s.*

Proof. Apply Theorem 3.2. ∎

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