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On a Bijection between Littlewood–Richardson Fillings of Conjugate Shape

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We present a new bijective proof of the equality between the number of Littlewood–Richardson fillings of a skew-shape λ/μ of weight ν, and those of the conjugate skew-shape λ^t/μ^t, of conjugate weight ν^t. The bijection is defined by means of a unique permutation $\alpha_{\lambda/\mu}$ associated to the skew-shape λ/μ. Our arguments use only well-established properties of Schensted insertion, and make no reference to *jeu de taquin*. © 1992 Academic Press, Inc.

INTRODUCTION

Given partitions μ of r and ν of s, it is well known that the product $s_\mu(x)\, s_\nu(x)$ of the corresponding Schur functions s_μ, s_ν, can be written as a non-negative integral linear combination of Schur functions:

$$s_\mu(x)\, s_\nu(x) = \sum_\lambda c_{\mu\nu}^{\lambda} s_\lambda(x), \tag{1}$$

where the sum runs over all partitions λ of $r+s$; the $c_{\mu\nu}^{\lambda}$ are the Littlewood–Richardson coefficients.

In [LR], Littlewood and Richardson give a combinatorial interpretation of the coefficients $c_{\mu\nu}^{\lambda}$, namely, as the number of lattice permutations of weight ν which fit the skew-shape λ/μ. (For a direct bijective proof, see [Tho]; a completed version of the proof in [LR] and in [L], appears in [Macd].) We refer to such lattice permutations as Littlewood–Richardson fillings of λ/μ, of weight ν.

If λ^t denotes the conjugate or transpose of the partition λ, then it follows (from the representation theory of the symmetric group S_n, for instance) that

$$c_{\mu\nu}^{\lambda} = c_{\mu^t\nu^t}^{\lambda^t} \tag{2}$$

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This may also be seen by applying to (1) the involutive algebra automorphism ω [Macd], defined on the ring of symmetric functions (for which the Schur functions form a Z-basis) by

$$\omega(s_\lambda(x)) = s_{\lambda'}(x).$$

In this paper we present a new bijection which establishes (2). A key element in our correspondence is a certain permutation $\alpha_{\lambda/\mu}$ associated to each skew-shape λ/μ. The permutation $\alpha_{\lambda/\mu}$ is shown to satisfy some interesting properties. In Section 1 we describe the motivation behind the search for our bijection. In Section 2, we set up more notation and record some basic theorems which we shall need to establish our results. The main theorem of the paper appears in Section 3, and further interesting properties of the bijection are developed in Section 4.

1

In this section we briefly outline the motivation behind our search for a canonical correspondence establishing (2). We start with some basic definitions.

DEFINITION 1.1. If λ/μ is a skew-shape, the Littlewood–Richardson (LR) labelling of the cells of λ/μ is an assignment of labels which orders the cells of λ/μ in increasing order from right to left along each row, and from top to bottom down the rows. Thus the rightmost cell in the first non-empty row of λ/μ has LR label 1. See Example 1.1.

EXAMPLE 1.1. Take $\lambda/\mu = (6, 4, 4, 1)/(3, 2, 2)$. The LR labels of λ/μ are

$$\begin{array}{cccccc} \cdot & \cdot & \cdot & \boxed{3} & \boxed{2} & \boxed{1} \\ \cdot & \cdot & \boxed{5} & \boxed{4} & & \\ \cdot & \cdot & \boxed{7} & \boxed{6} & & \\ \boxed{8} & & & & & \end{array}$$

DEFINITION 1.2. A lattice permutation $w = w_1 w_2 \cdots w_n$ is a word in $\{1, 2, ...,\}$ such that for every $j = 1, ..., n$, the initial segment $w_1 \cdots w_j$ of w contains at least as many occurrences of the letter i as it does of $(i+1)$, for every $i \geqslant 1$. The weight of a lattice permutation $w_1 w_2 \cdots w_n$ is the integer vector $(v_1, v_2, ..., v_n)$, where v_i is the number of i's in $w_1 \cdots w_n$. Clearly a lattice permutation always has partition weight.

EXAMPLE 1.2. The word 11221324 is a lattice permutation of weight (3, 3, 1, 1).

DEFINITION 1.3. If λ/μ is a skew-shape of size n, a word $w = w_1 \cdots w_n$ is said to fit λ/μ if the skew-tableau of shape λ/μ obtained by inserting w_i in the cell of λ/μ with LR label i, is semistandard, i.e., is weakly increasing along rows (left to right) and strictly increasing down the columns. We say w is a LR filling of λ/μ, of weight ν, if w is a lattice permutation of weight ν which fits λ/μ.

DEFINITION 1.4. Let T be any standard Young tableau of shape ν, where ν is a partition of n. Define $\mathrm{lp}(T)$ to be the word $w = w_1 w_2 \cdots w_n$, whose ith letter is w_i iff i appears in row w_i of T. Clearly $\mathrm{lp}(T)$ is a *lattice permutation* of weight ν, and conversely, given a lattice permutation w of weight ν, there is a unique standard Young tableau T of shape ν, such that $\mathrm{lp}(T) = w$.

EXAMPLE 1.4. If T =

1	2	3	6
4	5	9	
7	8		

then $\mathrm{lp}(T) = 111221332$.

Now suppose β is any partition whose Ferrers diagram has even columns (so $\beta = (\beta_1, \beta_2, ..., \beta_{2k})$, $\beta_{2i-1} = \beta_{2i}$, all $i = 1, ..., k$). Then there is a natural way to encode any LR filling of weight β, of a given skew-shape λ/μ, as a 1-factor on the LR-labelled cells of λ/μ, as illustrated by Example 1.5 below.

EXAMPLE 1.5. Let $\lambda/\mu = (5, 5, 2, 1)/(3, 2)$. The lattice permutation 11221324 gives the following LR filling of λ/μ:

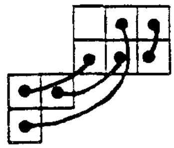
·	·	·	1	1
·	·	1	2	2
2	3			
4				

Also, 11221324 corresponds, via Definition 1.4, to the tableau

T=

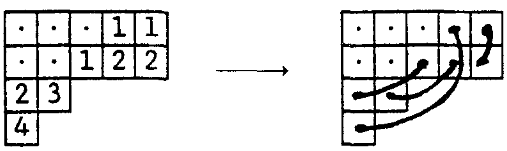
1	2	5
3	4	7
6		
8		

which in turn corresponds to the fixed-point free involution $w = (13)(46)(57)(28)$, via Schensted row-insertion. By putting an edge between the cells with LR labels i, j if (ij) is a transposition in w, we can encode the LR filling of λ/μ as the following 1-factor filling of λ/μ:

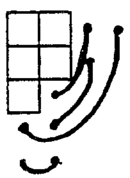


We had noticed that geometrically transposing such a diagram resulted in a 1-factor filling of the conjugate skew-shape λ^t/μ^t, which, apparently, could then be decoded into a LR filling of λ^t/μ^t, of conjugate weight β^t. We then surmised that this phenomenon must be a special case of a canonical bijection between the LR fillings of λ/μ, of weight ν, and those of λ^t/μ^t, of weight ν^t.

EXAMPLE 1.6. Continuing with Example 1.5, we have



which transposes to



This new 1-factor corresponds to the involution (42)(53)(61)(87), which gives the tableau

T^* =

1	3	7
2	5	8
4		
6		

Now

$(T^*)^t$ =

1	3	7
2	5	8
4		
6		

gives the lattice permutation 11212133, which is indeed a Littlewood–Richardson filling of the transpose of λ/μ:

·	·	1	1
·	·	2	
·	1		
1	2		
3	3		

Also note that the original LR filling of λ/μ had weight $\nu=(3, 3, 1, 1)$, while the LR filling of λ^t/μ^t obtained in this manner has weight $(4, 2, 2)=\nu^t$.

2

We shall assume that the reader is familiar with the elementary combinatorics of standard Young tableaux and in particular with Schensted row insertion and column insertion [S]. Following the notation in [S], for a permutation w in S_n, the symmetric group on the letters $\{1, 2, ..., n\}$, we write

$$(\phi \leftarrow w)=(P, Q)$$

to mean that row-inserting w_1, then w_2, ..., and finally w_n produces the pair of standard Young tableaux P, Q; we write

$$(w \rightarrow \phi)=(\bar{P}, \bar{Q})$$

to mean that column-inserting w_n, then w_{n-1}, ..., and finally w_1 produces the pair of tableaux $\bar{P}$, $\bar{Q}$.

Now suppose w is in S_n and ρ in S_n is the reversing permutation, i.e., $\rho(i)=n+1-i$. We list some well-known facts about row and column insertion:

THEOREM 2.1. (1) [S] *Row and column insertion are bijections between the set S_n and the set of all ordered pairs of standard Young tableaux of the same shape λ, where λ runs over all partitions of n.*

(2) [S] *If $(\phi \leftarrow w)=(P, Q)$ and $(w \rightarrow \phi)=(\bar{P}, \bar{Q})$, then $P=\bar{P}$.*

(3) [Sch] *If $(\phi \leftarrow w)=(P, Q)$ then $(\phi \leftarrow w^{-1})=(Q, P)$.*

(4) [Sch] *If $(\phi \leftarrow w)=(P, Q)$ then $(w \rightarrow \phi)=(P, Q_{\text{evac}})$, where Q_{evac} is obtained from Q by a process known as evacuation.*

(5) *If* $(\phi \leftarrow w) = (P, Q)$ *then* $(w\rho \rightarrow \phi) = (P^t, Q^t)$ *and* $(\phi \leftarrow \rho w \rho) = (P_{\text{evac}}, Q_{\text{evac}})$.

We will use this fact as a definition of the evacuation of a standard tableau.

(6) *Evacuation and transposing commute, i.e.,* $(Q_{\text{evac}})^t = (Q^t)_{\text{evac}}$.

(7) [Sch] *Evacuation is an involution, i.e.,* $(Q_{\text{evac}})_{\text{evac}} = Q$.

(8) *If* $(\phi \leftarrow w) = (P, Q)$ *then* $(\phi \leftarrow w\rho) = (P^t, Q^t_{\text{evac}})$ *and* $(\phi \leftarrow \rho w) = (P^t_{\text{evac}}, Q^t)$.

We now state the main theorem used to prove the results in the next section.

THEOREM 2.2 [Wh_1]. *Let* σ *be any permutation in* S_n, *and* λ/μ *a skew-shape of size n. If* $(\sigma \rightarrow \phi) = (S, T)$ *then* $\text{lp}(T)$ *fits* λ/μ *iff the permutation* $\sigma\rho$ *fits* λ/μ.

The theorem as stated above is, in fact, the special case of the main theorem in [Wh_1], restricted to permutations. White's result is, in effect, a bijective proof of the Schur function identity

$$s_{\lambda/\mu}(x) = \sum_{\nu} c^{\lambda}_{\mu\nu} s_{\nu}(x).$$

We shall need a slightly different version of this result:

COROLLARY 2.3. *Let* σ *be any permutation in* S_n *and* λ/μ *a skew-shape of size n. If* $(\phi \leftarrow \sigma) = (T, S)$, *then* $\text{lp}(T)$ *fits* λ/μ *iff the permutation* $\rho\sigma^{-1}$ *fits* λ/μ.

EXAMPLE 2.4. Take $\sigma = 784591236$ and $\lambda/\mu = (5, 4, 3)/(2.1)$. Then

$$(\phi \leftarrow \sigma) = \begin{pmatrix} 1236 & 1259 \\ 459 & 348 \\ 78 & 67 \end{pmatrix}.$$

Observe that

$$\text{lp}(T) = \text{lp}\begin{pmatrix} 1236 \\ 459 \\ 78 \end{pmatrix} = 111221332,$$

which gives the following LR filling of

$$\lambda/\mu : \quad \begin{array}{|c|c|c|c|c|} \hline \ & \ & 1 & 1 & 1 \\ \hline \ & 1 & 2 & 2 \\ \cline{1-4} 2 & 3 & 3 \\ \cline{1-3} \end{array} .$$

Also $\rho\sigma^{-1} = 432761985$ which fits λ/μ, giving the standard skew-tableau

		2	3	4
	1	6	7	
5	8	9		

Proof of Corollary 2.3. We have

$$(\phi \leftarrow \sigma) = (T, S)$$
$$\Leftrightarrow (\phi \leftarrow \rho\sigma\rho) = (T_{\text{evac}}, S_{\text{evac}}) \qquad \text{by Theorem 2.1(5)}$$
$$\Leftrightarrow (\phi \leftarrow \rho\sigma^{-1}\rho) = (S_{\text{evac}}, T_{\text{evac}})$$
$$\Leftrightarrow (\rho\sigma^{-1}\rho \rightarrow \phi) = (S_{\text{evac}}, T) \qquad \text{by Theorem 2.1(7).}$$

Hence by Theorem 2.2,

$$\text{lp}(T) \text{ fits } \lambda/\mu \Leftrightarrow (\rho\sigma^{-1}\rho)\,\rho = \rho\sigma^{-1} \text{ fits } \lambda/\mu.$$

3

DEFINITION 3.1. Given a skew-shape λ/μ of size n, define a permutation in S_n, $\alpha_{\lambda/\mu}$, as follows:

> $\alpha_{\lambda/\mu}$ is the permutation which takes the LR labelling of λ/μ to the LR labelling of the conjugate skew-shape λ^t/μ^t; i.e., if the cell (s, t) in λ/μ has LR label i, then the cell (t, s) in λ^t/μ^t has label $\alpha_{\lambda/\mu}(i)$.

EXAMPLE 3.2. If λ/μ is the skew-shape (4, 3, 2, 2,)/(3, 2) then the LR labels of λ/μ are

$$\begin{array}{l} \cdot\,\cdot\,\cdot\,1, \\ \cdot\,\cdot\,2 \\ 4\,3 \\ 6\,5 \end{array}$$

while those of λ^t/μ^t are

$$\begin{array}{l} \cdot\,\cdot\,2\;1, \\ \cdot\,\cdot\,4\;3 \\ \cdot\,5 \\ 6 \end{array}$$

so that $\alpha_{\lambda/\mu} = 654231$.

Remark. Observe that if we fill in the cells of λ/μ with the numbers $1, ..., n$, starting at the leftmost cell in the bottom row and moving up the columns, then over to the right to the next column, $\alpha_{\lambda/\mu}$ is precisely the permutation obtained by reading this filling of λ/μ according to its own LR labelling. In the above example,

$$\begin{array}{l} \alpha_{\lambda/\mu} = \cdot\ \cdot\ \cdot\ 6 \\ \qquad\quad \cdot\ \cdot\ 5 \\ \qquad\quad 2\ 4 \\ \qquad\quad 1\ 3 \end{array}$$

LEMMA 3.3. *If λ/μ is a skew-shape of size n then*

(1) $\alpha_{\lambda/\mu}(1) = n$, $\alpha_{\lambda/\mu}(n) = 1$.

(2) $\alpha_{\lambda^t/\mu^t} = \alpha^{-1}_{\lambda/\mu}$.

(3) *A permutation w fits λ^t/μ^t iff the permutation $w\alpha_{\lambda/\mu}$ fits λ/μ.*

Proof. (1) and (2) are clear.

For (3), observe that putting $w(i)$ into cell i of λ^t/μ^t is the same as putting $w\alpha_{\lambda/\mu}[\alpha^{-1}_{\lambda/\mu}(i)]$ into cell $\alpha^{-1}_{\lambda/\mu}(i)$ of λ/μ. Since the fillings are all standard, the result follows.

DEFINITION 3.4. Suppose T is a standard tableau of shape v, for some partition v of n, and suppose w is any permutation in S_n. By T^w we mean the array of the same shape v, obtained by replacing each entry i in T by $w(i)$. Note that T^w need not be a standard tableau.

EXAMPLE 3.5. Take

$$T = \begin{array}{lll} 1 & 3 & 4, \\ 2 & 6 & 7 \\ 5 \end{array} \qquad w = 4\ 1\ 6\ 3\ 2\ 7\ 5.$$

Then

$$T^w = \begin{array}{lll} 4 & 6 & 3 \\ 1 & 7 & 5 \\ 2 \end{array}$$

The next three lemmas, 3.6, 3.7, and 3.8, are essential to the main result of this paper (Theorem 3.14). The proofs are long, and, as is typical of arguments in this subject, somewhat intricate. Following the suggestions of the referee and the editor, we refer the reader to [HS] for all the unpleasant details of the proofs. A good understanding of the main ideas can also be obtained by studying the Remmel–Whitney algorithm for multiplying Schur functions [RW].

We emphasize that our arguments use only well-established properties of Schensted-insertions, with no reference whatsoever to *jeu de taquin.* For the remainder of this section, we fix a skew-shape λ/μ of size n. The permutation $\alpha_{\lambda/\mu}$ has the following interesting property:

LEMMA 3.6. *Given a skew-shape λ/μ, let T be any standard Young tableau such that* $\mathrm{lp}(T)$ *fits λ/μ. Then*

$$T^{\rho\alpha_{\lambda/\mu}} \text{ is a standard tableau.}$$

Proof. See [HS].

The next two lemmas will enable us to write the permutation which, when row-inserted, produces the pair of standard tableaux $(T^{\rho\alpha_{\lambda/\mu}}, T)$.

LEMMA 3.7. *Suppose T is a standard tableau such that* $\mathrm{lp}(T)$ *fits λ/μ. Let σ be a permutation such that $(\phi \leftarrow \sigma) = (T, S)$, for some standard tableau S (of the same shape as T). Suppose the entry in T corresponding to n in S is t_n. Then row-removing $\rho\alpha_{\lambda/\mu}(t_n)$ from $T^{\lambda/\mu}$ results in bumping out $\rho\alpha_{\lambda/\mu}(\sigma(n))$. In other words, if $(\phi \leftarrow w) = (T^{\rho\alpha_{\lambda/\mu}}, S)$, then $w(n) = \rho\alpha_{\lambda/\mu}(\sigma(n))$.*

Proof. See [HS].

LEMMA 3.8. *Suppose* $\mathrm{lp}(T)$ *fits λ/μ and σ is a permutation such that $(\phi \leftarrow \sigma) = (T, S)$. Let $T^{(n-1)}$ be the tableau obtained by inserting the first $(n-1)$ letters of σ (so $T = T^{(n-1)} \leftarrow \sigma(n)$). Then* $\mathrm{lp}(T^{(n-1)})$ *fits the skew-shape $\lambda/\mu^{(n-1)}$, where $\mu^{(n-1)} = \mu \cup \{$cell of λ/μ with LR label $\sigma(n)\}$.*

Proof. See [HS].

In fact, the proof shows

COROLLARY 3.9. *If* $\mathrm{lp}(T)$ *fits λ/μ and $(\phi \leftarrow \sigma) = (T, S)$, then there is a sequence of skew-shapes $\lambda/\mu = \lambda/\mu^{(n)} \subset \lambda/\mu^{(n-1)} \subset \cdots \subset \lambda/\mu^{(1)} \subset \lambda/\mu^{(0)} = \lambda$ such that* $\mathrm{lp}(T^{(i)})$ *fits $\lambda/\mu^{(n-i)}$, where $T^{(i)} = (\phi \leftarrow \sigma(1) \leftarrow \sigma(2) \leftarrow \cdots \leftarrow \sigma(i))$ and $\mu^{(i)} = \mu \cup \{$cells in λ/μ with LR labels $\sigma(n), \sigma(n-1), ..., \sigma(n-i+1)\} = \mu^{(i-1)} \cup \{$cell of λ/μ labelled $\sigma(n-i+1)\}$ (so $\mu \subset \mu^{(1)} \subset \cdots \subset \mu^{(n-1)} \subset \lambda$). In particular,*

(1) *the cell of λ/μ labelled $\sigma(n-i+1)$ is a corner cell of $\mu^{(i)}$ for all $i \leqslant n-1$ and*

(2) *$\sigma(1)$ is a corner cell of λ.*

An example illustrating the content of the above corollary follows.

EXAMPLE 3.10. Let σ be the permutation given by

$$\sigma=\begin{pmatrix}1&2&3&4&5&6&7&8&9\\7&8&4&5&9&1&2&3&6\end{pmatrix};$$

let λ/μ be the skew-shape $\lambda/\mu=(5,4,3)/(2,1)$. Figure 1 shows how the skew-shape grows (first column of Fig. 1), and how the lattice permutation evolves, as we go through the successive insertions of $\sigma(i)$.

We now deduce the following crucial result.

THEOREM 3.11. *Suppose T is a standard Young tableau such that* $\operatorname{lp}(T)$ *fits λ/μ, and σ is a permutation satisfying $(\phi\leftarrow\sigma)=(T,S)$ for some standard tableau S (of the same shape as T). Then $(\phi\leftarrow\rho\alpha_{\lambda/\mu}\sigma)=(T^{\rho\alpha_{\lambda/\mu}},S)$.*

Proof. If ω is the permutation giving the pair $(T^{\rho\alpha_{\lambda/\mu}},S)$, Lemma 3.7 shows that $\omega(n)=\rho\alpha_{\lambda/\mu}(\sigma(n))$.

To complete the proof, we continue to row-remove entries in $T^{\rho\alpha_{\lambda/\mu}}$, corresponding to entries in S. Let $(T^{\rho\alpha_{\lambda/\mu}})^{(n-1)}$ be the tableau obtained by row-removing from $T^{\rho\alpha_{\lambda/\mu}}$, the entry corresponding to n in S, i.e., $(T^{\rho\alpha_{\lambda/\mu}})^{(n-1)}$ is the result of bumping out $\rho\alpha_{\lambda/\mu}(\sigma(n))$.

Consider the pair $((T^{\rho\alpha_{\lambda/\mu}})^{(n-1)},S_{\leqslant n-1})$, where $S_{\leqslant n-1}$ is S with entry n erased. By Lemma 3.7, since bumping paths were "preserved," we have $(T^{\rho\alpha_{\lambda/\mu}})^{n-1}=(T^{(n-1)})^{\rho\alpha_{\lambda/\mu}}$ if $T^{(n-1)}$ denotes the tableau obtained by row-removing the entry corresponding to n in (T,S), i.e., $T^{(n-1)}$ is the result of the row-insertion $\phi\leftarrow\sigma(1)\leftarrow\cdots\leftarrow\sigma(n-1)$.

But the previous lemma says that the pair $(T^{(n-1)},S_{\leqslant n-1})$, which corresponds to the two-line array $\left\{\begin{smallmatrix}1&\cdots&n-1\\\sigma(1)&\cdots&\sigma(n-1)\end{smallmatrix}\right\}$, has the property that $\operatorname{lp}(T^{(n-1)})$ fits some sub-skew-shape $\lambda/\mu^{(n-1)}$ of λ/μ. Therefore the arguments of Lemma 3.7 apply to row-removing the entry corresponding to $(n-1)$ in S, in the pair

$$((T^{(n-1)})^{\rho\alpha_{\lambda/\mu}},S_{\leqslant n-1})=((T^{\rho\alpha_{\lambda/\mu}})^{(n-1)},S_{\leqslant n-1}),$$

giving $\rho\alpha_{\lambda/\mu}\sigma(n-1)$ as the bumped-out entry. Continuing this argument produces the two-line array

$$\left\{\begin{matrix}1&2&\cdots n-1&n\\\rho\alpha_{\lambda/\mu}\sigma(1)&\cdots&\rho\alpha_{\lambda/\mu}\sigma(n-1)&\rho\alpha_{\lambda/\mu}\sigma(n)\end{matrix}\right\}$$

as the result of row-removal in the pair $(T^{\rho\alpha_{\lambda/\mu}},S)$.

COROLLARY 3.12. *Suppose $(\phi\leftarrow\sigma)=(T,S)$ and* $\operatorname{lp}(T)$ *fits λ/μ. Then $(\phi\leftarrow(\alpha_{\lambda/\mu}\circ\sigma))=((T^{\rho\alpha_{\lambda/\mu}})^{t}_{\text{evac}},S^{t})$.*

Proof. This follows immediately from Theorems 2.1(8) and 3.11.

DEFINITION 3.13. Denote by $\mathscr{C}_{\nu}^{\lambda/\mu}$ the set of all standard tableaux T of shape ν, such that $\operatorname{lp}(T)$ is a LR filling of λ/μ.

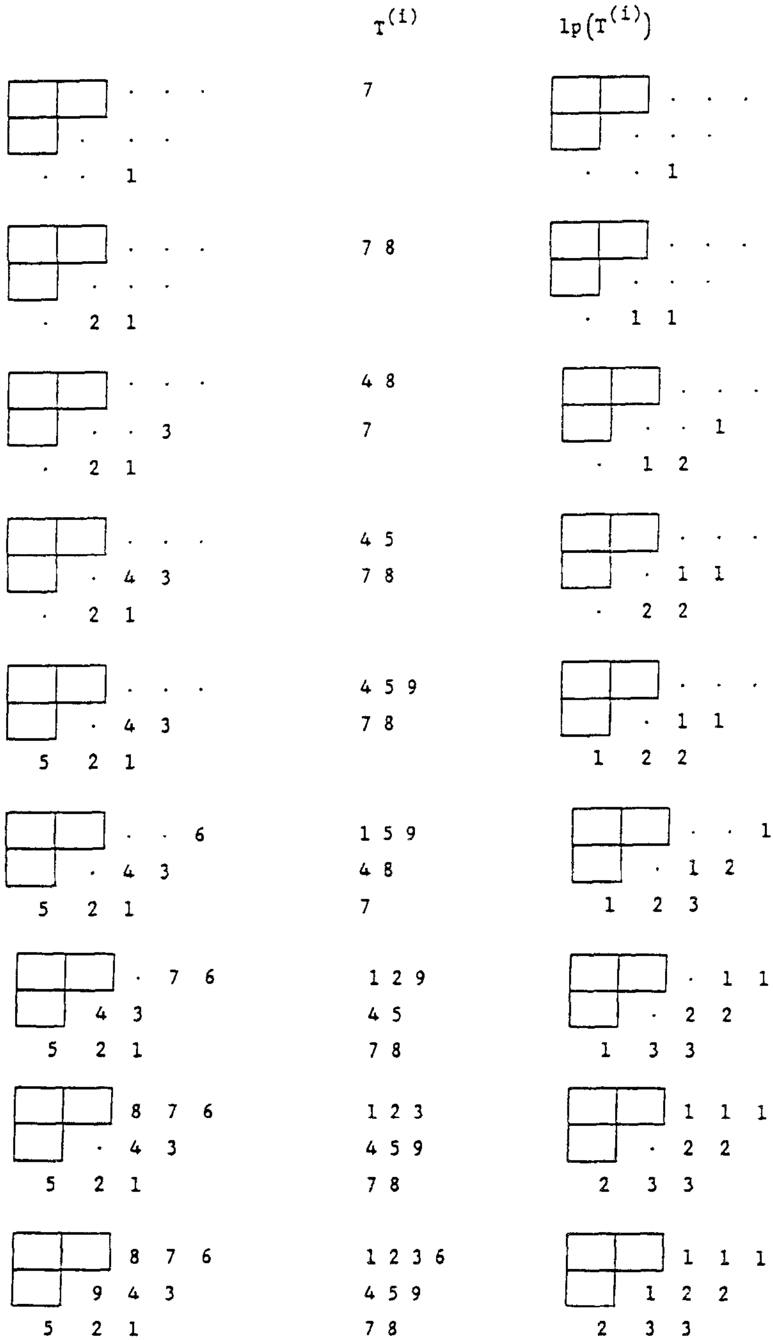


FIG. 1. $\mathrm{lp}(T^{(i)})$ is obtained as follows: In cell of λ/μ with LR label x, put the row number of x in $T^{(i)}$.

(Thus $|\mathscr{C}_\nu^{\lambda/\mu}|$ is just the LR coefficient $c_{\mu\nu}^{\lambda}$.) We are now ready to state the main result of this paper:

THEOREM 3.14. *Given a skew-shape λ/μ, define a map $\varphi_\nu^{\lambda/\mu}$ on $\mathscr{C}_\nu^{\lambda/\mu}$ by*

$$\varphi_\nu^{\lambda/\mu}(T) = [T^{\rho\sigma_{\lambda/\mu}}]_{\text{evac}}^t.$$

Then $\varphi_\nu^{\lambda/\mu}$ is a bijection from $\mathscr{C}_\nu^{\lambda/\mu}$ to $\mathscr{C}_{\nu^t}^{\lambda^t/\mu^t}$.

Proof. By Lemma 3.6, $\varphi_\nu^{\lambda/\mu}(T)$ is a standard tableau and has shape ν^t, since T has shape ν. Also clearly $\varphi_\nu^{\lambda/\mu}$ is 1–1 (evacuation is an involution).

Claim. $\text{lp}[(T^{\rho\alpha_{\lambda/\mu}})_{\text{evac}}^t]$ fits λ^t/μ^t.

Pick any permutation σ such that $(\phi \leftarrow \sigma) = (T, S)$ (for some tableau S of shape ν). We have $T \in \mathscr{C}_\nu^{\lambda/\mu} \Rightarrow \text{lp}(T)$ fits λ/μ, so by Corollary 3.12, $(\phi \leftarrow \alpha_{\lambda/\mu}\sigma) = ((T^{\rho\alpha_{\lambda/\mu}})_{\text{evac}}^t, S^t)$. Hence by Corollary 2.3, to establish our claim it suffices to show that $\rho(\alpha_{\lambda/\mu}\sigma)^{-1}$ fits λ^t/μ^t. But $\text{lp}(T)$ fits

$$\begin{aligned} \lambda/\mu &\Rightarrow \rho\sigma^{-1} \text{ fits } \lambda/\mu \qquad \text{(Corollary 2.3)} \\ &\Rightarrow (\rho\sigma^{-1})\,\alpha_{\lambda/\mu}^{-1} \text{ fits } \lambda^t/\mu^t \qquad \text{(Lemma 3.3)}, \end{aligned}$$

and the claim follows.

Thus $\varphi_\nu^{\lambda/\mu}$ is a 1–1 map from $\mathscr{C}_\nu^{\lambda/\mu}$ into $\mathscr{C}_{\nu^t}^{\lambda^t/\mu^t}$. Therefore $|\mathscr{C}_\nu^{\lambda/\mu}| \leqslant |\mathscr{C}_{\nu^t}^{\lambda^t/\mu^t}|$, and a symmetric argument shows that $|\mathscr{C}_{\nu^t}^{\lambda^t/\mu^t}| \leqslant |\mathscr{C}_\nu^{\lambda/\mu}|$; i.e., the two sets have the same cardinality.

Hence the map $\varphi_\nu^{\lambda/\mu} : \mathscr{C}_\nu^{\lambda/\mu} \to \mathscr{C}_{\nu^t}^{\lambda^t/\mu^t}$ is also onto.

EXAMPLE 3.15. If $\lambda/\mu = (5, 4, 3)/(2, 1)$ and

$$T = \begin{array}{cccc} 1 & 2 & 3 & 6, \\ 4 & 5 & 9 & \\ 7 & 8 & & \end{array}$$

we observe that $\text{lp}(T)$ gives the following LR filling of λ/μ:

•	•	1	1	1
•	1	2	2	
2	3	3		

From the labelling

$$\begin{array}{ccccc} \bullet & \bullet & 6 & 8 & 9 \\ \bullet & 3 & 5 & 7 & \\ 1 & 2 & 4 & & \end{array},$$

we obtain $\alpha_{\lambda/\mu} = 9\ 8\ 6\ 7\ 5\ 3\ 4\ 2\ 1$, so $\rho\alpha_{\lambda/\mu} = 1\ 2\ 4\ 3\ 5\ 7\ 6\ 8\ 9$. Then

$$T^{\rho\alpha_{\lambda/\mu}} = \begin{array}{cccc} 1 & 2 & 4 & 7 \\ 3 & 5 & 9 & \\ 6 & 8 & & \end{array} \quad \text{whose evacuation is} \quad \begin{array}{cccc} 1 & 2 & 4 & 9 \\ 3 & 5 & 7 & \\ 6 & 8 & & \end{array}.$$

Hence

$$(T^{\rho\alpha_{\lambda/\mu}})^t_{\text{evac}} = \begin{array}{ccc} 1 & 3 & 6 \\ 2 & 5 & 8 \\ 4 & 7 & \\ 9 & & \end{array},$$

which indeed gives a LR filling of λ/μ:

$$\begin{array}{ccc} \bullet & \bullet & 1 \\ \bullet & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & \\ 4 & & \end{array}.$$

We point out the effect of conjugating a permutation by $\alpha_{\lambda/\mu}$ in

COROLLARY 3.16. *Suppose T is a standard tableau such that* $\text{lp}(T)$ *is a LR filling of λ/μ, and suppose τ is an involution such that* $(\phi \leftarrow \tau) = (T, T)$. *Then* $(\phi \leftarrow \alpha_{\lambda/\mu}\tau\alpha^{-1}_{\lambda/\mu}) = ((T^{\rho\alpha_{\lambda/\mu}})_{\text{evac}}, (T^{\rho\alpha_{\lambda/\mu}})_{\text{evac}})$.

Proof. Applying Theorem 3.11 to τ, we obtain $(\phi \leftarrow \rho\alpha_{\lambda/\mu} \circ \tau) = (T^{\rho\alpha_{\lambda/\mu}}, T)$ and so $(\phi \leftarrow \tau\alpha^{-1}_{\lambda/\mu}\rho) = (T, T^{\rho\alpha_{\lambda/\mu}})$ (since $\tau = \tau^{-1}$). Now Theorem 3.11 applies to the permutation $\tau\alpha^{-1}_{\lambda/\mu}\rho$, giving $(\phi \leftarrow \rho\alpha_{\lambda/\mu}(\tau\alpha^{-1}_{\lambda/\mu}\rho)) = (T^{\rho\alpha_{\lambda/\mu}}, T^{\rho\alpha_{\lambda/\mu}})$. Hence $(\phi \leftarrow \alpha_{\lambda/\mu}\tau\alpha^{-1}_{\lambda/\mu}) = ((T^{\rho\alpha_{\lambda/\mu}})_{\text{evac}}, (T^{\rho\alpha_{\lambda/\mu}})_{\text{evac}})$, (by definition of evacuation; see Theorem 2.1).

4

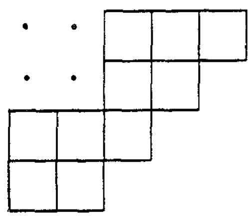
In this section we investigate further properties of the permutation $\alpha_{\lambda/\mu}$ and show that the map $\varphi_{\nu}^{\lambda/\mu}$ described at the end of the last section is the inverse of the "reverse" map $\varphi_{\nu'}^{\lambda'/\mu'}$.

DEFINITION 4.1. Given a skew-shape λ/μ, define its *complement*, denoted $\overline{\lambda/\mu}$, geometrically as follows:

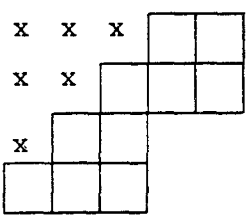
$\overline{\lambda/\mu}$ is obtained from λ/μ by reflecting the skew-shape once about the horizontal axis, then once about the vertical axis. Formally, if $\lambda=(\lambda_1 \geqslant \cdots \geqslant \lambda_l \geqslant 0)$, $\mu=(\mu_1 \geqslant \cdots \geqslant \mu_l \geqslant 0)$ then to obtain $\overline{\lambda/\mu}$, take $\bar{\mu}=(\lambda_1-\mu_l, \lambda_1-\mu_{l-1}, ..., \lambda_l-\mu_1)$ and $\bar{\lambda}=(\lambda_1-\lambda_l, \lambda_1-\lambda_{l-1}, ..., 0)$; then $\overline{(\lambda/\mu)}=\bar{\lambda}/\bar{\mu}$.

Note that $\bar{\mu}$, $\bar{\lambda}$ are respectively the complements of μ and λ in a λ_1 by l rectangle (λ_1 rows, l columns).

EXAMPLE 4.2. Take $\lambda/\mu=(5,4,3,2)/(2,2)$:



Then $\overline{\lambda/\mu}$ is simply λ/μ viewed from the northern boundary (row 1) of λ (turn the paper upside down to obtain $\overline{\lambda/\mu}$):

$\overline{\lambda/\mu}=$  $=\bar{\mu}/\bar{\lambda}=(5,5,3,3)/(3,2,1)$.

LEMMA 4.3. *Given λ/μ. Then*

(1) $\overline{\overline{(\lambda/\mu)}}=\lambda/\mu$

(2) $\alpha_{\overline{\lambda/\mu}}=\rho\alpha_{\lambda/\mu}\rho$

(3) *A permutation ω fits $\lambda/\mu \Leftrightarrow \rho\omega\rho$ fits $\overline{\lambda/\mu}$*

(4) *Transposing and complementing commute, i.e., $(\overline{\lambda/\mu})^t=\overline{\lambda^t/\mu^t}$ (where the transpose of λ/μ is defined to be λ^t/μ^t).*

Proof. (1) is obvious.

(2) Recall that to obtain $\alpha_{\lambda/\mu}$, we fill λ/μ with 1, 2, ..., working up the column, starting at the bottom of the leftmost column, and then we read the filling according to the LR labels for λ/μ (i.e., right to left, top to bottom along rows).

Now observe that $\alpha_{\lambda/\mu}\rho$ is this filling read in reverse order, i.e., in the order of the LR labelling for $\bar{\mu}/\bar{\lambda}$, and that the elements in $\alpha_{\lambda/\mu}\rho$ are exactly the complements of the elements in $\alpha_{\bar{\mu}/\bar{\lambda}}$ ($\alpha_{\bar{\mu}/\bar{\lambda}}$ would fill the cells of λ/μ with 1 2, ..., from top to bottom *down* the columns, and starting at the rightmost column). Thus $\alpha_{\lambda/\mu}\rho = \rho\alpha_{\bar{\mu}/\bar{\lambda}}$.

(3) The same geometric reasoning shows that if ω fits λ/μ, then the operations of complementing and reversing ω produce a permutation which fits $\overline{\lambda/\mu}$. But this is precisely $\rho\omega\rho$. Now use the fact that $\rho^2 = 1$ and the first statement of this lemma to obtain the reverse impliction.

(4) $(\overline{\lambda/\mu})^t = (\bar{\mu}/\bar{\lambda})^t = \bar{\mu}^t/\bar{\lambda}^t = \overline{\mu^t}/\overline{\lambda^t} = \overline{\lambda^t/\mu^t}$.

LEMMA 4.4. $T \in \mathscr{C}_\nu^{\lambda/\mu} \Leftrightarrow T_{\text{evac}} \in \mathscr{C}_\nu^{\bar{\mu}/\bar{\lambda}}$; *i.e.*, $\text{lp}(T)$ *fits* $\lambda/\mu \Leftrightarrow \text{lp}(T_{\text{evac}})$ *fits* $\overline{\lambda/\mu}$.

Proof. Let τ be the involution such that $(\phi \leftarrow \tau) = (T, T)$. Then $\rho\tau\rho$ is the corresponding involution for T_{evac}:

$$\begin{aligned} \text{lp}(T) \text{ fits } \lambda/\mu &\Leftrightarrow \rho\tau^{-1} = \rho\tau \text{ fits } \lambda/\mu && \text{(Corollary 2.3)} \\ &\Leftrightarrow \rho(\rho\tau)\,\rho = \tau\rho \text{ fits } \overline{\lambda|\mu} && \text{(Lemma 4.3)} \\ &\Leftrightarrow \text{lp}(T_{\text{evac}}) \text{ fits } \overline{\lambda/\mu} && \text{by Theorem 2.2,} \end{aligned}$$

since $(\rho\tau\rho)^{-1} = \rho\tau\rho$.

Remark. This lemma says that the skew-Schur functions $s_{\lambda/\mu}$ and $s_{\bar{\mu}/\bar{\lambda}}$ are equal. This is easy to see by means of a direct bijection between the associated semistandard skew-tableaux.

THEOREM 4.5. *Suppose* $T \in \mathscr{C}_\nu^{\lambda/\mu}$. *Then* $[(T_{\text{evac}})^{\alpha_{\lambda/\mu}\rho}]^t \in \mathscr{C}_{\nu^t}^{\lambda^t/\mu^t}$ *and* $(T_{\text{evac}})^{\alpha_{\lambda/\mu}\rho} = (T^{\rho\alpha_{\lambda/\mu}})_{\text{evac}}$.

Proof. $T \in \mathscr{C}_\nu^{\lambda/\mu} \Leftrightarrow \text{lp}(T)$ fits $\lambda/\mu \Leftrightarrow \text{lp}(T_{\text{evac}})$ fits $\overline{\lambda/\mu} = \bar{\mu}/\bar{\lambda} \Rightarrow \text{lp}([(T_{\text{evac}})^{\rho\alpha_{\overline{\lambda/\mu}}}]^t_{\text{evac}})$ fits $(\overline{\lambda/\mu})^t = \bar{\mu}^t/\bar{\lambda}^t$ by Theorem 3.14 applied to the skew-shape $\overline{\lambda/\mu}$. But $((T_{\text{evac}})^{\rho\alpha_{\overline{\lambda/\mu}}})^t_{\text{evac}} = ((T_{\text{evac}})^{\alpha_{\lambda/\mu}\rho})^t_{\text{evac}}$ by Lemma 4.3 (In particular, this says $(T_{\text{evac}})^{\alpha_{\lambda/\mu}\rho} = (T_{\text{evac}})^{\rho\alpha_{\overline{\lambda/\mu}}}$ is standard.) So we obtain that $\text{lp}((T_{\text{evac}})^{\alpha_{\lambda/\mu}\rho})^t_{\text{evac}}$ fits $\bar{\mu}^t/\bar{\lambda}^t$. Hence (by Lemma 4.4 again) $\text{lp}[(T_{\text{evac}})^{\alpha_{\lambda/\mu}\rho}]^t$ fits $\overline{\bar{\mu}^t/\bar{\lambda}^t} = \lambda^t/\mu^t$. Also, clearly $[(T_{\text{evac}})^{\alpha_{\lambda/\mu}\rho}]^t$ has shape ν^t, if T has shape ν.

Finally, if τ is the involution such that $(\phi \leftarrow \tau) = (T, T)$, then $\rho\tau\rho$ is the corresponding involution for T_{evac}.

By Corollary 3.16, since $\text{lp}(T_{\text{evac}})$ fits $\overline{\lambda/\mu}$,

$$(\{(T_{\text{evac}})^{\rho\alpha_{\overline{\lambda/\mu}}}\}_{\text{evac}}, \{(T_{\text{evac}})^{\rho\alpha_{\overline{\lambda/\mu}}}\}_{\text{evac}}) = (\phi \leftarrow \alpha_{\overline{\lambda/\mu}}(\rho\tau\rho)\,\alpha_{\overline{\lambda/\mu}}^{-1}) = (\phi \leftarrow \rho\alpha_{\lambda/\mu}\tau\alpha_{\lambda/\mu}^{-1}\rho) \qquad \text{by Lemma 4.3,}$$

so $(\phi \leftarrow \alpha_{\lambda/\mu}\tau\alpha_{\lambda/\mu}^{-1}) = ((T_{\text{evac}})^{\rho\alpha_{\overline{\lambda/\mu}}}, (T_{\text{evac}})^{\rho\alpha_{\overline{\lambda/\mu}}})$. But then again by Corollary 3.16, $(T^{\rho\alpha_{\lambda/\mu}})_{\text{evac}} = (T_{\text{evac}})^{\rho\alpha_{\overline{\lambda/\mu}}} = (T_{\text{evac}})^{\alpha_{\lambda/\mu}\rho}$.

COROLLARY 4.6. *The map* $\varphi_\nu^{\lambda/\mu} : \mathscr{C}_\nu^{\lambda/\mu} \to \mathscr{C}_{\nu^t}^{\lambda^t/\mu^t}$ *and the map* $\varphi_{\nu^t}^{\lambda^t/\mu^t} : \mathscr{C}_{\nu^t}^{\lambda^t/\mu^t} \to \mathscr{C}_\nu^{\lambda/\mu}$ *are inverses. Equivalently,* $\text{lp}(T)$ *fits* λ/μ *iff* $\text{lp}((T^{\rho\alpha_{\lambda/\mu}})^t_{\text{evac}})$ *fits* λ^t/μ^t.

Proof.

$$\begin{aligned}
\varphi_{\nu^t}^{\lambda^t/\mu^t}\varphi_\nu^{\lambda/\mu}\cdot(T) &= \varphi_{\nu^t}^{\lambda^t/\mu^t}((T^{\rho\alpha_{\lambda/\mu}})^t_{\text{evac}}) = \{[(T^{\rho\alpha_{\lambda/\mu}})^t_{\text{evac}}]^{\rho\alpha_{\lambda^t/\mu^t}}\}^t_{\text{evac}} \\
&= \{[(T^{\rho\alpha_{\lambda/\mu}})_{\text{evac}}]^{\rho\alpha^{-1}_{\lambda^t/\mu^t}}\}_{\text{evac}} \\
&\qquad \text{(transposing commutes with everything)} \\
&= [\{(T_{\text{evac}})^{\alpha_{\lambda/\mu}\rho}_{\text{evac}}\}^{\rho\alpha^{-1}_{\lambda/\mu}}]_{\text{evac}} \qquad \text{(by Theorem 4.5)} \\
&= (T_{\text{evac}})_{\text{evac}} = T.
\end{aligned}$$

The above result enables us to deduce easily the following.

THEOREM 4.7. *Suppose* τ *is an involution. Then*

$$\begin{aligned}
\rho\tau \textit{ fits } \lambda/\mu &\Leftrightarrow \alpha_{\lambda/\mu}\tau \textit{ fits } \lambda/\mu \\
&\Leftrightarrow \alpha_{\lambda/\mu}\tau\alpha_{\lambda/\mu}^{-1} \textit{ fits } \lambda^t/\mu^t.
\end{aligned}$$

Proof. Suppose T is the standard tableau obtained by row-inserting τ, i.e., $(\phi \leftarrow \tau) = (T, T)$. Then $\rho\tau = \rho\tau^{-1}$ fits $\lambda/\mu \Leftrightarrow \text{lp}(T)$ fits λ/μ (by Corollary 2.3) $\Leftrightarrow \text{lp}[(T^{\rho\alpha_{\lambda/\mu}})^t_{\text{evac}}]$ fits λ^t/μ^t.

But by Corollary 3.16,

$$(\phi \leftarrow (\alpha_{\lambda/\mu}\tau\alpha_{\lambda/\mu}^{-1})\,\rho) = ((T^{\alpha\rho_{\lambda/\mu}})^t_{\text{evac}}, (T^{\rho\alpha_{\lambda/\mu}})^t).$$

Hence

$$\begin{aligned}
\rho\tau \text{ fits } \lambda/\mu &\Leftrightarrow \rho[(\alpha_{\lambda/\mu}\tau\alpha_{\lambda/\mu}^{-1})\,\rho]^{-1} \text{ fits } \lambda^t/\mu^t \\
&\qquad \text{(again by Corollary 2.3)} \\
&\Leftrightarrow \alpha_{\lambda/\mu}\tau^{-1}\alpha_{\lambda/\mu}^{-1} = \alpha_{\lambda/\mu}\tau\alpha_{\lambda/\mu}^{-1} \text{ fits } \lambda^t/\mu^t \\
&\Leftrightarrow \alpha_{\lambda/\mu}\tau \text{ fits } \lambda/\mu \qquad \text{(by Lemma 3.3).}
\end{aligned}$$

The reader will observe that the statement of this theorem does not involve LR fillings: the authors can, in fact, prove this result by a direct argument similar in flavour to the arguments in Section 3.

We now record the implications of these results for the case when $\mu = \phi$; i.e., when the skew-shape is, in fact, a full shape λ. In this case we note that there is a unique standard tableau R_λ giving the unique LR filling of λ, which is the row-superstandard tableau of shape λ, obtained by filling the cells of λ with 1, 2, ..., starting at the top row and moving left to right along the rows.

The column superstandard tableau C_λ of shape λ is defined analogously; we note that $(C_{\lambda^t})^t = R_\lambda$.

EXAMPLE 4.8. If $\lambda = (5, 4, 1, 1)$,

$$R_\lambda = \begin{matrix} 1 & 2 & 3 & 4 & 5, \\ 6 & 7 & 8 & 9 & \\ 10 & & & & \\ 11 & & & & \end{matrix} \qquad C_\lambda = \begin{matrix} 1 & 5 & 7 & 9 & 11. \\ 2 & 6 & 8 & 10 & \\ 3 & & & & \\ 4 & & & & \end{matrix}$$

COROLLARY 4.9. *If λ is a shape then*

$$[(R_\lambda)^{\rho\alpha_\lambda}]_{\text{evac}} = C_\lambda = \{(R_\lambda)_{\text{evac}}\}^{\alpha_\lambda \rho},$$

where $\alpha_\lambda = \alpha_{\lambda/\phi}$.

Proof. Immediate from the above remarks.

APPENDIX

The authors have learned that White [Wh_2] has found a bijection between the sets $\mathscr{C}_\nu^{\lambda/\mu}$ and $\mathscr{C}_{\nu^t}^{\lambda^t/\mu^t}$ using a *jeu de taquin* approach, which apparently produces the same output as the mapping $\varphi_\nu^{\lambda/\mu}$ of this paper. The connection between the two bijections is not at all obvious from the definitions and has as yet been unexplored.

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