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ScienceDirect
Stochastic Processes and their Applications 120 (2010) 873–900
stochastic processes and their applications
www.elsevier.com/locate/spa

Law of large numbers for the maximal flow through tilted cylinders in two-dimensional first passage percolation

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Received 29 July 2009; received in revised form 14 January 2010; accepted 9 February 2010
Available online 16 February 2010

Abstract

Equip the edges of the lattice $\mathbb{Z}^2$ with i.i.d. random capacities. We prove a law of large numbers for the maximal flow crossing a rectangle in $\mathbb{R}^2$ when the side lengths of the rectangle go to infinity. The value of the limit depends on the asymptotic behaviour of the ratio of the height of the cylinder over the length of its basis. This law of large numbers extends the law of large numbers obtained in Grimmett and Kesten (1984) [6] for rectangles of particular orientation.

MSC: primary 60K35; secondary 82B43

Keywords: First passage percolation; Maximal flow; Law of large numbers

1. Introduction

The model of maximal flow in a randomly porous medium with independent and identically distributed capacities has been introduced by Chayes and Chayes [4] and Kesten [11]. The purpose of this model is to understand the behaviour of the maximum amount of flow that can cross the medium from one part to another.

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0304-4149/$ - see front matter
doi:10.1016/j.spa.2010.02.005

All the precise definitions will be given in Section 2, but let us draw the general picture in dimension d. The random medium is represented by the lattice $\mathbb{Z}^d$. We see each edge as a microscopic pipe which the fluid can flow through. To each edge e, we attach a non-negative capacity $t(e)$ which represents the amount of fluid (or the amount of fluid per unit of time) that can effectively go through the edge e. Capacities are then supposed to be random, identically and independently distributed with common distribution function F. Let A be some hyperrectangle in $\mathbb{R}^d$ and n an integer. The portion of medium that we will look at is a box B_n of basis nA and of height $2h(n)$, which nA splits into two boxes of equal volume. The boundary of B_n is thus split into two parts, A_n^1 and A_n^2. There are two protagonists in this play, two types of flows through B_n: the maximal flow τ_n for which the fluid can enter the box through A_n^1 and leave it through A_n^2, and the maximal flow ϕ_n for which the fluid enters B_n only through its bottom side and leaves it through its top side. The first quality of τ_n is that it is (almost) a subadditive quantity, whereas ϕ_n is not. The main question now is: "How do ϕ_n and τ_n behave when n is large?".

In this paper, we shall understand this question as "Is there a law of large numbers for ϕ_n and τ_n?", and let us say that such results do indeed already exist. However, it is important to stress that the orientation of A plays an important role in these results. Indeed, the first ones were obtained for "straight" boxes, i.e., when A is of the form $\prod_{i=1}^{d-1}[0, a_i] \times \{0\}$. Especially concerning the study of ϕ_n, this simplifies considerably the task. Let us draw a precise state of the art. The law of large numbers for τ_n were proved under mild hypotheses: in [11] for straight boxes and in [12] for general boxes. These results follow essentially from the subadditivity property already alluded to. Suppose that $t(e)$ has finite expectation, $\vec{v}$ denotes a unit vector orthogonal to a hyperrectangle A containing the origin of the graph, and $h(n)$ goes to infinity. Then there is a function ν defined on S^{d-1} such that:

$$\nu(\vec{v}) = \lim_{n\to\infty} \frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \quad \text{a.s. and in } L^1,$$

where $\mathcal{H}^{d-1}(nA)$ is the $(d-1)$-dimensional Hausdorff measure of nA. If the height function $h(n)$ is negligible compared to n, ϕ_n satisfies the same law of large numbers as n (see for example [12]). Otherwise, the law of large numbers for ϕ_n was proved only for straight boxes, with suboptimal assumptions on the height h, the moments of F and on $F(\{0\})$, in [11]. In dimension 2, this was first studied in [6]. The assumption on $F(\{0\})$ was optimized in [14,15]. The assumptions on the moments of F and the height h have been improved in [12]. A specificity of the lattice $\mathbb{Z}^d$, namely its invariance under reflexions with respect to integer coordinate hyperplanes, implies that the law of large numbers is the same for ϕ_n and τ_n in straight cylinders (provided $\log h(n)$ does not grow too fast).

Summarizing, τ_n is fairly well studied concerning laws of large numbers, but for ϕ_n, nothing is known when the boxes are not straight, except when the height is small compared to n (note however a related result by Garet [5], cf. also Remark 2.12). This paper aims at filling this gap, although we can do so only in dimension 2. For instance, suppose that $2h(n)/(nl(A))$ goes to $\tan(\alpha)$ when n goes to infinity, with $\alpha \in [0, \frac{\pi}{2}]$ and $l(A)$ denoting the length of the line segment A. Our main results imply, under some conditions on F and A, that:

$$\frac{\phi_n}{nl(A)} \xrightarrow[n\to\infty]{} \inf_{\widetilde{\theta}\in[\theta-\alpha,\theta+\alpha]} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)} \quad \text{a.s. and in } L^1, \tag{1}$$

where we re-encoded the function ν as follows: $\nu_{\widetilde{\theta}} := \nu(\vec{v}(\widetilde{\theta}))$ when $\vec{v}(\widetilde{\theta})$ makes an angle $\widetilde{\theta}$ with $(1, 0)$. Notice that there is no reason for the limit in (1) to be identical to ν_θ. Thus, something different happens when the boxes are not straight. Notice that this fact can already be observed when F is concentrated on one point. For instance, if $t(e) = 1$ deterministically and $2h(n)/(nl(A))$ goes to $\tan(\alpha)$ when n goes to infinity, with $\alpha > \frac{\pi}{4}$, then one may easily compute that $\nu_\theta = |\cos\theta| + |\sin\theta|$, whereas the limit of $\phi_n/(nl(A))$ is $\min\{1/|\cos\theta|, 1/|\sin\theta|\}$. Moreover the moment conditions on F that we need to prove (1) are very weak.

The paper is organized as follows. In Section 2, we give the precise definitions and state the main result of the paper. Section 3 is devoted to a deviation result for ϕ_n. In Section 4, we prove the convergence of the rescaled expectation of ϕ_n. Finally, we complete the proof of the law of large numbers for ϕ_n in Section 5.

2. Notations, background and main results

The most important notations are gathered in Sections 2.1–2.3, the relevant background is described in Section 2.4 while our main results are stated in Section 2.5.

2.1. Maximal flow on a graph

First, let us define the notion of a flow on a finite unoriented graph $G = (V, \mathcal{E})$ with set of vertices V and set of edges $\mathcal{E}$. Let $t = (t(e))_{e\in\mathcal{E}}$ be a collection of non-negative real numbers, which are called *capacities*. It means that $t(e)$ is the maximal amount of fluid that can go through the edge e per unit of time. To each edge e, one may associate two oriented edges, and we shall denote by $\overrightarrow{\mathcal{E}}$ the set of all these oriented edges. Let A and Z be two finite, disjoint, non-empty sets of vertices of G: A denotes the source of the network, and Z the sink. A function θ on $\overrightarrow{\mathcal{E}}$ is called *a flow from* A to Z *with strength* $\|\theta\|$ *and capacities* t if it is antisymmetric, i.e. $\theta_{\overrightarrow{xy}} = -\theta_{\overrightarrow{yx}}$, if it satisfies the node law at each vertex x of $V \setminus (A \cup Z)$:

$$\sum_{y\sim x} \theta_{\overrightarrow{xy}} = 0,$$

where $y \sim x$ means that y and x are neighbours on G, if it satisfies the capacity constraints:

$$\forall e \in \mathcal{E}, \quad |\theta(e)| \le t(e),$$

and if the "flow in" at A and the "flow out" at Z equal $\|\theta\|$:

$$\|\theta\| = \sum_{a\in A} \sum_{\substack{y\sim a \\ y\notin A}} \theta(\overrightarrow{ay}) = \sum_{z\in Z} \sum_{\substack{y\sim z \\ y\notin Z}} \theta(\overrightarrow{yz}).$$

The *maximal flow from* A *to* Z, denoted by $\phi_t(G, A, Z)$, is defined as the maximum strength of all flows from A to Z with capacities t. We shall in general omit the subscript t when it is understood from the context. The *max-flow min-cut theorem* (see [2] for instance) asserts that the maximal flow from A to Z equals the minimal capacity of a cut between A and Z. Precisely, let us say that $E \subset \mathcal{E}$ is a cut between A and Z in G if every path from A to Z borrows at least one edge of E. Define $V(E) = \sum_{e\in E} t(e)$ to be the capacity of a cut E. Then,

$$\phi_t(G, A, Z) = \min\{V(E) \text{ s.t. } E \text{ is a cut between } A \text{ and } Z \text{ in } G\}. \tag{2}$$

By convention, if A or Z is empty, we shall define $\phi_t(G, A, Z)$ to be zero.

2.2. On the square lattice

We shall always consider G as a piece of $\mathbb{Z}^2$. More precisely, we consider the graph $\mathbb{L} = (\mathbb{Z}^2, \mathbb{E}^2)$ having for vertices $\mathbb{Z}^2$ and for edges $\mathbb{E}^2$, the set of pairs of nearest neighbours for the standard L^1 norm. The notation $\langle x, y\rangle$ corresponds to the edge with endpoints x and y. To each edge e in $\mathbb{E}^2$ we associate a random variable $t(e)$ with values in $\mathbb{R}^+$. *We suppose that the family $(t(e), e \in \mathbb{E}^2)$ is independent and identically distributed, with a common distribution function F.* More formally, we take the product measure $\mathbb{P} = F^{\otimes\Omega}$ on $\Omega = \prod_{e\in\mathbb{E}^2}[0,\infty[$, and we write its expectation $\mathbb{E}$. If G is a subgraph of $\mathbb{L}$, and A and Z are two subsets of vertices of G, *we shall denote by $\phi(G, A, Z)$ the maximal flow in G from A to Z*, where G is equipped with capacities t. When B is a subset of $\mathbb{R}^2$, and A and Z are subsets of $\mathbb{Z}^2 \cap B$, we shall denote by $\phi(B, A, Z)$ again the maximal flow $\phi(G, A, Z)$ where G is the induced subgraph of $\mathbb{Z}^2$ with set of vertices $\mathbb{Z}^2 \cap B$.

We denote by $\overrightarrow{e}_1$ (resp. $\overrightarrow{e}_2$) the vector $(1,0) \in \mathbb{R}^2$ (resp. $(0,1)$). Let A be a non-empty line segment in $\mathbb{R}^2$. We shall denote by $l(A)$ its (euclidean) length. All line segments will be supposed to be closed in $\mathbb{R}^2$. We denote by $\vec{v}(\theta)$ the vector of unit euclidean norm orthogonal to $\mathrm{hyp}(A)$, the hyperplane spanned by A, and such that there is $\theta \in [0,\pi[$ such that $\vec{v}(\theta) = (\cos\theta, \sin\theta)$. Define $\vec{v}^{\perp}(\theta) = (\sin\theta, -\cos\theta)$ and denote by a and b the endpoints of A such that $(b-a).\vec{v}^{\perp}(\theta) > 0$. For h a positive real number, *we denote by* $\mathrm{cyl}(A, h)$ *the cylinder of basis A and height $2h$*, i.e., the set

$$\mathrm{cyl}(A, h) = \{x + t\vec{v}(\theta) \mid x \in A, t \in [-h, h]\}.$$

We define also *the r-neighbourhood $\mathcal{V}(H, r)$ of a subset H of* $\mathbb{R}^d$ as

$$\mathcal{V}(H, r) = \{x \in \mathbb{R}^d \mid d(x, H) < r\},$$

where the distance is the euclidean one ($d(x, H) = \inf\{\|x - y\|_2 \mid y \in H\}$).

Now, $D(A, h)$ denotes the set of *admissible boundary conditions* on $\mathrm{cyl}(A, h)$ (see Fig. 1):

$$D(A,h) = \left\{(k, \tilde{\theta}) \mid k \in [0,1] \text{ and } \tilde{\theta} \in \left[\theta - \arctan\left(\frac{2hk}{l(A)}\right), \theta + \arctan\left(\frac{2h(1-k)}{l(A)}\right)\right]\right\}.$$

The meaning of an element $\kappa = (k, \tilde{\theta})$ of $D(A, h)$ is the following. We define

$$\vec{v}(\widetilde{\theta}) = (\cos\widetilde{\theta}, \sin\widetilde{\theta}) \quad \text{and} \quad \vec{v}^{\perp}(\widetilde{\theta}) = (\sin\widetilde{\theta}, -\cos\widetilde{\theta}).$$

In $\mathrm{cyl}(nA, h(n))$, we may define two points c and d such that c is "at height $2kh$ on the left side of $\mathrm{cyl}(A, h)$", and d is "on the right side of $\mathrm{cyl}(A, h)$" by

$$c = a + (2k-1)h\vec{v}(\theta), \quad (d - c) \text{ is orthogonal to } \vec{v}(\widetilde{\theta}) \quad \text{and}$$
$$d \text{ satisfies } \vec{cd} \cdot \vec{v}^{\perp}(\widetilde{\theta}) > 0.$$

Then we see that $D(A, h)$ is exactly the set of parameters so that c and d remain "on the sides of $\mathrm{cyl}(A, h)$".

We define also $\mathcal{D}(A, h)$, the set of angles $\tilde{\theta}$ such that there is an admissible boundary condition with angle $\tilde{\theta}$:

$$\mathcal{D}(A, h) = \left[\theta - \arctan\left(\frac{2h}{l(A)}\right), \theta + \arctan\left(\frac{2h}{l(A)}\right)\right].$$

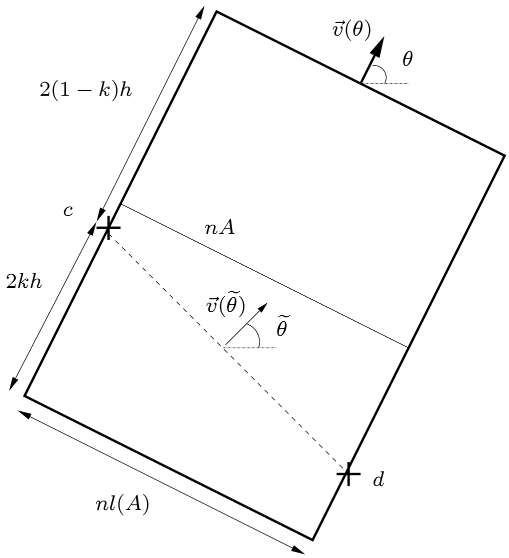

Fig. 1. An admissible boundary condition $(k, \tilde{\theta})$.

It will be useful to define the *left side (resp. right side) of* $\mathrm{cyl}(A, h)$: let $\mathrm{left}(A)$ (resp. $\mathrm{right}(A)$) be the set of vertices in $\mathrm{cyl}(A, h) \cap \mathbb{Z}^2$ such that there exists $y \notin \mathrm{cyl}(A, h)$, $\langle x, y\rangle \in \mathbb{E}^d$ and $[x, y[$, the segment that includes x and excludes y, intersects $a + [-h, h].\vec{v}(\theta)$ (resp. $b + [-h, h].\vec{v}(\theta)$).

Now, the set $\mathrm{cyl}(A, h) \setminus (c + \mathbb{R}(d - c))$ has two connected components, which we denote by $\mathcal{C}_1(A, h, k, \tilde{\theta})$ and $\mathcal{C}_2(A, h, k, \tilde{\theta})$. For $i = 1, 2$, let $A_i^{h,k,\tilde{\theta}}$ be the set of the points in $\mathcal{C}_i(A, h, k, \tilde{\theta}) \cap \mathbb{Z}^2$ which have a nearest neighbour in $\mathbb{Z}^2 \setminus \mathrm{cyl}(A, h)$:

$$A_i^{h,k,\tilde{\theta}} = \{x \in \mathcal{C}_i(A, h, k, \tilde{\theta}) \cap \mathbb{Z}^2 \mid \exists y \in \mathbb{Z}^2 \setminus \mathrm{cyl}(A, h),\ \|x - y\|_1 = 1\}.$$

We define *the flow in* $\mathrm{cyl}(A, h)$ *constrained by the boundary condition* $\kappa = (k, \tilde{\theta})$ as:

$$\phi^\kappa(A, h) := \phi\left(\mathrm{cyl}(A, h), A_1^{h,k,\tilde{\theta}}, A_2^{h,k,\tilde{\theta}}\right).$$

A special role is played by the condition $\kappa = (1/2, \theta)$, and we shall denote:

$$\tau(A, h) = \tau(\mathrm{cyl}(A, h), \vec{v}(\theta)) = \phi^{(1/2,\theta)}(A, h).$$

Let $T(A, h)$ (respectively $B(A, h)$) be the top (respectively the bottom) of $\mathrm{cyl}(A, h)$, i.e.,

$$T(A, h) = \{x \in \mathrm{cyl}(A, h) \mid \exists y \notin \mathrm{cyl}(A, h), \langle x, y\rangle \in \mathbb{E}^d \text{ and } \langle x, y\rangle \text{ intersects } A + h\vec{v}(\theta)\}$$

and

$$B(A, h) = \{x \in \mathrm{cyl}(A, h) \mid \exists y \notin \mathrm{cyl}(A, h),\ \langle x, y\rangle \in \mathbb{E}^d \text{ and } \langle x, y\rangle \text{ intersects } A - h\vec{v}(\theta)\}.$$

We shall denote the flow in $\mathrm{cyl}(A, h)$ from the top to the bottom as:

$$\phi(A, h) = \phi(\mathrm{cyl}(A, h), \vec{v}(\theta)) = \phi(\mathrm{cyl}(A, h), T(A, h), B(A, h)).$$

2.3. Duality

The main reason why dimension 2 is easier to deal with than dimension $d \geq 3$ is duality. Planar duality implies that there are only $O(h^2)$ admissible boundary conditions on $\mathrm{cyl}(A, h)$. Let us go a bit into the details.

The dual lattice $\mathbb{L}^*$ of $\mathbb{L}$ is constructed as follows: place a vertex in the centre of each face of $\mathbb{L}$ and join two vertices in $\mathbb{L}^*$ if and only if the corresponding faces of $\mathbb{L}$ share an edge. To each edge e^* of $\mathbb{L}^*$, we assign the time coordinate $t(e)$, where e is the unique edge of $\mathbb{E}^2$ crossed by e^*. Now, let A be a line segment in $\mathbb{R}^2$. Let G_A be the induced subgraph of $\mathbb{L}$ with set of vertices $\mathrm{cyl}(A, h) \cap \mathbb{Z}^2$. Let G_A^* be the planar dual of G_A in the following sense: G_A^* has set of edges $\{\mathrm{e}^* \text{ s.t. } e \in G_A\}$, and set of vertices those vertices which belong to this set of edges. Now, we define $\mathrm{left}^*(A)$ (resp. $\mathrm{right}^*(A)$) as the set of vertices v of G_A^* which have at least one neighbour in $\mathbb{L}^*$ which is not in G_A and such that there exists an edge e^* in G_A^* with $v \in \mathrm{e}^*$ and $\mathrm{e}^* \cap \mathrm{left}(A) \neq \emptyset$ (resp. $\mathrm{e}^* \cap \mathrm{right}(A) \neq \emptyset$).

It is well known that the (planar) dual of a cut between the top and the bottom of $\mathrm{cyl}(A, h)$ is a self-avoiding path from "left" to "right". Furthermore, if the cut is minimal for the inclusion, the dual self-avoiding path has only one vertex on the left boundary of the dual of $A \cap \mathbb{Z}^2$ and one vertex on the right boundary. The following lemma is a formulation in our setting of those classical duality results (see for instance [6] p.358 and [2], p.47).

Lemma 2.1. *Let A be a line segment $\mathbb{R}^2$ and h be a positive real number. If E is a set of edges, let*

$$E^* = \{\mathrm{e}^* \mid e \in E\}.$$

If E is a cut between $B(A, h)$ and $T(A, h)$, minimal for the inclusion, then E^* is a self-avoiding path from $\mathrm{left}^*(A)$ *to* $\mathrm{right}^*(A)$ *such that exactly one point of E^* belongs to* $\mathrm{left}^*(A)$*, exactly one point of E^* belongs to* $\mathrm{right}^*(A)$*, and these two points are the endpoints of the path.*

An immediate consequence of this planar duality is the following.

Lemma 2.2. *Let A be any line segment in $\mathbb{R}^2$ and h a positive real number. Then,*

$$\phi(A, h) = \min_{\kappa \in D(A,h)} \phi^{\kappa}(A, h).$$

Notice that the condition κ belongs to the non-countable set $D(A, h)$, but the graph is discrete so $\phi^{\kappa}(A, h)$ takes only a finite number of values when $\kappa \in D(A, h)$. Precisely, there is a finite subset $\tilde{D}(A, h)$ of $D(A, h)$, such that:

$$\mathrm{card}(\tilde{D}(A, h)) \leq C_4 h^2, \tag{3}$$

for some universal constant C_4, and:

$$\phi(A, h) = \min_{\kappa \in \tilde{D}(A,h)} \phi^{\kappa}(A, h).$$

2.4. Background

First, let us recall some facts concerning the behaviour of $\tau(nA, h(n))$ when n and $h(n)$ go to infinity. Using a subadditive argument and deviation inequalities, Rossignol and Théret have proved in [12] that $\tau(nA, h(n))$ satisfies a law of large numbers:

Theorem 2.3. *We suppose that*

$$\int_{[0,\infty[} x \, \mathrm{d}F(x) < \infty.$$

For every unit vector $\vec{v}(\theta) = (\cos\theta, \sin\theta)$, there exists a constant ν_θ depending on F, d and θ, such that for every non-empty line segment A orthogonal to $\vec{v}(\theta)$ and of euclidean length $l(A)$, for every height function $h : \mathbb{N} \to \mathbb{R}^+$ satisfying $\lim_{n\to\infty} h(n) = +\infty$, we have

$$\lim_{n\to\infty} \frac{\tau(nA, h(n))}{nl(A)} = \nu_\theta \quad \text{in } L^1.$$

Moreover, if the origin of the graph belongs to A, or if

$$\int_{[0,\infty[} x^2 \, \mathrm{d}F(x) < \infty,$$

then

$$\lim_{n\to\infty} \frac{\tau(nA, h(n))}{nl(A)} = \nu_\theta \quad a.s.$$

Under the added assumption that $\lim_{n\to\infty} h(n)/n = 0$, the variable $\phi(nA, h(n))$ satisfies the same law of large numbers as $\tau(nA, h(n))$, under the same conditions.

This law of large numbers holds in fact for every dimension $d \geq 2$. Let us remark that (in dimension two) ν_θ is equal to $\mu(\vec{v}^\perp(\theta)) = \mu(\vec{v}(\theta))$, where $\mu(.)$ is the time-constant function of first passage percolation as defined in [10], (3.10) p. 158. This equality follows from the duality considerations of Section 2.3 and standard first passage percolation techniques (see also Theorem 5.1 in [6]) that relate cylinder passage times to unrestricted passage times (as in [7], Theorem 4.3.7 for instance). Boivin has also proved a very similar law of large numbers (see Theorem 6.1 in [1]). Notice that for the definition of $\mu(.)$, Kesten requires only the existence of the first moment of the law F in the proof from [10], and it can also be defined under the weaker condition $\int_0^\infty (1 - F(x))^4 \, \mathrm{d}x < \infty$.

One consequence of this equality between ν and μ is that $\theta \mapsto \nu_\theta$ is either constant equal to zero, or always non-zero. In fact the following property holds (cf. [10], Theorem 6.1 and Remark 6.2 p. 218):

Proposition 2.4. *We suppose that $\int_{[0,+\infty[} x \, \mathrm{d}F(x) < \infty$. Then ν_θ is well defined for all θ, and we have*

$$\nu_\theta > 0 \iff F(0) < 1/2.$$

There exists a law of large numbers for the variable $\phi(nA, h(n))$ when the rectangle we consider is straight, i.e., $\theta = 0$. It has been proved in [6], Corollary 4.2, that:

Theorem 2.5. *Suppose that $A = [0, 1] \times \{0\}$, $\int_{[0,+\infty[} x \, \mathrm{d}F(x) < \infty$,*

$$h(n) \xrightarrow[n\to\infty]{} \infty \quad \text{and} \quad \frac{\log h(n)}{n} \xrightarrow[n\to\infty]{} 0.$$

Then,

$$\frac{\phi(nA, h(n))}{n} \xrightarrow[n\to\infty]{a.s} \nu_0.$$

Remark 2.6. Notice that in [6], the condition on F is in fact weakened to $\int_0^\infty (1 - F(x))^4 \, dx < \infty$, obtaining the convergence to $\mu((0, 1))$. However, our definition of ν_θ requires a moment of order 1.

Finally, let us remark that Garet [5] proved a law of large numbers for the maximal flow between a compact convex set $\Lambda \subset \mathbb{R}^2$ and infinity. This is somewhat related to our main result, Theorem 2.8, see Remark 2.12. Before stating Garet's result, we need some notations. For every convex bounded set $\Lambda \subset \mathbb{R}^2$, we denote by $\partial^* \Lambda$ the set of all the points x of the boundary $\partial \Lambda$ of Λ where Λ admits a unique outer normal, which is denoted by $\vec{v}_\Lambda(x)$. We denote the coordinates of $\vec{v}_\Lambda(x)$ by $(\cos(\theta(\Lambda, x)), \sin(\theta(\Lambda, x)))$ for every x in $\partial^* \Lambda$. We denote by $\sigma(\Lambda)$ the maximal flow from Λ to infinity. Let $\mathcal{H}^1$ be the one-dimensional Hausdorff measure. Theorem 2.1 in [5] is the following:

Theorem 2.7. *We suppose that* $F(0) < 1/2$ *and that*

$$\exists \gamma > 0 \quad \int_{[0,+\infty[} e^{\gamma t(e)} \, dF(x) < \infty.$$

Then, for each bounded convex set $\Lambda \subset \mathbb{R}^2$ *with the origin of the graph 0 in its interior, we have*

$$\lim_{n\to\infty} \frac{\sigma(n\Lambda)}{n} = \int_{\partial^* \Lambda} \nu_{\theta(\Lambda,x)} d\mathcal{H}^1(x) = \mathcal{I}(\Lambda) > 0. \tag{4}$$

2.5. Main result

We recall that for all $n \in \mathbb{N}$, we have defined

$$\mathcal{D}(nA, h(n)) = \left[\theta - \arctan\left(\frac{2h(n)}{nl(A)}\right), \theta + \arctan\left(\frac{2h(n)}{nl(A)}\right)\right].$$

We may now state our main result.

Theorem 2.8. *Let* A *be a non-empty line segment in* $\mathbb{R}^2$, *with euclidean length* $l(A)$. *Let* $\theta \in [0, \pi[$ *be such that* $(\cos\theta, \sin\theta)$ *is orthogonal to* A *and* $(h(n))_{n\geq 0}$ *be a sequence of positive real numbers such that:*

$$\begin{cases} h(n) \xrightarrow[n\to\infty]{} +\infty, \\ \dfrac{\log h(n)}{n} \xrightarrow[n\to\infty]{} 0. \end{cases} \tag{5}$$

Define:

$$\overline{\mathcal{D}} = \limsup_{n\to\infty} \mathcal{D}(nA, h(n)) = \bigcap_{N\geq 1} \bigcup_{n\geq N} \mathcal{D}(nA, h(n)),$$

and

$$\underline{\mathcal{D}} = \liminf_{n\to\infty} \mathcal{D}(nA, h(n)) = \bigcup_{N\geq 1} \bigcap_{n\geq N} \mathcal{D}(nA, h(n)).$$

Suppose that F has a finite moment of order 1:

$$\int_0^\infty x \, \mathrm{d}F(x) < \infty. \tag{6}$$

Then,

$$\liminf_{n\to\infty} \frac{\mathbb{E}[\phi(nA, h(n))]}{nl(A)} = \inf\left\{ \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)} \mid \widetilde{\theta} \in \overline{\mathcal{D}} \right\} \tag{7}$$

and

$$\limsup_{n\to\infty} \frac{\mathbb{E}[\phi(nA, h(n))]}{nl(A)} = \inf\left\{ \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)} \mid \widetilde{\theta} \in \underline{\mathcal{D}} \right\}. \tag{8}$$

Moreover, if

$$\int_0^\infty x^2 \, \mathrm{d}F(x) < \infty, \tag{9}$$

or if:

$$0 \textit{ is the middle of } A, \tag{10}$$

then

$$\liminf_{n\to\infty} \frac{\phi(nA, h(n))}{nl(A)} = \inf\left\{ \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)} \mid \widetilde{\theta} \in \overline{\mathcal{D}} \right\} \quad a.s.$$

and

$$\limsup_{n\to\infty} \frac{\phi(nA, h(n))}{nl(A)} = \inf\left\{ \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)} \mid \widetilde{\theta} \in \underline{\mathcal{D}} \right\} \quad a.s.$$

Remark 2.9. It is likely that condition (6) can be weakened to $\int_0^\infty (1 - F(x))^4 \, \mathrm{d}x$, as in Theorem 2.5. This would require to define ν a bit differently.

Corollary 2.10. *We suppose that conditions (5) on h are satisfied. We suppose also that there is some $\alpha \in \left[0, \frac{\pi}{2}\right]$ such that:*

$$\frac{2h(n)}{nl(A)} \xrightarrow[n\to\infty]{} \tan\alpha.$$

Then, if condition (6) on F is satisfied, we have

$$\lim_{n\to\infty} \frac{\phi(nA, h(n))}{nl(A)} = \inf\left\{ \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)} \mid \widetilde{\theta} \in [\theta - \alpha, \theta + \alpha] \right\} \quad in\ L^1.$$

Moreover, if condition (9) or (10) are satisfied, then

$$\lim_{n\to\infty} \frac{\phi(nA, h(n))}{nl(A)} = \inf\left\{ \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)} \mid \widetilde{\theta} \in [\theta - \alpha, \theta + \alpha] \right\} \quad a.s.$$

It has already been remarked in [13] (see the discussion after Theorem 2) that the condition on h is the good one to have positive speed when one allows edge capacities to be null with positive probability.

Remark 2.11. Notice that Theorem 2.8 is consistent with Theorem 2.5, the existing law of large numbers for $\phi(nA, h(n))$ in the straight case. Indeed, it is known that ν satisfies the weak triangle inequality (see Section 4.4 in [12]), and for symmetry reasons, it implies that when $\theta \in \{0, \pi/2\}$, the function $\widetilde{\theta} \mapsto \nu_{\widetilde{\theta}}/\cos(\widetilde{\theta} - \theta)$ is minimum for $\widetilde{\theta} = \theta$ and thus, Theorem 2.8 implies that $\phi(nA, h(n))/(nl(A))$ converges to ν_0, the limit of $\tau(nA, h(n))/(nl(A))$, when $\mathrm{cyl}(nA, h(n))$ is a straight cylinder. In fact, the same phenomenon occurs for any θ such that there is a symmetry axis of direction θ for the lattice $\mathbb{Z}^2$. These directions in $[0, \pi[$ are of course $\{0, \pi/4, \pi/2, 3\pi/4\}$. Also, Corollary 2.10 is consistent with the fact that for general boxes, when $h(n)$ is small with respect to n, $\phi(nA, h(n))/(nl(A))$ and $\tau(nA, h(n))/(nl(A))$ have the same limit.

Remark 2.12. Theorems 2.8 and 2.7are related. First, they are stated in dimension two only, because both proofs use the duality of the planar graph to define the path which is the dual of a cutset, and then the fact that such paths can be glued together if they have a common endpoint. These properties hold only in dimension two: the dual of an edge in dimension greater than three is a unit surface, and it is much more difficult to study the boundary of a surface. This is the reason why these theorems are not yet generalized in higher dimensions (see also Remark 5.1). Moreover, the expressions of the limits $\mathcal{I}(\Lambda)$ and $\eta_{\theta,h}$ appearing in these theorems are very similar. On one hand, the constant $\eta_{\theta,h}$ is the infimum of the integral of ν along the segments that cut the top from the bottom of $\mathrm{cyl}(A, h(n)/n)$ for large n. Since ν satisfies the weak triangle inequality, $\eta_{\theta,h}$ is also equal to infimum of the integral of ν along the polyhedral curves that have the same property of cutting. On the other hand, Garet only has to consider the case of a polyhedral convex set Λ during his proof, and he proves the important following property: if $\Lambda \subset \Lambda'$, where Λ and Λ' are polyhedral and Λ is convex, then $\mathcal{I}(\Lambda) \leq \mathcal{I}(\Lambda')$. Thus, for a polyhedral convex set Λ, $\mathcal{I}(\Lambda)$ is the infimum of the integral of ν along the polyhedral curves that cut Λ from infinity.

2.6. Sketch of the proof

We suppose that A is a non-empty line segment in $\mathbb{R}^2$. To shorten the notations, we shall write $D_n = D(nA, h(n))$, the set of all admissible conditions for $(nA, h(n))$:

$$D_n = \Bigg\{(k, \tilde{\theta}) \mid k \in [0, 1] \text{ and } \tilde{\theta} \in \left[\theta - \arctan\left(\frac{2h(n)k}{nl(A)}\right), \theta + \arctan\left(\frac{2h(n)(1-k)}{nl(A)}\right)\right]\Bigg\},$$

and

$$\mathcal{D}_n = \left[\theta - \arctan\left(\frac{2h(n)}{nl(A)}\right), \theta + \arctan\left(\frac{2h(n)}{nl(A)}\right)\right].$$

Also, we shall use:

$$\phi_n = \phi(nA, h(n)), \qquad \phi_n^{\kappa} = \phi^{\kappa}(nA, h(n)) \quad \text{and} \quad \tau_n = \tau(nA, h(n)).$$

First, notice that $0 \leq \phi_n \leq \tau_n$. If $F(0) \geq \frac{1}{2}$, then τ_n/n converges to zero, and so does ϕ_n, so Theorem 2.8 is trivially true. *We shall therefore make the following hypothesis in the rest of the article:*

$$F(0) < \frac{1}{2}. \tag{11}$$

Now, let us draw a sketch of the proof of Theorem 2.8. Recall that from Lemma 2.2,

$$\phi_n = \min_{\kappa \in D_n} \phi_n^\kappa .$$

First, we shall study the asymptotics of $\mathbb{E}(\phi_n)$ (Section 4):

Step 1. By a subadditive argument (see Fig. 2), we show in Section 4.1 that

$$\limsup_{n\to\infty} \frac{\mathbb{E}[\phi_n]}{nl(A)} \leq \inf_{\widetilde{\theta}\in\underline{\mathcal{D}}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)}$$

and

$$\liminf_{n\to\infty} \frac{\mathbb{E}[\phi_n]}{nl(A)} \leq \inf_{\widetilde{\theta}\in\overline{\mathcal{D}}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)}.$$

Step 2. On the other hand, by a similar subadditive argument (see Fig. 3), we show in Section 4.2 that

$$\liminf_{n\to\infty} \inf_{\kappa\in D_n} \frac{\mathbb{E}[\phi_n^\kappa]}{nl(A)} \geq \inf_{\widetilde{\theta}\in\overline{\mathcal{D}}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)}$$

and

$$\limsup_{n\to\infty} \inf_{\kappa\in D_n} \frac{\mathbb{E}[\phi_n^\kappa]}{nl(A)} \geq \inf_{\widetilde{\theta}\in\underline{\mathcal{D}}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)}.$$

Step 3. Using deviation results for the variables ϕ_n^κ (Section 3), we prove in Section 4.3 that $\mathbb{E}[\phi_n]$ is equivalent to $\inf_{\kappa\in D_n} \mathbb{E}[\phi_n^\kappa]$, and this ends the study of the asymptotic behaviour of $\mathbb{E}[\phi_n]$.

Next, we relate ϕ_n and $\mathbb{E}(\phi_n)$ to show the almost sure asymptotics (Section 5):

Step 4. A deviation result for ϕ_n obtained in Section 3 shows that almost surely, asymptotically, ϕ_n/n is at least as large as $\mathbb{E}(\phi_n)/n$.

Step 5. Finally, we use again the subadditive argument of the first step of the proof to prove that almost surely,

$$\limsup_{n\to\infty} \frac{\phi_n}{nl(A)} \leq \inf_{\widetilde{\theta}\in\underline{\mathcal{D}}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)}$$

and

$$\liminf_{n\to\infty} \frac{\phi_n}{nl(A)} \leq \inf_{\widetilde{\theta}\in\overline{\mathcal{D}}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)}.$$

3. Deviation properties of the maximal flows

The following proposition, due to Kesten, allows to control the size of the minimal cut, and is of fundamental importance in the study of First Passage Percolation.

Proposition 3.1 (*Proposition 5.8 in [10]*). *Suppose that $F(0) < \frac{1}{2}$. Then, there are constants ε, C_1 and C_2, depending only on F, such that:*

$$\mathbb{P}\left(\begin{array}{c}\exists \text{ a self-avoiding path } \gamma \text{ in } \mathbb{L}^*, \text{ starting at } \left(\frac{1}{2},\frac{1}{2}\right), \\ \text{with } \operatorname{card}(\gamma) \geq m \text{ and } \sum_{\mathrm{e}^* \in \gamma} t(\mathrm{e}^*) \leq \varepsilon m\end{array}\right) \leq C_1 \mathrm{e}^{-C_2 m}.$$

Thanks to Proposition 3.1 and general deviation inequalities due to [3], we obtain the following deviation result for the maximal flows ϕ_n and ϕ_n^κ. The proof is exactly the same as the proof of Proposition 4.3 in [12], using Proposition 3.1 instead of Zhang's result. We reproduce it here for the sake of completeness.

Proposition 3.2. *Suppose that hypotheses (6) and (11) hold. Then, for any $\eta \in]0,1]$, there are strictly positive constants $C(\eta, F)$, $K_1(F)$ and $K_2(F)$, such that, for every $n \in \mathbb{N}^*$, and every non-degenerate line segment A,*

$$\max_{\kappa \in D_n} \mathbb{P}(\phi_n^\kappa < \mathbb{E}(\phi_n^\kappa)(1-\eta)) \leq K_1 \mathrm{e}^{-C(\eta,F) \min_\kappa \mathbb{E}(\phi_n^\kappa)} \tag{12}$$

and:

$$\mathbb{P}(\phi_n \leq \mathbb{E}(\phi_n)(1-\eta)) \leq K_2 h(n)^2 \mathrm{e}^{-C(\eta,F)\mathbb{E}(\phi_n)}. \tag{13}$$

Proof. Let us fix A, $n \in \mathbb{N}^*$ and $\kappa = (k, \tilde{\theta}) \in D_n$. First, we prove the result for ϕ_n^κ. We shall denote by $E_{\phi_n^\kappa}$ a cut whose capacity achieves the minimum in the dual definition (2) of ϕ_n^κ. Since $\mathbb{P}\left(\phi_n^\kappa \leq \mathbb{E}(\phi_n^\kappa)(1-\eta)\right)$ is a decreasing function of η, it is enough to prove the result for all η less than or equal to some absolute $\eta_0 \in]0,1[$. We use this remark to exclude the case $\eta = 1$ in our study, thus, from now on, let η be a fixed real number in $]0,1[$.

We order the edges in $\operatorname{cyl}(nA, h(n))$ as $e_1, \ldots, e_{m_n}$. For every hyperrectangle A, we denote by $\mathcal{N}(A,h)$ the minimal number of edges in A that can disconnect A_1^h from A_2^h in $\operatorname{cyl}(A,h)$. For any real number $r \geq \mathcal{N}(nA, h(n))$, we define:

$$\psi_n^r = \min\left\{\begin{array}{c} V(E) \text{ s.t. } \operatorname{card}(E) \leq r \text{ and } E \text{ cuts} \\ (nA)_1^{h(n),k,\tilde{\theta}} \text{ from } (nA)_2^{h(n),k,\tilde{\theta}} \text{ in } \operatorname{cyl}(nA,h(n))\end{array}\right\}.$$

Now, suppose that hypotheses (6) and (11) hold, let ε, C_1 and C_2 be as in Proposition 3.1, and define $r = (1-\eta)\mathbb{E}(\phi_n^\kappa)/\varepsilon$. Suppose first that $r < \mathcal{N}(nA, h(n))$. Then,

$$\begin{aligned}\mathbb{P}(\phi_n^\kappa \leq (1-\eta)\mathbb{E}(\phi_n^\kappa)) &= \mathbb{P}(\phi_n^\kappa \leq (1-\eta)\mathbb{E}(\phi_n^\kappa) \text{ and } \operatorname{card}(E_{\phi_n^\kappa}) \geq (1-\eta)\mathbb{E}(\phi_n^\kappa)/\varepsilon), \\ &\leq C_1 \mathrm{e}^{-C_2(1-\eta)\mathbb{E}(\phi_n^\kappa)/\varepsilon},\end{aligned}$$

from Proposition 3.1, and the desired inequality is obtained. Suppose now that we have $r \geq \mathcal{N}(nA, h(n))$. Then,

$$\begin{aligned}
&\mathbb{P}(\phi_n^\kappa \leq (1-\eta)\mathbb{E}(\phi_n^\kappa)) \\
&\quad = \mathbb{P}(\phi_n^\kappa \leq (1-\eta)\mathbb{E}(\phi_n^\kappa) \text{ and } \psi_n^r \neq \phi_n^\kappa) + \mathbb{P}(\psi_n^r \leq (1-\eta)\mathbb{E}(\phi_n^\kappa)), \\
&\quad \leq C_1 \mathrm{e}^{-C_2 r} + \mathbb{P}(\psi_n^r \leq (1-\eta)\mathbb{E}(\psi_n^r)), \qquad (14)
\end{aligned}$$

from Proposition 3.1 and the fact that $\phi_n^\kappa \leq \psi_n^r$. Now, we truncate our variables $t(e)$. Let a be a positive real number to be chosen later, and define $\tilde{t}(e) = t(e) \wedge a$. Let:

$$\tilde{\psi}_n^r = \min \left\{ \begin{array}{c} \sum_{e \in E} \tilde{t}(e) \text{ s.t. } \operatorname{card}(E) \leq r \text{ and } E \text{ cuts} \\ (nA)_1^{h(n),k,\tilde{\theta}} \text{ from } (nA)_2^{h(n),k,\tilde{\theta}} \text{ in } \operatorname{cyl}(nA, h(n)) \end{array} \right\}.$$

Notice that $\tilde{\psi}_n^r \leq \psi_n^r$. We shall denote by $E_{\tilde{\psi}_n^r}$ a cutset whose capacity achieves the minimum in the definition of $\tilde{\psi}_n^r$. If there are more than one, we use a deterministic method to select a unique one with the minimal number of edges among these. Then,

$$\begin{aligned}
0 \leq \mathbb{E}(\psi_n^r) - \mathbb{E}(\tilde{\psi}_n^r) &\leq \mathbb{E}\left[\sum_{e \in E_{\tilde{\psi}_n^r}} t(e) - \sum_{e \in E_{\tilde{\psi}_n^r}} \tilde{t}(e) \right], \\
&\leq \mathbb{E}\left[\sum_{e \in E_{\tilde{\psi}_n^r}} t(e) \mathbb{1}_{t(e) \geq a} \right], \\
&= \sum_{i=1}^{m_n} \mathbb{E}(t(e_i) \mathbb{1}_{t(e_i) \geq a} \mathbb{1}_{e_i \in E_{\tilde{\psi}_n^r}}), \\
&= \sum_{i=1}^{m_n} \mathbb{E}\left[\mathbb{E}\left(t(e_i) \mathbb{1}_{t(e_i) \geq a} \mathbb{1}_{e_i \in E_{\tilde{\psi}_n^r}} | (t(e_j))_{j \neq i} \right) \right].
\end{aligned}$$

Now, when $(t(e_j))_{j \neq i}$ is fixed, $t(e_i) \mapsto \mathbb{1}_{e_i \in E_{\tilde{\psi}_n^r}}$ is a non-increasing function and $t(e_i) \mapsto t(e_i)\mathbb{1}_{t(e_i) \geq a}$ is of course non-decreasing. Furthermore, since the variables $(t(e_i))$ are independent, the conditional expectation $\mathbb{E}\left(.|(t(e_j))_{j \neq i}\right)$ corresponds to expectation over $t(e_i)$, keeping $(t(e_j))_{j \neq i}$ fixed. Thus, Chebyshev's association inequality (see [8], p. 43) implies:

$$\begin{aligned}
&\mathbb{E}\left(t(e_i) \mathbb{1}_{t(e_i) \geq a} \mathbb{1}_{e_i \in E_{\tilde{\psi}_n^r}} | (t(e_j))_{j \neq i} \right) \\
&\quad \leq \mathbb{E}\left(t(e_i) \mathbb{1}_{t(e_i) \geq a} | (t(e_j))_{j \neq i} \right) \mathbb{E}\left(\mathbb{1}_{e_i \in E_{\tilde{\psi}_n^r}} | (t(e_j))_{j \neq i} \right), \\
&\quad = \mathbb{E}\left(t(e_1) \mathbb{1}_{t(e_1) \geq a} \right) \mathbb{E}\left(\mathbb{1}_{e_i \in E_{\tilde{\psi}_n^r}} | (t(e_j))_{j \neq i} \right).
\end{aligned}$$

Thus,

$$0 \leq \mathbb{E}(\psi_n^r) - \mathbb{E}(\tilde{\psi}_n^r) \leq \mathbb{E}\left(t(e_1) \mathbb{1}_{t(e_1) \geq a} \right) \mathbb{E}(\operatorname{card}(E_{\tilde{\psi}_n^r})) \leq r \mathbb{E}\left(t(e_1) \mathbb{1}_{t(e_1) \geq a} \right). \qquad (15)$$

Now, since F has a finite moment of order 1, we can choose $a = a(\eta, F, d)$ such that:

$$\frac{1-\eta}{\varepsilon}\mathbb{E}\left(t(e_1)\mathbb{1}_{t(e_1)\geq a}\right) \leq \frac{\eta}{2},$$

to get:

$$\mathbb{E}(\psi_n^r) - \mathbb{E}(\tilde{\psi}_n^r) \leq \frac{\eta}{2}\mathbb{E}(\phi_n^\kappa) \leq \frac{\eta}{2}\mathbb{E}(\psi_n^r),$$

$$\mathbb{P}(\psi_n^r \leq (1-\eta)\mathbb{E}(\psi_n^r)) \leq \mathbb{P}\left(\tilde{\psi}_n^r \leq \mathbb{E}(\tilde{\psi}_n^r) - \frac{\eta}{2}\mathbb{E}(\psi_n^r)\right). \tag{16}$$

Now, we shall use Corollary 3 in [3]. To this end, we need some notations. We take $\tilde{t}'$ an independent collection of capacities with the same law as $\tilde{t} = (\tilde{t}(e_i))_{i=1,\ldots,m_n}$. For each edge $e_i \in \mathrm{cyl}(A, h)$, we denote by $\tilde{t}^{(i)}$ the collection of capacities obtained from $\tilde{t}$ by replacing $\tilde{t}(e_i)$ by $\tilde{t}'(e_i)$, and leaving all other coordinates unchanged. Define:

$$V_- := \mathbb{E}\left[\sum_{i=1}^{m_n}(\tilde{\psi}_n^r(t) - \tilde{\psi}_n^r(t^{(i)}))_-^2 \middle| t\right],$$

where $\tilde{\psi}_n^r(t)$ is the maximal flow through $\mathrm{cyl}(nA, h(n))$ when capacities are given by t. We shall denote by $R_{\tilde{\psi}_n^r}$ the intersection of all the cuts whose capacity achieves the minimum in the definition of $\tilde{\psi}_n^r$. Observe that:

$$\tilde{\psi}_n^r(t^{(i)}) - \tilde{\psi}_n^r(t) \leq (\tilde{t}'(e_i) - \tilde{t}(e_i))\mathbb{1}_{e_i \in R_{\tilde{\psi}_n^r}},$$

and thus,

$$V_- \leq a^2\mathbb{E}[\mathrm{card}(R_{\tilde{\psi}_n^r})] \leq a^2 r = a^2(1-\eta)\mathbb{E}(\phi_n^\kappa)/\varepsilon.$$

Thus, Corollary 3 in [3] implies that, for every $\eta \in]0, 1[$,

$$\mathbb{P}\left(\tilde{\psi}_n^r \leq \mathbb{E}(\tilde{\psi}_n^r) - \frac{\eta}{2}\mathbb{E}(\psi_n^r)\right) \leq \mathrm{e}^{-\frac{\mathbb{E}(\psi_n^r)^2\eta^2\varepsilon}{16a^2(1-\eta)\mathbb{E}(\phi_n^\kappa)}} \leq \mathrm{e}^{-\frac{\mathbb{E}(\phi_n^\kappa)\eta^2\varepsilon}{16a^2(1-\eta)}}.$$

Using inequalities (16) and (14) and taking the maximum over $\kappa \in D_n$, this ends the proof of Inequality (12).

To see that (13) holds, notice that $\mathbb{E}(\phi_n) \leq \min_{\kappa \in D_n} \mathbb{E}(\phi_n^\kappa)$. Thus, (13) is a consequence of inequalities (12) and (3). □

4. Asymptotic behaviour of the expectation of the maximal flow

4.1. Upper bound

From now on, we suppose that the conditions (6) on F and (5) on h are satisfied. We consider a line segment A, of orthogonal unit vector $\vec{v}(\theta) = (\cos\theta, \sin\theta)$ for $\theta \in [0, \pi[$, and a function $h : \mathbb{N} \to \mathbb{R}^+$ satisfying $\lim_{n\to\infty} h(n) = +\infty$. Recall that $\mathcal{D}_n = \mathcal{D}(nA, h(n))$. For all $\widetilde{\theta} \in \mathcal{D}_n$, we define

$$k_n = \frac{1}{2} + \frac{nl(A)\tan(\widetilde{\theta} - \theta)}{4h(n)},$$

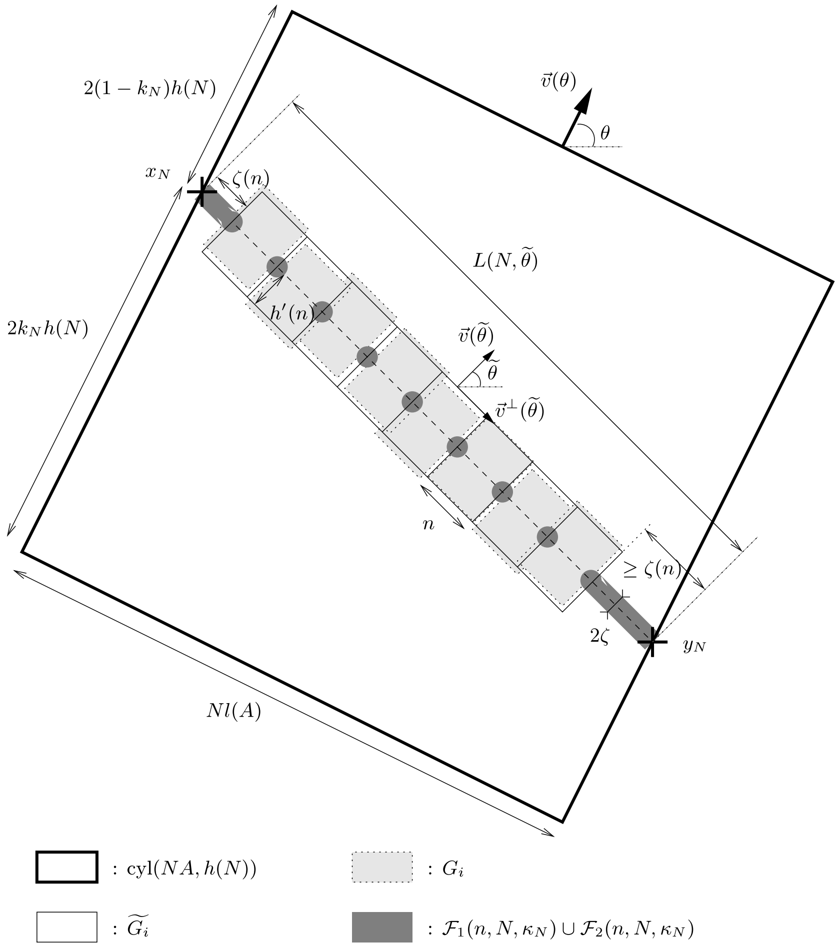

Fig. 2. The cylinders $\mathrm{cyl}(NA, h(N))$ and G_i, for $i = 1, \ldots, \mathcal{M}$.

and thus $\kappa_n = (k_n, \widetilde{\theta}) \in D_n$. We want to compare $\phi_n^{\kappa_n}$ with the maximal flow τ in a cylinder inside $\mathrm{cyl}(nA, h(n))$ and oriented towards the direction $\widetilde{\theta}$. In fact, we must use the subadditivity of τ and compare $\phi_n^{\kappa_n}$ with a sum of such variables τ.

We consider n and N in $\mathbb{N}$, with N a lot bigger than n. The following definitions can seem a little bit complicated, but Fig. 2 is more explicit. We choose two functions $h', \zeta : \mathbb{N} \to \mathbb{R}^+$ such that

$$\lim_{n\to\infty} h'(n) = \lim_{n\to\infty} \zeta(n) = +\infty,$$

and

$$\lim_{n\to\infty} \frac{h'(n)}{\zeta(n)} = 0. \tag{17}$$

We consider a fixed $\widetilde{\theta} \in \mathcal{D}_N$. Let

$$\vec{v}(\widetilde{\theta}) = (\cos\widetilde{\theta}, \sin\widetilde{\theta}) \quad \text{and} \quad \vec{v}^{\perp}(\widetilde{\theta}) = (\sin\widetilde{\theta}, -\cos\widetilde{\theta}).$$

In $\mathrm{cyl}(NA, h(N))$, we denote by x_N and y_N the two points corresponding to the boundary conditions κ_N, such that $\overrightarrow{x_N y_N} \cdot \vec{v}^{\perp}(\widetilde{\theta}) > 0$. Notice that according to our choice of k_N, the segments $[x_N, y_N]$ and NA cut each other in their middle. If we denote by $L(N, \widetilde{\theta})$ the distance between x_N and y_N, we have:

$$L(N, \widetilde{\theta}) = \frac{Nl(A)}{\cos(\widetilde{\theta} - \theta)}.$$

We define

$$\mathrm{cyl}'(n) = \mathrm{cyl}([0, n\vec{v}^{\perp}(\widetilde{\theta})], h'(n)).$$

We will translate $\mathrm{cyl}'(n)$ numerous times inside $\mathrm{cyl}(NA, h(N))$. We define

$$t_i = x_N + (\zeta(n) + (i-1)n)\, \vec{v}^{\perp}(\widetilde{\theta}),$$

for $i = 1, \ldots, \mathcal{M}$, where

$$\mathcal{M} = \mathcal{M}(n, N) = \left\lfloor \frac{L(N, \widetilde{\theta}) - 2\zeta(n)}{n} \right\rfloor.$$

Of course we consider only N large enough to have $\mathcal{M} \geq 2$. For $i = 1, \ldots, \mathcal{M}$, we denote by $\widetilde{G_i}$ the image of $\mathrm{cyl}'(n)$ by the translation of vector $\overrightarrow{0t_i}$. For n (and thus N) sufficiently large, thanks to condition (17), we know that $\widetilde{G_i} \subset \mathrm{cyl}(NA, h(N))$ for all i. We can translate $\widetilde{G_i}$ again by a vector of norm strictly smaller than 1 to obtain an integer translate of $\mathrm{cyl}'(n)$ (i.e., a translate by a vector whose coordinates are in $\mathbb{Z}^2$) that we will call G_i. Now we want to glue together cutsets of boundary condition $(1/2, \widetilde{\theta})$ in the cylinders G_i. We define:

$$\mathcal{F}_1(n, N, \kappa_N) = \left(\bigcup_{i=1}^{\mathcal{M}} \mathcal{V}(t_i, \zeta_0) \right) \bigcap \mathrm{cyl}(NA, h(N)),$$

where ζ_0 is a fixed constant larger than 4, and:

$$\mathcal{F}_2(n, N, \kappa_N) = \mathcal{V}\left([x_N, x_N + \zeta(n)\vec{v}^{\perp}(\widetilde{\theta})] \cup [z_{\mathcal{M}}, y_N], \zeta_0 \right) \bigcap \mathrm{cyl}(NA, h(N)).$$

Let $F_1(n, N, \kappa_N)$ (respectively $F_2(n, N, \kappa_N)$) be the set of the edges included in $\mathcal{F}_1(n, N, \kappa_N)$ (respectively $\mathcal{F}_2(n, N, \kappa_N)$). If for every $i = 1, \ldots, \mathcal{M}$, $\mathcal{G}_i$ is a cutset of boundary condition $(1/2, \widetilde{\theta})$ in G_i, then

$$\bigcup_{i=1}^{\mathcal{M}} \mathcal{G}_i \cup F_1(n, N, \kappa_N) \cup F_2(n, N, \kappa_N)$$

contains a cutset of boundary conditions κ_N in $\mathrm{cyl}(NA, h(N))$. We obtain:

$$\phi_N^{\kappa_N} \leq \sum_{i=1}^{\mathcal{M}} \tau(G_i, \vec{v}(\widetilde{\theta})) + V(F_1(n, N, \kappa_N) \cup F_2(n, N, \kappa_N)), \tag{18}$$

and so,

$$\forall \widetilde{\theta} \in \mathcal{D}_N \quad \phi_N \le \phi_N^{\kappa_N} \le \sum_{i=1}^{\mathcal{M}} \tau(G_i, \vec{v}(\widetilde{\theta})) + V(F_1(n, N, \kappa_N) \cup F_2(n, N, \kappa_N)). \tag{19}$$

There exists a constant C_5 such that:

$$\mathrm{card}(F_1(n, N, \kappa_N)) \le C_5 \mathcal{M} \quad \text{and} \quad \mathrm{card}(F_2(n, N, \kappa_N)) \le C_5 \left(\zeta(n) + n\right),$$

and since the set of edges $F_1(n, N, \kappa_N) \cup F_2(n, N, \kappa_N)$ is deterministic,

$$\mathbb{E}[V(F_1(n, N, \kappa_N) \cup F_2(n, N, \kappa_N))] \le C_5 \mathbb{E}(t) \left(\mathcal{M} + \zeta(n) + n\right).$$

So

$$\forall \widetilde{\theta} \in \mathcal{D}_N \qquad \frac{\mathbb{E}(\phi_N)}{Nl(A)} \le \frac{\mathcal{M}n}{Nl(A)} \times \frac{\mathbb{E}[\tau(\mathrm{cyl}'(n), \vec{v}(\widetilde{\theta}))]}{n} + \frac{C_5 \mathbb{E}(t) \left(\mathcal{M} + \zeta(n) + n\right)}{Nl(A)}. \tag{20}$$

We want to send N to infinity. First, let $\widetilde{\theta} \in \underline{\mathcal{D}}$. Then for all N large enough, $\widetilde{\theta} \in \mathcal{D}_N$, and thus for all n large enough we have

$$\limsup_{N\to\infty} \frac{\mathbb{E}(\phi_N)}{Nl(A)} \le \frac{1}{\cos(\widetilde{\theta} - \theta)} \frac{\mathbb{E}[\tau(\mathrm{cyl}'(n), \vec{v}(\widetilde{\theta}))]}{n} + \frac{C_5 \mathbb{E}(t)}{n \cos(\widetilde{\theta} - \theta)}.$$

Sending n to infinity, thanks to Theorem 2.3, we obtain that

$$\limsup_{N\to\infty} \frac{\mathbb{E}(\phi_N)}{Nl(A)} \le \inf_{\widetilde{\theta} \in \underline{\mathcal{D}}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)}. \tag{21}$$

We now suppose that $\widetilde{\theta} \in \overline{\mathcal{D}}$. Let $\psi : \mathbb{N} \to \mathbb{N}$ be strictly increasing and such that for all N, $\widetilde{\theta} \in \mathcal{D}_{\psi(N)}$. Then thanks to Eq. (20), sending first N to infinity and then n to infinity, we obtain that

$$\liminf_{N\to\infty} \frac{\mathbb{E}(\phi_N)}{Nl(A)} \le \limsup_{N\to\infty} \frac{\mathbb{E}(\phi_{\psi(N)})}{\psi(N)l(A)} \le \inf_{\widetilde{\theta} \in \overline{\mathcal{D}}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)}. \tag{22}$$

4.2. *Lower bound*

We do the symmetric construction of the one done in Section 4.1. We consider n and N in $\mathbb{N}$ and take N a lot bigger than n. We choose functions $\zeta', h'' : \mathbb{N} \to \mathbb{R}^+$ such that

$$\lim_{n\to\infty} \zeta'(n) = \lim_{n\to\infty} h''(n) = +\infty,$$

and

$$\lim_{n\to\infty} \frac{h(n)}{\zeta'(n)} = 0. \tag{23}$$

We consider $\kappa = (k, \widetilde{\theta}) \in D_n$. Keeping the same notations as in Section 4.1, we define

$$\mathrm{cyl}''(N) = \mathrm{cyl}\left([0, N\vec{v}^{\perp}(\widetilde{\theta})], h''(N)\right).$$

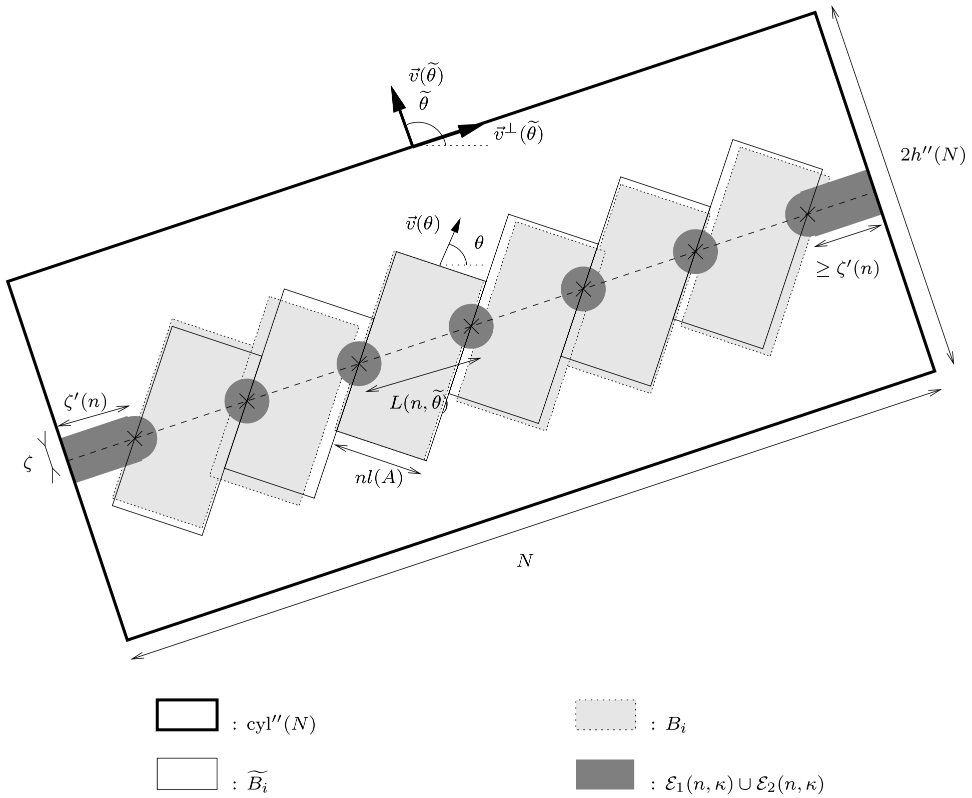

Fig. 3. The cylinders $\mathrm{cyl}''(N)$ and B_i, for $i = 1, \dots, \mathcal{N}$.

We will translate $\mathrm{cyl}(nA, h(n))$ numerous times in $\mathrm{cyl}''(N)$. The Fig. 3 is more explicit than the following definitions. The condition κ defines two points x_n and y_n on the boundary of $\mathrm{cyl}(nA, h(n))$ (see Section 4.1). As in Section 4.1, we denote by $L(n, \widetilde{\theta})$ the distance between x_n and y_n, and we have

$$L(n, \widetilde{\theta}) = \frac{nl(A)}{\cos(\widetilde{\theta} - \theta)}.$$

We define

$$z_i = \left(\zeta'(n) + (i-1)L(n, \widetilde{\theta})\right) \vec{v}^{\perp}(\widetilde{\theta}),$$

for $i = 1, \dots, \mathcal{N}$, where

$$\mathcal{N} = \left\lfloor \frac{N - 2\zeta'(n)}{L(n, \widetilde{\theta})} \right\rfloor.$$

Of course we consider only N large enough to have $\mathcal{N} \geq 2$. For $i = 1, \dots, \mathcal{N}$, we denote by $\widetilde{B_i}$ the image of $\mathrm{cyl}(nA, h(n))$ by the translation of vector $\overrightarrow{x_n z_i}$. For N sufficiently large, thanks to condition (23), we know that $\widetilde{B_i} \subset \mathrm{cyl}''(N)$ for all i. We can translate $\widetilde{B_i}$ again by a vector of norm strictly smaller than 1 to obtain an integer translate of $\mathrm{cyl}(nA, h(n))$ (i.e., a translate by a vector whose coordinates are in $\mathbb{Z}^2$) that we will call B_i. Now we want to glue together cutsets of boundary condition κ in the different B_i's. We define:

$$\mathcal{E}_1(n, N, \kappa) = \left(\bigcup_{i=1}^{\mathcal{N}} \mathcal{V}(z_i, \zeta)\right) \bigcap \text{cyl}''(N),$$

where ζ is still a fixed constant bigger than 4, and:

$$\mathcal{E}_2(n, N, \kappa) = \mathcal{V}\left([0, \zeta'(n)\vec{v}^{\perp}(\widetilde{\theta})] \cup [z_{\mathcal{N}}, N\vec{v}^{\perp}(\widetilde{\theta})], \zeta\right) \bigcap \text{cyl}''(N).$$

Let $E_1(n, N, \kappa)$ (respectively $E_2(n, N, \kappa)$) be the set of the edges included in $\mathcal{E}_1(n, N, \kappa)$ (respectively $\mathcal{E}_2(n, N, \kappa)$). Then, still by gluing cutsets together, we obtain:

$$\tau(\text{cyl}''(N), \vec{v}(\widetilde{\theta})) \leq \sum_{i=1}^{\mathcal{N}} \phi^{\kappa}(B_i, \vec{v}(\theta)) + V(E_1(n, N, \kappa) \cup E_2(n, N, \kappa)). \tag{24}$$

On one hand, there exists a constant C_6 (independent of κ) such that:

$$\text{card}(E_1(n, N, \kappa) \cup E_2(n, N, \kappa)) \leq C_6 \left(\mathcal{N} + \zeta'(n) + L(n, \widetilde{\theta})\right),$$

and since the sets $E_1(n, N, \kappa)$ and $E_2(n, N, \kappa)$ are deterministic, we deduce:

$$\mathbb{E}[V(E_1(n, N, \kappa) \cup E_2(n, N, \kappa))] \leq C_6 \mathbb{E}(t) \left(\mathcal{N} + \zeta'(n) + L(n, \widetilde{\theta})\right).$$

On the other hand, the variables $(\phi^{\kappa}(B_i))_{i=1,\ldots,\mathcal{N}}$ are identically distributed, with the same law as ϕ_n^{κ} (because we only consider integer translates), so (24) leads to

$$\mathbb{E}[\tau(\text{cyl}''(N), \vec{v}(\widetilde{\theta}))] \leq \mathcal{N}\mathbb{E}[\phi_n^{\kappa}] + C_6 \mathbb{E}(t) \left(\mathcal{N} + \zeta'(n) + L(n, \widetilde{\theta})\right).$$

Dividing by N and sending N to infinity, we get, thanks to Theorem 2.3:

$$\nu_{\widetilde{\theta}} \leq \frac{\mathbb{E}[\phi_n^{\kappa}]}{L(n, \widetilde{\theta})} + \frac{C_6 \mathbb{E}(t)}{L(n, \widetilde{\theta})},$$

and so:

$$\frac{\mathbb{E}[\phi_n^{\kappa}]}{nl(A)} \geq \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)} - \frac{C_6 \mathbb{E}(t)}{nl(A)}.$$

Since C_6 is independent of κ,

$$\inf_{\kappa \in D_n} \frac{\mathbb{E}[\phi_n^{\kappa}]}{nl(A)} \geq \inf_{\widetilde{\theta} \in \mathcal{D}_n} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)} - \frac{C_6 \mathbb{E}(t)}{nl(A)}.$$

First, we affirm:

$$\liminf_{n\to\infty} \inf_{\widetilde{\theta} \in \mathcal{D}_n} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)} \geq \inf_{\widetilde{\theta} \in \overline{\mathcal{D}}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)}, \tag{25}$$

and thus:

$$\liminf_{n\to\infty} \inf_{\kappa \in D_n} \frac{\mathbb{E}[\phi_n^{\kappa}]}{nl(A)} \geq \inf_{\widetilde{\theta} \in \overline{\mathcal{D}}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)}. \tag{26}$$

We also claim that:

$$\limsup_{n\to\infty} \inf_{\widetilde{\theta} \in \mathcal{D}_n} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)} \geq \inf_{\widetilde{\theta} \in \underline{\mathcal{D}}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)}, \tag{27}$$

and therefore:

$$\limsup_{n\to\infty} \inf_{\kappa\in D_n} \frac{\mathbb{E}[\phi_n^\kappa]}{nl(A)} \geq \inf_{\widetilde{\theta}\in\underline{\mathcal{D}}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)}. \tag{28}$$

Let us prove Inequality (25). In fact, we will state a more general result:

Lemma 4.1. *Let* $\theta\in[0,\pi[$, *and* f *be a lower semi-continuous function from* $[\theta-\pi/2,\theta+\pi/2]$ *to* $\mathbb{R}^+\cup\{+\infty\}$. *Then we have*

$$\liminf_{n\to\infty} \inf_{\widetilde{\theta}\in\mathcal{D}_n} f(\widetilde{\theta}) \geq \inf_{\widetilde{\theta}\in ad(\overline{\mathcal{D}})} f(\widetilde{\theta}),$$

where $\mathrm{ad}(\overline{\mathcal{D}})$ *is the adherence of* $\overline{\mathcal{D}}$.

Proof. We consider a positive ε. For all n, since f is lower semi-continuous and $\mathcal{D}_n$ is compact, there exists $\widetilde{\theta}_n\in\mathcal{D}_n$ such that $f(\widetilde{\theta}_n)=\inf_{\widetilde{\theta}\in\mathcal{D}_n} f(\widetilde{\theta})$. Up to extracting a subsequence, we can suppose that the sequence $(\inf_{\widetilde{\theta}\in\mathcal{D}_n} f(\widetilde{\theta}))_{n\geq 0}$ converges towards $\liminf_{n\to\infty}\inf_{\widetilde{\theta}\in\mathcal{D}_n} f(\widetilde{\theta})$, and so:

$$\lim_{n\to\infty} f(\widetilde{\theta}_n) = \liminf_{n\to\infty}\inf_{\widetilde{\theta}\in\mathcal{D}_n} f(\widetilde{\theta}).$$

The sequence $(\widetilde{\theta}_n)_{n\geq 0}$ (in fact the previous subsequence) takes values in the compact $[\theta-\pi/2,\theta+\pi/2]$, so up to extracting a second subsequence we can suppose that $(\widetilde{\theta}_n)_{n\geq 0}$ converges towards a limit $\widetilde{\theta}_\infty$ in this compact. Since f is lower semi-continuous,

$$f(\widetilde{\theta}_\infty) \leq \lim_{n\to\infty} f(\widetilde{\theta}_n) = \liminf_{n\to\infty}\inf_{\widetilde{\theta}\in\mathcal{D}_n} f(\widetilde{\theta}),$$

and we just have to prove that $\widetilde{\theta}_\infty$ belongs to $\mathrm{ad}(\overline{\mathcal{D}})$. Indeed, for all positive ε, $\widetilde{\theta}_n\in[\widetilde{\theta}_\infty-\varepsilon,\widetilde{\theta}_\infty+\varepsilon]$ for an infinite number of n. We remember that all the $\mathcal{D}_n$ are closed intervals centered at θ. If $\widetilde{\theta}_\infty=\theta$, the result is obvious. We suppose that $\widetilde{\theta}_\infty>\theta$ for example, and thus, for ε small enough, $\widetilde{\theta}_\infty-\varepsilon>\theta$. Then $[\theta,\widetilde{\theta}_\infty-\varepsilon]$ is included in an infinite number of $\mathcal{D}_n$, so $\widetilde{\theta}_\infty-\varepsilon$ belongs to $\overline{\mathcal{D}}$, and then $\widetilde{\theta}_\infty$ belongs to $\mathrm{ad}(\overline{\mathcal{D}})$. The same holds if $\widetilde{\theta}_\infty<\theta$. This ends the proof of Lemma 4.1. □

We use Lemma 4.1 with $f(\widetilde{\theta})=\nu_{\widetilde{\theta}}/\cos(\widetilde{\theta}-\theta)$. Here f is lower semi-continuous, because $\widetilde{\theta}\to\nu_{\widetilde{\theta}}$ is continuous since it satisfies the weak triangle inequality. Indeed, it is obvious in dimension 2 because $\nu_{\widetilde{\theta}}=\mu(\vec{v}(\widetilde{\theta}))$ which satisfies the (ordinary) triangle inequality, but it has also been proved in any dimension $d\geq 2$ (see section 4.4 in [12]). Moreover we know that f is finite and continuous on $]\theta-\pi/2,\theta+\pi/2[$, infinite at $\theta+\pi/2$ and $\theta-\pi/2$ and

$$\lim_{\widetilde{\theta}\to\theta+\pi/2} f(\widetilde{\theta}) = \lim_{\widetilde{\theta}\to\theta-\pi/2} f(\widetilde{\theta}) = +\infty,$$

so we can even say in this case:

$$\inf_{\widetilde{\theta}\in\mathrm{ad}(\overline{\mathcal{D}})} f(\widetilde{\theta}) = \inf_{\widetilde{\theta}\in\overline{\mathcal{D}}} f(\widetilde{\theta}),$$

and we obtain Inequality (25).

Let us now prove Inequality (27). We state again a more general result:

Lemma 4.2. *Let* $\theta \in [0, \pi[$, *and* f *be a lower semi-continuous function from* $[\theta - \pi/2, \theta + \pi/2]$ *to* $\mathbb{R}^+ \cup \{+\infty\}$. *Then we have*

$$\limsup_{n\to\infty} \inf_{\widetilde{\theta}\in\mathcal{D}_n} f(\widetilde{\theta}) \geq \inf_{\widetilde{\theta}\in \mathrm{ad}(\underline{\mathcal{D}})} f(\widetilde{\theta}),$$

where $\mathrm{ad}(\underline{\mathcal{D}})$ *is the adherence of* $\underline{\mathcal{D}}$.

Proof. We denote $\mathrm{ad}(\underline{\mathcal{D}})$ by $[\theta - \alpha, \theta + \alpha]$. For all integer $p \geq 1$, there exists $n_p \geq n_{p-1}$ ($n_0 = 1$) such that:

$$\theta + \alpha + 1/p \notin \mathcal{D}_{n_p} \quad \text{and} \quad \theta - \alpha - 1/p \notin \mathcal{D}_{n_p},$$

thus

$$\mathcal{D}_{n_p} \subset]\theta - \alpha - 1/p, \theta + \alpha + 1/p[,$$

then

$$\begin{aligned}\limsup_{n\to\infty} \inf_{\widetilde{\theta}\in\mathcal{D}_n} f(\widetilde{\theta}) &\geq \limsup_{p\to\infty} \inf_{\widetilde{\theta}\in\mathcal{D}_{n_p}} f(\widetilde{\theta}) \\ &\geq \limsup_{p\to\infty} \inf_{\widetilde{\theta}\in[\theta-\alpha-1/p,\theta+\alpha+1/p]} f(\widetilde{\theta}).\end{aligned}$$

The function f is lower semi-continuous and $[\theta - \alpha - 1/p, \theta + \alpha + 1/p]$ is compact, so for all integers p there exists $\widetilde{\theta}_p \in [\theta - \alpha - 1/p, \theta + \alpha + 1/p]$ such that $f(\widetilde{\theta}_p) = \inf_{\widetilde{\theta}\in[\theta-\alpha-1/p,\theta+\alpha+1/p]} f(\widetilde{\theta})$. Up to extraction, we can suppose that $(\widetilde{\theta}_p)_{p\geq 1}$ converges towards a limit $\widetilde{\theta}_\infty$, that belongs obviously to $[\theta - \alpha, \theta + \alpha]$. Finally, because f is lower semi-continuous,

$$\inf_{\widetilde{\theta}\in[\theta-\alpha,\theta+\alpha]} f(\widetilde{\theta}) \leq f(\widetilde{\theta}_\infty) \leq \limsup_{p\to\infty} f(\widetilde{\theta}_p) \leq \limsup_{n\to\infty} \inf_{\widetilde{\theta}\in\mathcal{D}_n} f(\widetilde{\theta}),$$

so Lemma 4.2 is proved. □

As previously, we use Lemma 4.2 with $f(\widetilde{\theta}) = \nu_{\widetilde{\theta}} / \cos(\widetilde{\theta} - \theta)$. Again, we have:

$$\inf_{\widetilde{\theta}\in \mathrm{ad}(\underline{\mathcal{D}})} f(\widetilde{\theta}) = \inf_{\widetilde{\theta}\in\underline{\mathcal{D}}} f(\widetilde{\theta}),$$

and Eq. (27) is proved.

4.3. End of the study of the mean

Now, we are able to conclude the proof of (7) and (8). First, we show that $\mathbb{E}(\phi_n)$ and $\min_\kappa \mathbb{E}(\phi_n^\kappa)$ are of the same order.

Lemma 4.3. *Let* A *be a line segment in* $\mathbb{R}^2$. *Suppose that conditions (5) and (6) are satisfied. Then,*

$$\lim_{n\to\infty} \frac{\mathbb{E}(\phi_n)}{\min_{\kappa\in D_n} \mathbb{E}(\phi_n^\kappa)} = 1.$$

Proof. Notice that $\mathbb{E}(\phi_n) \leq \min_{\kappa\in D_n} \mathbb{E}(\phi_n^\kappa)$, and thus it is sufficient to show that:

$$\liminf_{n\to\infty} \frac{\mathbb{E}(\phi_n)}{\min_{\kappa\in D_n} \mathbb{E}(\phi_n^\kappa)} \geq 1.$$

Recall from (3) and Lemma 2.2 that there is a finite subset $\tilde{D}_n$ of D_n, such that:

$$\operatorname{card}(\tilde{D}_n) \leq C_4 h(n)^2,$$

for some constant C_4 and every n, and

$$\phi_n = \min_{\kappa \in \tilde{D}_n} \phi_n^\kappa. \tag{29}$$

Thus, for η in $]0, 1[$,

$$\begin{aligned}\mathbb{P}(\min_{\kappa\in\tilde{D}_n} \phi_n^\kappa \geq \min_{\kappa\in\tilde{D}_n} \mathbb{E}(\phi_n^\kappa)(1-\eta)) &= 1 - \mathbb{P}(\exists \kappa \in \tilde{D}_n,\ \phi_n^\kappa < \min_{\kappa\in\tilde{D}_n} \mathbb{E}(\phi_n^\kappa)(1-\eta)),\\ &\geq 1 - |\tilde{D}_n| \max_{\kappa\in\tilde{D}_n} \mathbb{P}(\phi_n^\kappa < \min_{\kappa\in\tilde{D}_n} \mathbb{E}(\phi_n^\kappa)(1-\eta)),\\ &\geq 1 - C_4 h(n)^2 \max_{\kappa\in\tilde{D}_n} \mathbb{P}(\phi_n^\kappa < \mathbb{E}(\phi_n^\kappa)(1-\eta)).\end{aligned}$$

Now, Proposition 3.2 implies that for η in $]0, 1[$,

$$\mathbb{P}(\min_{\kappa\in D_n} \phi_n^\kappa \geq \min_{\kappa\in D_n} \mathbb{E}(\phi_n^\kappa)(1-\eta)) \geq 1 - C_4 K_1 h(n)^2 \mathrm{e}^{-C(\eta,F) \min_{\kappa\in D_n} \mathbb{E}(\phi_n^\kappa)},$$

where $C(\eta, F)$ is strictly positive. Now, let η_0 be fixed in $]0, 1/2[$.

$$\begin{aligned}\mathbb{E}(\min_{\kappa\in D_n} \phi_n^\kappa) &= \int_0^{+\infty} \mathbb{P}(\min_{\kappa\in D_n} \phi_n^\kappa \geq t)\mathrm{d}t,\\ &\geq \int_0^{\min_{\kappa\in D_n} \mathbb{E}(\phi_n^\kappa)} \mathbb{P}(\min_{\kappa\in D_n} \phi_n^\kappa \geq \min_{\kappa\in D_n} \mathbb{E}(\phi_n^\kappa) - u)\mathrm{d}u,\\ &\geq \min_{\kappa\in D_n} \mathbb{E}(\phi_n^\kappa) \int_{\eta_0}^{(1-\eta_0)} \mathbb{P}(\min_{\kappa\in D_n} \phi_n^\kappa \geq \min_{\kappa\in D_n} \mathbb{E}(\phi_n^\kappa)(1-\eta))\mathrm{d}\eta,\\ &\geq \min_{\kappa\in D_n} \mathbb{E}(\phi_n^\kappa)(1-2\eta_0)\left(1 - C_4 K_1 h(n)^2 \mathrm{e}^{-C(1-\eta_0,F)\min_{\kappa\in D_n}\mathbb{E}(\phi_n^\kappa)}\right).\end{aligned}$$

Thanks to Inequality (26), we know that there is a strictly positive constant $C(A)$ such that:

$$\liminf_{n\to\infty} \frac{\min_{\kappa\in\tilde{D}_n} \mathbb{E}(\phi_n^\kappa)}{n} \geq C(A).$$

Thus, using assumption (5), namely the fact that $\log h(n)$ is small compared to n,

$$\liminf_{n\to\infty} \frac{\mathbb{E}(\phi_n)}{\min_{\kappa\in D_n} \mathbb{E}(\phi_n^\kappa)} \geq 1 - 2\eta_0.$$

Since this is true for any $\eta_0 \in]0, 1/2[$, this finishes the proof of Lemma 4.3. □

Now, inequalities (21), (28) and Lemma 4.3 give:

$$\limsup_{n\to\infty} \frac{\mathbb{E}[\phi_n]}{nl(A)} = \inf_{\widetilde{\theta}\in\underline{\mathcal{D}}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)} \tag{30}$$

which is (8). Similarly, inequalities (22), (26) and Lemma 4.3 give:

$$\liminf_{n\to\infty} \frac{\mathbb{E}[\phi_n]}{nl(A)} = \inf_{\widetilde{\theta}\in\overline{\mathcal{D}}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)}. \tag{31}$$

which is (7).

5. Proof of the law of large numbers

Using Borel–Cantelli's Lemma and Proposition 3.2, we obtain that

$$\liminf_{n\to\infty} \frac{\phi_n - \mathbb{E}[\phi_n]}{nl(A)} \geq 0,$$

and thus, using Eqs. (30) and (31), that

$$\liminf_{n\to\infty} \frac{\phi_n}{nl(A)} \geq \liminf_{n\to\infty} \frac{\mathbb{E}[\phi_n]}{nl(A)} = \inf_{\widetilde{\theta}\in\overline{\mathcal{D}}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)} \tag{32}$$

and

$$\limsup_{n\to\infty} \frac{\phi_n}{nl(A)} \geq \limsup_{n\to\infty} \frac{\mathbb{E}[\phi_n]}{nl(A)} = \inf_{\widetilde{\theta}\in\underline{\mathcal{D}}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)}. \tag{33}$$

It can seem a bit strange to bound $\limsup_{n\to\infty} \phi_n/(nl(A))$ from below in the study of the asymptotic behaviour of ϕ_n. The reason is the following: we do not only want to prove the convergence of the rescaled flow ϕ_n in some cases, we want to obtain a necessary and sufficient condition for this convergence to hold. Thus we need to know exactly the values of $\limsup_{n\to\infty} \phi_n/(nl(A))$ and $\liminf_{n\to\infty} \phi_n/(nl(A))$. We will prove the converse of Inequalities (32) and (33). For that purpose we use again the geometrical construction performed in Section 4.1. Suppose only for the moment that

$$\int_{[0,+\infty[} x \,\mathrm{d}F(x) < \infty.$$

Let $\widetilde{\theta}_1 \in \underline{\mathcal{D}}$ be such that

$$\frac{\nu_{\widetilde{\theta}_1}}{\cos(\widetilde{\theta}_1-\theta)} = \inf_{\widetilde{\theta}\in\underline{\mathcal{D}}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)}.$$

Such a $\widetilde{\theta}_1$ exists, since

$$\inf_{\widetilde{\theta}\in\underline{\mathcal{D}}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)} = \inf_{\widetilde{\theta}\in\mathrm{ad}(\underline{\mathcal{D}})} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta}-\theta)}$$

as stated in Section 4.2, $\mathrm{ad}(\underline{\mathcal{D}})$ is compact and the function $\widetilde{\theta} \mapsto \nu_{\widetilde{\theta}}/\cos(\widetilde{\theta}-\theta)$ is lower semi-continuous. For all N large enough, $\widetilde{\theta}_1 \in \mathcal{D}_N$, and we only consider such large N. First suppose that 0, the origin of the graph, is the middle of A. Then consider $\kappa_N = (k_N, \widetilde{\theta}_1)$ as defined in Section 4.1. We performed the geometrical construction of Section 4.1: we consider several integer translates G_i, for $i = 1, \ldots, \mathcal{M}(n, N)$, of $\mathrm{cyl}'(n)$ inside $\mathrm{cyl}(NA, h(N))$. Since 0 belongs to $[x_N, y_N]$, we can construct the cylinders G_i and the sets of edges $F_1(n, N, \kappa_N)$ and $F_2(n, N, \kappa_N)$ in such a way that

$$\forall N_1 \leq N_2 \quad (G_i)_{i=1,\ldots,\mathcal{M}(n,N_1)} \subset (G_i)_{i=1,\ldots,\mathcal{M}(n,N_2)}$$
$$\text{and} \quad F_1(n, N_1, \kappa_{N_1}) \subset F_1(n, N_2, \kappa_{N_2}).$$

We use again Inequality (19) to obtain that:

$$\frac{\phi_N}{Nl(A)} \leq \frac{n\mathcal{M}}{Nl(A)} \frac{1}{\mathcal{M}} \sum_{i=1}^{\mathcal{M}} \frac{\tau(G_i, \vec{v}(\widetilde{\theta}_1))}{n} + \frac{V(F_1(n, N, \kappa_N))}{Nl(A)} + \frac{V(F_2(n, N, \kappa_N))}{Nl(A)}. \tag{34}$$

The variables $(\tau(G_i, \widetilde{\theta}_1), i = 1, \dots, \mathcal{M}(n, N))$ are not independent. However, each cylinder G_i can intersect at most the two other cylinders that are its neighbours, thus we can divide the family $(\tau(G_i, \widetilde{\theta}_1), i = 1, \dots, \mathcal{M}(n, N))$ into two families $(\tau(G_i, \widetilde{\theta}_1), i \in \{1, \dots, \mathcal{M}(n, N)\} \cap P_j)$ for $j = 1, 2$, $P_1 = 2\mathbb{N}$ and $P_2 = 2\mathbb{N} + 1$, such that for each $j \in \{1, 2\}$, the family $(\tau(G_i, \widetilde{\theta}_1), i \in \{1, \dots, \mathcal{M}(n, N)\} \cap P_j)$ is i.i.d. Since

$$\int_{[0,+\infty[} x \, \mathrm{d}F(x) < \infty,$$

it is easy to see that the variable $\tau(\mathrm{cyl}'(n), \widetilde{\theta}_1)$ is integrable (we can compare this variable with the capacity of a deterministic cutset), and we can apply the strong law of large numbers to each of the two families of variables described above to finally obtain that

$$\lim_{N\to\infty} \frac{n\mathcal{M}}{Nl(A)} \frac{1}{\mathcal{M}} \sum_{i=1}^{\mathcal{M}} \frac{\tau(G_i, \vec{v}(\widetilde{\theta}_1))}{n} = \frac{1}{\cos(\widetilde{\theta}_1 - \theta)} \frac{\mathbb{E}[\tau(\mathrm{cyl}'(n), \widetilde{\theta}_1)]}{n} \quad \text{a.s.} \tag{35}$$

Up to increasing a little the sets $F_1(n, N, \kappa_N)$, we can suppose that for all N, we have

$$\mathrm{card}(F_1(n, N, \kappa_N)) = C_5 \mathcal{M}(n, N),$$

and thus, by the strong law of large numbers, we obtain that

$$\lim_{N\to\infty} \frac{V(F_1(n, N, \kappa_N))}{Nl(A)} = \frac{C_5 \mathbb{E}[t(e)]}{n \cos(\widetilde{\theta}_1 - \theta)} \quad \text{a.s.} \tag{36}$$

Moreover, we know that

$$\mathrm{card}(F_2(n, N, \kappa_N)) \le C_5(n + \zeta(n)),$$

thus for all $\eta > 0$ we have

$$\begin{aligned} \sum_{N\in\mathbb{N}^*} \mathbb{P}[V(F_2(n, N, \kappa_N)) \ge \eta Nl(A)] &\le \sum_{N\in\mathbb{N}^*} \mathbb{P}\left[\sum_{i=1}^{C_5(n+\zeta(n))} t_i \ge \eta Nl(A)\right] \\ &\le \mathbb{E}\left[1 + \frac{1}{\eta l(A)} \sum_{i=1}^{C_5(n+\zeta(n))} t_i\right] < \infty \end{aligned}$$

where $(t_i, i \in \mathbb{N})$ is a family of i.i.d. variables with distribution function F. By a simple Borel–Cantelli's Lemma, we conclude that

$$\lim_{N\to\infty} \frac{V(F_2(n, N, \kappa_N))}{Nl(A)} = 0 \quad \text{a.s.} \tag{37}$$

Combining Eqs. (34)–(37), and sending n to infinity, thanks to Theorem 2.3 we obtain that

$$\limsup_{N\to\infty} \frac{\phi_N}{Nl(A)} \le \frac{\nu_{\widetilde{\theta}_1}}{\cos(\widetilde{\theta}_1 - \theta)} = \inf_{\widetilde{\theta}\in\underline{\mathcal{D}}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)} \quad \text{a.s.}$$

Similarly, we can choose $\widetilde{\theta}_2 \in \overline{\mathcal{D}}$ satisfying

$$\frac{\nu_{\widetilde{\theta}_2}}{\cos(\widetilde{\theta}_2 - \theta)} = \inf_{\widetilde{\theta}\in\overline{\mathcal{D}}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)}.$$

We consider a subsequence $(\psi(N), N \in \mathbb{N})$ of $\mathbb{N}$ such that for all N, $\widetilde{\theta}_2 \in \mathcal{D}_{\psi(N)}$. If 0 is the middle of A, for every N we consider $k_{\psi(N)}$ as defined in Section 4.1, and which is such that $\kappa_{\psi(N)} = (k_{\psi(N)}, \widetilde{\theta}_2) \in D_{\psi(N)}$ and 0 belongs to the segments $[x_N, y_N]$ determined by the boundary condition κ_N. Then we obtain exactly by the same methods that

$$\liminf_{N\to\infty} \frac{\phi_N}{Nl(A)} \leq \limsup_{N\to\infty} \frac{\phi_{\psi(N)}}{\psi(N)l(A)} \leq \frac{\nu_{\widetilde{\theta}_2}}{\cos(\widetilde{\theta}_2 - \theta)} = \inf_{\widetilde{\theta}\in\overline{\mathcal{D}}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)} \qquad \text{a.s.}$$

If the condition on the origin 0 of the graph is not satisfied, we suppose that

$$\int_{[0,+\infty[} x^2 \,\mathrm{d}F(x) < \infty.$$

To obtain Eqs. (35) and (36) in the case where 0 is the middle of A, we have used the strong law of large numbers. If 0 is not the middle of A we may not construct the cylinders $(G_i, i \in \{1, \dots, \mathcal{M}(n, N)\})$ such that the same G_i's appear for different N. Thus we obtain cylinders $(G_i(N), i \in \{1, \dots, \mathcal{M}(n, N)\})$ that depend on N. The sets $(\tau(G_i(N), \widetilde{\theta}_1), i \in \{1, \dots, \mathcal{M}(n, N)\} \cap P_j)$ (resp. $(t(e), e \in F_1(n, N, \kappa_N))$) are families of i.i.d. random variables for a given N, and $\tau(G_i(N))$ (resp. $t(e)$) has the same law whatever the value of i and N (resp. whatever e and N), but we are not in the conditions of application of the strong law of large numbers: we consider the behaviour of a sequence of the form

$$\left(\frac{\sum_{i=1}^{n} X_i^{(n)}}{n}, n \in \mathbb{N} \right),$$

where $(X_i^{(j)})_{i,j}$ is an array of i.d. random variables such that for each n, the variables $(X_1^{(n)}, \dots, X_n^{(n)})$ are independent. Thanks to Theorem 3 in [9], we know that such a sequence converges a.s. towards $\mathbb{E}(X_1^{(1)})$ as soon as $\mathbb{E}[(X_1^{(1)})^2] < \infty$. This theorem is based on a result of complete convergence (see Theorem 1 in [9]) and a Borel–Cantelli's Lemma. If $t(e)$ admits a moment of order 2, the same holds for $\tau(G_i(N), \widetilde{\theta}_1)$, thus we can use Theorem 3 in [9] to get Eqs. (35) and (36) again. This ends the proof of Theorem 2.8.

Obviously, the condition

$$\inf_{\widetilde{\theta}\in\underline{\mathcal{D}}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)} = \inf_{\widetilde{\theta}\in\overline{\mathcal{D}}} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)} := \eta_{\theta,h}, \tag{38}$$

necessary and sufficient for the convergence a.s. of $(\phi_n/(nl(A)))_{n\geq 0}$, is closely linked to the asymptotic behaviour of $h(n)/n$. Indeed we know that

$$\mathcal{D}_n = [\theta - \alpha_n, \theta + \alpha_n],$$

where $\alpha_n = \arctan\left(\frac{2h(n)}{nl(A)}\right)$. If $\lim_{n\to\infty} 2h(n)/(nl(A))$ exists in $\mathbb{R}^+ \cup \{+\infty\}$, and we denote it by $\tan\alpha$ ($\alpha \in [0, \pi/2]$), then $\underline{\mathcal{D}}$ and $\overline{\mathcal{D}}$ are equal to $[\theta - \alpha, \theta + \alpha]$ or $]\theta - \alpha, \theta + \alpha[$, and we obtain that $\eta_{\theta,h}$ exists and

$$\eta_{\theta,h} = \inf_{\widetilde{\theta}\in[\theta-\alpha,\theta+\alpha]} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)}.$$

As previously, we do not care keeping $\theta + \alpha$ and $\theta - \alpha$ in the infimum. Then we obtain the a.s. convergence appearing in Corollary 2.10. Obviously, if there exists a $\widetilde{\theta_0}$ such that

$$\frac{\nu_{\widetilde{\theta_0}}}{\cos(\widetilde{\theta_0} - \theta)} = \inf_{\widetilde{\theta} \in [\theta - \pi/2, \theta + \pi/2]} \frac{\nu_{\widetilde{\theta}}}{\cos(\widetilde{\theta} - \theta)}$$

and if

$$\liminf_{n \to \infty} \frac{2h(n)}{nl(A)} \geq |\tan(\widetilde{\theta_0} - \theta)|,$$

then $\eta_{\theta,h}$ also exists (and equals $\nu_{\widetilde{\theta_0}} / \cos(\widetilde{\theta_0} - \theta)$) and is the limit of $(\phi_n/(nl(A)))_{n \in \mathbb{N}}$ almost surely, even if $\lim_{n \to \infty} h(n)/n$ does not exist.

To complete the proof of Corollary 2.10, it remains to prove the convergence of $\phi_n/nl(A)$ in L^1. Suppose first that the condition (10) is satisfied. Then, one can find a sequence of sets of edges $(E(n))_{n \in \mathbb{N}}$ such that for each n, $E(n)$ is a cut between $T(nA, h(n))$ and $B(nA, h(n))$, $E(n) \subset E(n+1)$ and:

$$\lim_{n \to \infty} \frac{\text{card}(E(n))}{nl(A)} \quad \text{exists,}$$

cf. Lemma 4.1 in [12], for instance. Now, define:

$$f_n = \frac{\phi_n}{nl(A)} \quad \text{and} \quad g_n = \frac{1}{nl(A)} \sum_{e \in E(n)} t(e).$$

We know that $(g_n)_{n \in \mathbb{N}}$ converges almost surely and in L^1, thanks to the usual law of large numbers, thus the family $(g_n)_{n \in \mathbb{N}}$ is equi-integrable. Since $0 \leq f_n \leq g_n$ for every n, the family $(f_n)_{n \in \mathbb{N}}$ is equi-integrable too, so its almost sure convergence towards $\eta_{\theta,h}$ implies its convergence in L^1 towards the same limit.

It remains to show the convergence in L^1 without the condition (10). Let A'' be the translate of A such that $0 \in A''$, and 0 is the centre of A'', thus condition (10) holds for A''. For any fixed n, there exists a segment A'_n which is a translate of nA by an integer vector and such that $d_\infty(0, nA'_n) < 1$ and $d_\infty(nA'', A'_n) < 1$, where d_∞ denotes the distance induced by $\|.\|_\infty$. We want to compare the maximal flow through $\text{cyl}(nA'', h(n))$ with the maximal flow through $\text{cyl}(A'_n, h(n))$. We have to distort a little bit the cylinder $\text{cyl}(nA'', h(n))$. We only consider n large enough so that $h(n) > 1$. Thus the following inclusions hold:

$$\text{cyl}\left(\left(n - \left\lceil \frac{2}{l(A)} \right\rceil\right) A'', h(n) - 1\right) \subset \text{cyl}(A'_n, h(n))$$
$$\text{cyl}(A'_n, h(n)) \subset \text{cyl}\left(\left(n + \left\lceil \frac{2}{l(A)} \right\rceil\right) A'', h(n) + 1\right),$$

where $\lceil x \rceil$ is the smallest integer bigger than or equal to x. We get

$$\phi\left(\left(n - \left\lceil \frac{2}{l(A)} \right\rceil\right) A'', h(n) + 1\right) \leq \phi(A'_n, h(n)) \leq \phi\left(\left(n + \left\lceil \frac{2}{l(A)} \right\rceil\right) A'', h(n) - 1\right),$$

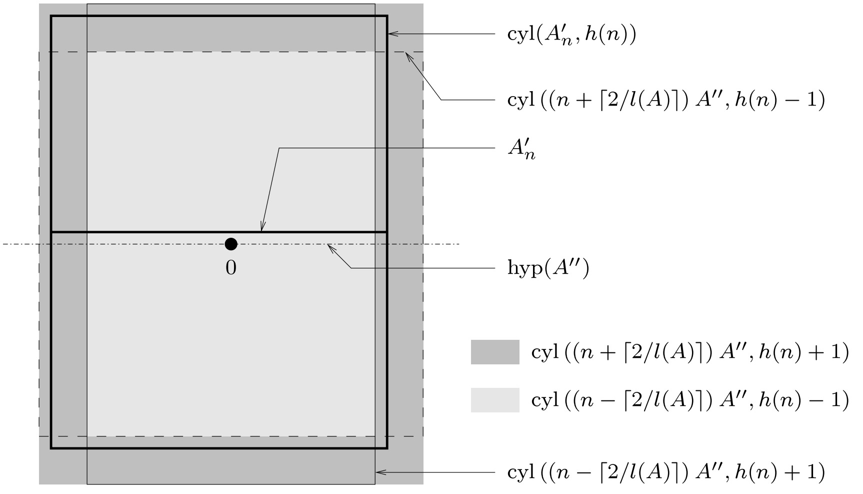

Fig. 4. The cylinder cyl$(A'_n, h(n))$.

(see Fig. 4). Using the convergence in L^1 for A'' which satisfies the condition (10), we see that

$$\frac{\phi\left(\left(n - \left\lceil \frac{2}{l(A)} \right\rceil\right) A'', h(n) + 1\right)}{nl(A)} \quad \text{and} \quad \frac{\phi\left(\left(n + \left\lceil \frac{2}{l(A)} \right\rceil\right) A'', h(n) - 1\right)}{nl(A)}$$

converge to $\eta_{\theta,h}$ in L^1 as n goes to infinity. It is obvious that the small difference in the parameters n and $h(n)$ does not change the value of the limit $\eta_{\theta,h}$. We get the convergence of $\tau(A'_n, h(n))/(nl(A))$ to $\eta_{\theta,h}$ in L^1. But since A'_n is an integer translate of nA, it implies the convergence of $\tau(nA, h(n))/(nl(A))$ to $\eta_{\theta,h}$ in L^1.

Remark 5.1. In dimension $d \geq 3$, if we denote by $\vec{v}$ a unit vector orthogonal to a non-degenerate hyperrectangle A and by $\overrightarrow{\mathcal{D}_n(A)}$ the set of all admissible directions for the cylinder cyl$(nA, h(n))$, i.e., the set of the vectors $\vec{v}'$ in S^{d-1} such that there exists a hyperplane $\mathcal{P}$ orthogonal to $\vec{v}'$ that intersects cyl$(nA, h(n))$ only on its "vertical faces", and if $\lim_{n\to\infty} h(n)/n$ exists (thus $\overrightarrow{\mathcal{D}(A)} = \text{ad}(\underline{\overrightarrow{\mathcal{D}(A)}}) = \text{ad}(\overline{\overrightarrow{\mathcal{D}(A)}})$ exists), we conjecture that

$$\lim_{n\to\infty} \frac{\phi(nA, h(n))}{n^{d-1}\mathcal{H}^{d-1}(A)} = \inf_{\vec{v}' \in \overrightarrow{\mathcal{D}(A)}} \frac{\nu(\vec{v}')}{|\vec{v} \cdot \vec{v}'|} \quad \text{a.s.,}$$

under assumptions (6) on F and if $h(n)$ goes to infinity with n in such a way that we have $\lim_{n\to\infty} \log h(n)/n^{d-1} = 0$. We could not prove this conjecture, because we are not able to prove that $\phi(nA, h(n))$ behaves asymptotically like $\min_{\kappa\in K} \phi^{\kappa}(nA, h(n))$, where K is the set of the flat boundary conditions, i.e., the boundary conditions given by the intersection of a hyperplane with the vertical faces of cyl$(nA, h(n))$.

Acknowledgement

The first author was supported by the Swiss National Science Foundation grants 200021-1036251/1 and 200020-112316/1.

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