1. INTRODUCTION AND STATEMENT OF RESULTS

Let us consider a finite Coxeter system $(W, S)$. This means that $W$ is a finite group and $S \subset W$ is a set of generators such that the following relations form a set of defining relations for $W$,

$$s^2 = 1 \text{ for all } s \in S \quad \text{and} \quad (st)^{m(s, t)} = 1 \text{ for all } s \neq t,$$

where $m(s, t)$ denotes the order of $st \in W$. We will assume throughout that $(W, S)$ is irreducible.
It is the purpose of this paper to prove that the isomorphism problem for integral group rings has a positive solution for $W$, and to classify the automorphisms of the (abstract) group $W$, its integral group ring $\mathbb{Z}W$, and the associated generic Iwahori–Hecke algebra $H$. All of our results can be stated in general terms with one exception: this is the actual classification of group automorphisms of $W$, where type $A_n$ does not fit into a general scheme (see Theorem 1.4). This might also be a reason why our proofs have to depend to some extent on a case by case examination according to the classification of finite irreducible Coxeter systems. We shall now describe our main results in more detail.

Every group automorphism of $W$ induces a permutation of the classes and of the irreducible complex characters of $W$. This defines a map $\text{Aut}(W) \to \text{AutCT}(W)$ where $\text{Aut}(W)$ is the automorphism group of $W$ and $\text{AutCT}(W)$ denotes the group of character table automorphisms of $W$ (see the precise definition in Definition 2.6). The kernel of this map clearly contains all inner automorphisms of $W$. The relation between group and character table automorphisms is completely described by the following result.

**Theorem 1.1.** The sequence $1 \to \text{Inn}(W) \to \text{Aut}(W) \to \text{AutCT}(W) \to 1$ is exact.

There is a standard reflection representation of $W$ by orthogonal transformations on an Euclidean vectorspace $V$ of dimension $|S|$. Thus, we can regard $W$ as a subgroup of $\text{GL}(V)$ where the elements in $S$ are reflections. We will denote throughout by $\rho$ the character of this standard reflection representation of $W$. Since $(W, S)$ is assumed to be irreducible, we have $\rho \in \text{Irr}(W)$. We use [5; 16; 9, Sect. 64] as references for such results on finite Coxeter groups; the construction of $\rho$ can be found in [9, Theorem 64.28], for example. For uniqueness questions concerning this representation see Proposition 5.3.

It will be essential for us to know precisely which automorphisms fix $\rho$. This is given by the following result.

**Theorem 1.2.** A character table automorphism of $W$ fixes the reflection character $\rho$ if and only if it is induced by a graph automorphism.

Note that $\alpha \in \text{Aut}(W)$ is called a graph automorphism if $\alpha(S) = S$.

Now we consider the integral group ring $\mathbb{Z}W$. Let $G \subseteq \mathbb{Z}W$ be a group basis of $\mathbb{Z}W$, i.e., $G$ is a $\mathbb{Z}$-basis of $\mathbb{Z}W$ and a normalized subgroup of the unit group of $\mathbb{Z}W$. Normalized means that all elements in $G$ have augmentation 1. By classical results of Glauberman, Passman, and Saksonov, there exists a bijection between the classes of $G$ and $W$ and a bijection between the irreducible characters of $G$ and $W$ such that the respective character tables together with their powermaps are equal. We will use this to show
that $\rho$ can also be interpreted as the character of a reflection representation of $G$. Since $G$ and $W$ have the same order we can in fact conclude that $G$ and $W$ are isomorphic. (The details of this argument will be given in Section 5.) In particular, the isomorphism problem for integral group rings has a positive solution for $W$. Combining this with the surjectivity of the map $\text{Aut}(W) \to \text{Aut}_\mathbb{T}(W)$ from Theorem 1.1, we can establish the following conjecture of Zassenhaus in the case of Coxeter groups.

**Theorem 1.3** (The Zassenhaus Z2-Conjecture, cf. [32, (2.12)]). Let $G$ be a group basis of $\mathbb{Z}W$. Then there exists a group isomorphism $\alpha: W \to G$ such that the composition of $\alpha$ with the embedding $G \subset \mathbb{Z}W$ induces a central automorphism of $\mathbb{Z}W$.

Note that a central automorphism of any ring is an automorphism which fixes all elements in the center of this ring. In the case of an integral group ring $\mathbb{Z}G$ the Noether–Skolem theorem shows that central automorphisms are precisely those which are given by a conjugation with a unit of $\mathbb{Q}G$.

If the Zassenhaus Z2-Conjecture is valid for $W$ then this implies that every normalized ring automorphism $\sigma$ of $\mathbb{Z}W$ admits a so-called Zassenhaus decomposition. This means that $\sigma$ can be written as the composition of a central automorphism of $\mathbb{Z}W$ and a $\mathbb{Z}$-linearly extended group automorphism of $W$.

By [20, (5.4c)], it follows that the Z2-Conjecture even holds for all finite Coxeter groups (since they are direct products of irreducible ones). Note that the Z2-Conjecture does not hold for arbitrary finite groups. A first counter example was found by Roggenkamp and Scott; see [24] for such a construction and [21, 32, 35] for a survey of known results. The results proved in [2] indicate that it might hold for all simple groups of Lie type. Theorem 1.3 should be seen also in this context because a finite crystallographic Coxeter group provides much information about the representation theory of the associated group of Lie type.

The proofs of the above results will be obtained in Sections 2–5. The logical structure is roughly given as follows. The first and crucial step is to establish Theorem 1.2. After some general preparations in Section 2 concerning class correspondences and their compatibility with certain character theoretic constructions, this will be done in Sections 3 and 4. For this argument we will need a case by case consideration but our proofs are reasonably self-contained in the sense that, both for the classical types and even for the large exceptional types, we only use general information which is available in the tables in Carter’s book [7], for example. All of the remaining assertions and the proof of Theorem 1.3 then follow from rather general arguments about finite reflection groups, see (4.6) and Section 5.

We remark that our approach provides new (and, in our opinion, more conceptual) proofs for some existing results. For example, the character
table automorphisms of the symmetric group $\mathfrak{S}_n$ (the Coxeter group of type $A_{n-1}$) have been determined by Peterson [31]. The reader should compare Peterson’s argument with our approach described in Subsection 2.7, which also provides a good illustration of our methods in general. See also Remark 5.6 concerning known results about the isomorphism problem and the Zassenhaus $Z_2$-Conjecture.

The above results will be completed by the following full description of all character table automorphisms of $W$, and hence of $\text{Aut}(W)/\text{Inn}(W)$ by Theorem 1.1.

Let $I \in \text{GL}(V)$ denote the identity and assume that $-I \in W \subset \text{GL}(V)$. We say that $\tau \in \text{Aut}(W)$ is a duality automorphism if it is induced by a group automorphism of the form $s \mapsto (\pm I)s$ for all $s \in S$. (This will be discussed in more detail in Lemma 3.3.)

We say that $\tau \in \text{Aut}(W)$ is a field automorphism if it is given by algebraic conjugation in the field extension of $\mathbb{Q}$ containing the values of all irreducible characters of $W$.

With this notation we can now state (the proof will also be given in Sections 3 and 4):

**Theorem 1.4.** If $(W, S)$ is not of type $A_n$ then every character table automorphism of $W$ can be written as the composition of an automorphism fixing $\rho$, a duality, and a field automorphism.

In the exceptional case of type $A_1$ it is well known that the group $W$ is isomorphic to the symmetric group $\mathfrak{S}_2$ and $\text{Aut}(W)$ is cyclic of order 2, generated by an automorphism which maps the class of cycle type $(2)$ to the class of cycle type $(2, 2, 2)$.

At the end of Section 4 we give a table with the orders of the groups $\text{Aut}(W)$ for all types of $(W, S)$ and describe their structure.

The main application of the above results is a classification of the automorphisms of the generic Iwahori–Hecke algebra $H$ associated with $(W, S)$. This algebra can be viewed as a deformation of the integral group ring of $W$ over the ring $\mathbb{A} = \mathbb{Z}[v, v^{-1}]$ of Laurent polynomials in one indeterminate. The algebra $H$ is a free $\mathbb{A}$-module with basis $(T_w | w \in W)$, and the multiplication is given by the rules

$$T_w T_w' = T_{ww'} \quad \text{if} \quad l(ww') = l(w) + l(w'),$$
$$T_s^2 = uT_1 + (u - 1)T_s \quad \text{for} \quad s \in S,$$

where $u := v^2$ and $l: W \to \mathbb{N}_0$ is the usual length function on $W$. We call the element $u$ the parameter of $H$. (For the construction of this algebra see [9, Sect. 68A], for example.)
The first step in classifying the \( \mathcal{A} \)-algebra automorphisms of \( H \) is to define an analogue of the augmentation map on \( \mathbb{Z}W \) and to show that every automorphism of \( H \) can be assumed without loss of generality to be normalized.

**Definition 1.5.** The 1-dimensional representation of \( H \) given by \( \text{ind}: \mathbb{Z} \to \mathbb{Z}H \) is called the augmentation map of \( H \). An arbitrary \( \mathcal{A} \)-algebra automorphism \( \sigma: H \to H \) will be called normalized if \( \text{ind} \circ \sigma = \text{ind} \).

There is a distinguished, non-normalized \( \mathcal{A} \)-algebra automorphism of \( H \). This will be called Alvis–Curtis duality. It is defined as the unique \( \mathcal{A} \)-algebra automorphism \( \ast: H \to H \) which maps \( T_s \mapsto -uT_s^{-1} \) for all \( s \in S \). The composition of the representation \( \text{ind} \) with this duality operation gives the 1-dimensional representation \( \text{sgn}: T_s \mapsto -1 \ (s \in S) \), which corresponds to the sign representation of \( W \).

Let \( K \) be the field of fractions of \( \mathcal{A} \). If \( F \) is any field of characteristic 0 and \( q \in F \) is any non-zero element which has a square root in \( F \) there exists a unique ring homomorphism \( f: \mathcal{A} \to F \) such that \( f(v) = q^{1/2} \). Then we can regard \( F \) as an \( \mathcal{A} \)-module via \( f \) and obtain a corresponding specialized algebra \( FH := F \otimes_q H \). Every such specialization induces a decomposition map between the Grothendieck groups of irreducible representations of \( KH \) and \( FH \) (see [11]); this is completely analogous to the \( p \)-modular decomposition map for a finite group \( G \) and a prime \( p \). As in [4] for integral group rings an \( \mathcal{A} \)-algebra automorphism of \( H \) induces automorphisms of \( KH \) and \( FH \) and it must be compatible with the decomposition map. The action of the \( \mathcal{A} \)-algebra automorphisms on quotients of \( FH \) which are Brauer tree algebras provides good information about the nature of these automorphisms. This is again completely analogous to the methods used in [4]. It turns out that we obtain in particular strong obstructions to the existence of \( \mathcal{A} \)-algebra automorphisms of \( H \) by using the specializations \( v \mapsto 1 \), in which case \( FH \) is the group algebra of \( W \), and \( v \mapsto \zeta_{2h} \), where \( h \) is the Coxeter number of \( W \) and \( \zeta_{2h} \) is a primitive \( 2h \)-root of unity. In the latter case, we determine explicitly the corresponding decomposition numbers in Theorem 6.6. This will allow us to prove the following result which was actually the main motivation for this article.

**Theorem 1.6.** If \( \sigma: H \to H \) is an \( \mathcal{A} \)-algebra automorphism of \( H \), then either \( \sigma \) or the composition of \( \sigma \) with the Alvis–Curtis duality is normalized. Moreover, if \( \sigma \) is normalized, then \( \sigma \) is the composition of a graph automorphism and a central automorphism.

The group of \( \mathcal{A} \)-algebra automorphisms of \( H \) is a semidirect product

\[
\text{Aut}_\mathcal{A}(H) \cong \text{Aut}_{\text{cen}}(H) \rtimes (\Gamma \times \langle \ast \rangle),
\]
where $\text{Aut}_{\text{cen}}(H)$ denotes the group of central $A$-algebra automorphisms of $H$, $\Gamma$ is the group of outer graph automorphisms, and $\ast$ is the Alvis–Curtis duality. $\Gamma = 1$ if and only if $H$ is of type $A_n$, $E_6$, $E_7$, $E_8$, $H_3$, $H_4$, $B_n$ with $n \geq 3$, $D_n$ with $n$ odd, or $I_m(m)$ with $m$ odd.

A graph automorphism of $H$ is by definition an automorphism such that $T_{\alpha} \mapsto T_{\alpha^{(w)}}$, where $\alpha$ is a graph automorphism of $W$. Such an automorphism is called outer if the corresponding graph automorphism of $W$ is an outer automorphism (i.e., not given by conjugation with some element in $W$). It is clear that for Hecke algebras, Theorem 1.6 is the analogue to the existence of a Zassenhaus decomposition for every normalized ring automorphism of $ZG$ for some finite group $G$. For generic Hecke algebras the specific set of generators $\{T_w, w \in W\}$ play the role of the group $G$ for integral group rings $ZG$. If $\Gamma = 1$ all normalized $A$-algebra automorphisms of $H$ are central. This certainly justifies us in saying that such generic Iwahori–Hecke algebras are rigid.

In Proposition 6.11 we deduce from Theorem 1.6 a classification of the $A$-algebra automorphisms of generic Iwahori–Hecke algebras $H$ with unequal parameters. In case that the Coxeter diagrams are not simply laced, the graph automorphisms do not induce $A$-algebra automorphisms of $H$. Thus, if unequal parameters are permitted all generic Iwahori–Hecke algebras which are not of type $D_2n$ are rigid.

It is likely that at least some of the above results extend to the case where $W$ is a finite complex reflection group. This will be discussed elsewhere.

### 2. Character Table Automorphisms and Class Correspondences

In this section we consider two finite groups $H$ and $G$ and bijections from the set of conjugacy classes of $H$ to the set of conjugacy classes of $G$ (if such bijections exist at all). Among all such bijections we take a closer look at what we would like to call central Brauer class correspondences which appear in problems concerning isomorphisms between the integral group rings of $H$ and $G$ (cf. Section 5). Definitions 2.1, 2.6 and the (easy) results in Lemma 2.2, Corollary 2.3 are usually formulated under the assumption that a class correspondence is induced by a normalized isomorphism of the groups rings of $H$ and $G$. Our point here is to develop the theory starting with Definition 2.1. The main applications are Subsection 2.4 and Corollary 2.5, where compatibility properties of these class correspondences with the construction of symmetrizations of characters are considered. These properties are the key tools in the proofs of the results stated in Section 1.
Let us first introduce some notation. For any finite group $G$ we denote by $\text{Cl}(G)$ the set of conjugacy classes of $G$. For $g \in G$ we let $C_g$ be the conjugacy class of $g$ in $G$, and $\bar{C}_g$ the corresponding class sum in $\mathbb{Z}G$. For any integer $k \geq 1$ the $k$th powermap is defined as the map $\text{Cl}(G) \to \text{Cl}(G)$, $C \mapsto C^{[k]}$, where $C^{[k]}$ is the class of an $k$th power of an element in $C$.

Let $\text{CF}(G)$ denote the space of class functions $f: C \to \mathbb{C}$. We extend such a function linearly to $\text{Cl}(G)$ and denote it by the same symbol. In order to simplify notation we shall also regard a class function as a map $f: \text{Cl}(G) \to \mathbb{C}$ so that we can write $f(C_g)$ instead of $f(g)$. For $f_1, f_2 \in \text{CF}(G)$ we let

$$(f_1, f_2)_G = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)} = \frac{1}{|G|} \sum_{C \in \text{Cl}(G)} |C| f_1(C) \overline{f_2(C)}$$

denote their usual hermitian scalar product.

Definition 2.1. Let $H$ and $G$ be two groups such that $\text{Cl}(G)$ and $\text{Cl}(H)$ have the same cardinality. Let $\tau: \text{Cl}(H) \to \text{Cl}(G)$ be a bijection. We say that $\tau$ is central if the induced $\mathbb{Z}$-linear map

$$\hat{\tau}: \mathbb{Z}(\mathbb{Z}H) \to \mathbb{Z}(\mathbb{Z}G), \quad \hat{C} \mapsto \tau(C)$$

is a ring homomorphism which preserves the identity elements. If, moreover, $\tau$ commutes with all powermaps on $\text{Cl}(H)$ and $\text{Cl}(G)$ we say that $\tau$ is a central Brauer class correspondence.

Any bijection $\tau: \text{Cl}(H) \to \text{Cl}(G)$ induces a map $\tau^*: \text{CF}(G) \to \text{CF}(H)$, $f \mapsto f \circ \tau$ (note our above conventions). We say that $\tau$ preserves characters if $\tau^*$ maps irreducible characters of $G$ to irreducible characters of $H$.

The following result will show that the conditions “central” and “character preserving” are equivalent. Hence the existence of a central Brauer class correspondence between $H$ and $G$ is equivalent to the property that $H$ and $G$ have the same character tables and also the same powermaps. Two groups with the latter property are usually called a “Brauer pair,” and this is why we use the name “Brauer” in Definition 2.1; see also [23] for a closely related study of class correspondences between finite groups.

Lemma 2.2. A bijection $\tau: \text{Cl}(H) \to \text{Cl}(G)$ is central if and only if it preserves irreducible characters. In this case, $\tau$ preserves the lengths of classes and, in particular, $G$ and $H$ have the same order. Moreover, $\tau^*: \text{CF}(G) \to \text{CF}(H)$ is an isometry.
Proof. For $C_1, C_2 \in \text{Cl}(H)$ we have

$$\hat{C}_1 \hat{C}_2 = \sum_{C \in \text{Cl}(H)} a(C_1, C_2, C) \hat{C},$$

where $a(C_1, C_2, C) = \| (x, y) \in C_1 \times C_2 \mid xy = z \|$ for some fixed element $z \in C$. The coefficients are determined from the character table of $H$ by the formula

$$a(C_1, C_2, C) = \left| \frac{C_1}{C} \right| \frac{\chi(C_1) \chi(C_2) \overline{\chi(C)}}{\chi(1)}.$$  

Assume now that $\tau$ preserves characters. Since the lengths of the conjugacy classes of a finite group are determined by the second orthogonality relations for the irreducible characters of that group, we have $|\tau(C)| = |C|$ for all $C \in \text{Cl}(H)$. We can then replace each $\chi$ by $\tau^*(\chi)$ on the right hand side of the above formula and obtain that

$$a(C_1, C_2, C) = a(\tau(C_1), \tau(C_2), \tau(C)).$$

This proves that $\hat{\tau}$ is a ring homomorphism. Since the trivial class is the only class such that all character values are positive real numbers, the identity elements are preserved under $\hat{\tau}$.

Conversely, assume that $\hat{\tau}$ is a ring homomorphism. At first we claim that then $\tau$ preserves class lengths. Indeed, let $x \in H$ and consider the product

$$\hat{C}_x \hat{C}_{x^{-1}} = \sum_{C \in \text{Cl}(H)} a(C_x, C_{x^{-1}}, C) \hat{C}.$$  

The above formula for the coefficients shows that $a(C_x, C_{x^{-1}}, (1)) = |C_x|$. Since $\hat{\tau}$ preserves the identity elements, the class $(1)$ also appears with multiplicity $|C_x|$ in the product of the images of $\hat{C}_x$ and $\hat{C}_{x^{-1}}$ under $\hat{\tau}$. In particular, this multiplicity is non-zero and so $\tau(C_{x^{-1}})$ is the class containing the inverses of the elements in the class $\tau(C_x)$. This implies our claim.

Let $\chi \in \text{Irr}(G)$ and $\omega_\chi : Z(CG) \to \mathbb{C}$ the corresponding central character. Then the composition of $\hat{\tau}$ (extended $\mathbb{C}$-linearly to $Z(C \text{Cl}(H))$ with $\omega_\chi$ is an algebra homomorphism $Z(C \text{Cl}(H)) \to \mathbb{C}$. Hence it must be of the form $\omega_\psi$ for a unique $\psi \in \text{Irr}(H)$. The formula

$$\omega_\chi(\hat{C}) = \left| \frac{C}{C} \right| \frac{\chi(C)}{\chi(1)} \quad (\text{for } C \in \text{Cl}(G))$$

then implies that $\tau^*(\chi) = \psi$, hence $\tau$ preserves irreducible characters.
The above arguments prove the asserted equivalence of $\tau$ being central and $\tau$ being character preserving. We have also seen that then $\tau$ preserves the lengths of classes. This implies that

$$(f_1, f_2)_G = (\tau^*(f_1), \tau^*(f_2))_H$$

for all $f_1, f_2 \in \text{CF}(G)$,

hence $\tau$ defines an isometry from $\text{CF}(G)$ to $\text{CF}(H)$. $\blacksquare$

We will now derive some restrictions on the possible effect of a central Brauer class correspondence $\tau: \text{Cl}(H) \to \text{Cl}(G)$. First we draw conclusions from the property that $\tau$ is central. The first part of the following result was stated in Peterson [31, Lemma 2.1] under the assumption that $\tau$ is induced by a normalized isomorphism between the integral group rings of $H$ and $G$. The second part appears in a similar context in Passman [30, p. 567].

**Corollary 2.3.** Let $\tau: \text{Cl}(H) \to \text{Cl}(G)$ be a central bijection.

(a) Let $x, y \in H, u, v \in G$, and assume that $\tau(C_x) = C_u$, $\tau(C_y) = C_v$. Then there exists some $g \in G$ such that $\tau(C_{xy}) = C_{ug}^{-1}v^g$.

(b) Let $C_1, \ldots, C_k \in \text{Cl}(H)$ such that $N := C_1 \cup \cdots \cup C_k \subseteq H$ is a normal subgroup. Then $\tau(N) := \tau(C_1) \cup \cdots \cup \tau(C_k) \subseteq G$ also is a normal subgroup. Moreover $\tau$ induces an isomorphism between the lattices of normal subgroups of $H$ and $G$.

**Proof.** See [31, (2.1); 30, p. 567]; the proofs given there only use the fact that $\tau$ is central. Part (a) follows easily from the definitions. To prove (b), let $e \in \mathbb{Z}H$ be the sum of the elements in $N$ and use the equation $e^2 = |N|e$. $\blacksquare$

Next we consider implications of the property that a central Brauer class correspondence is also powermap preserving.

**2.4. Symmetrizations of Characters.** Let $V$ be a $CG$-module, affording the character $\chi$ say. Consider the $r$-fold tensor product $V^{\otimes r}$, for some integer $r \geq 1$. The symmetric group $S_r$ acts on $V^{\otimes r}$ by permuting the factors in tensors of the form $v_1 \otimes \cdots \otimes v_r$ (where $v_j \in V$). By convention, we also let $V^{\otimes 0} = \mathbb{C}$ be the trivial module and $S_0 = (1)$ be the trivial group. Now $V^{\otimes r}$ is a $(CG, C[S_r])$-bimodule. The irreducible characters of $S_r$, are parametrized by partitions of $r$, and we denote by $[\lambda]$ the character corresponding to the partition $\lambda$. For each such partition $\lambda$ we consider the $[\lambda]$-isotypic part in $V^{\otimes r}$ and denote it by $V[\lambda]$. Its character is given
by the formula (see [19, (5.2.13)])

\[
(\chi \star [\lambda])(g) = \frac{1}{r!} \sum_{\pi \in \mathfrak{S}_r} [\lambda](\pi) \prod_{i=1}^r \chi(g^i)^{a_i(\pi)} \quad \text{for } g \in G,
\]

where \(a_i(\pi)\) denotes the number of cycles of length \(i\) in the permutation \(\pi \in \mathfrak{S}_r\). For each \(r \geq 0\) and each partition \(\lambda\) of \(r\), we have thus defined an operation, \(\chi \mapsto \chi \star [\lambda]\), on the set of actual characters of \(G\). We can use the above formula to define a class function \(f \star [\lambda]\) for every \(f \in \text{CF}(G)\).

Given a powermap preserving bijection \(\tau \colon \text{Cl}(H) \rightarrow \text{Cl}(G)\), it then follows that

\[
\tau^*(f \star [\lambda]) = \tau^*(f) \star [\lambda]
\]

for all \(f \in \text{CF}(G)\) and all partitions \(\lambda\) of \(r \geq 0\). Hence, in particular, the \(\star\)-operation is compatible with central Brauer class correspondences. Similar compatibility properties have also been studied in [3] in a slightly different context.

We can use this to obtain the following result about characteristic polynomials: as above let \(V\) be a \(CG\)-module affording the character \(\chi\). For \(g \in G\) we denote by

\[
\text{char.pol}(C_g, \chi) := \det(X \cdot \text{id} - \varphi_g) \in \mathbb{C}[X]
\]

the characteristic polynomial of the endomorphism \(\varphi_g \colon V \rightarrow V\) given by the action of \(g\). (Note that this only depends on \(C_g\) and \(\chi\).)

**Corollary 2.5.** Let \(\tau \colon \text{Cl}(H) \rightarrow \text{Cl}(G)\) be a central Brauer class correspondence and \(\chi\) a character of \(G\). Then \(\text{char.pol}(\tau(C), \chi) = \text{char.pol}(C, \tau^*(\chi))\) for all \(C \in \text{Cl}(H)\).

**Proof.** Let \(V\) be a vectorspace and \(\varphi \colon V \rightarrow V\) any endomorphism. Denote by \(\varphi^\wedge r\) the induced endomorphism on the \(r\)th exterior power of \(V\) (for \(r \geq 0\)). Then

\[
\text{char.pol}(\varphi, V) = \sum_{r=0}^{\dim V} (-1)^r \text{trace}(\varphi^\wedge r) X^{n-r}.
\]

Now let \(V\) be a \(CG\)-module affording \(\chi\) and \(\varphi_g \colon V \rightarrow V\) the endomorphism given by the action of \(g \in G\). Then, by definition, \(\text{trace}(\varphi_g^\wedge r) = (\chi \star [\lambda_g])(g)\) with \(\lambda_g = (1^r)\). So the coefficients in \(\text{char.pol}(C_g, \chi)\) are given by the character values \((\chi \star [\lambda_g])(g)\), for \(0 \leq r \leq \dim V\). This completes the proof, since the \(\star\)-construction is compatible with \(\tau^*\) by Subsection 2.4. \(\blacksquare\)
Definition 2.6. If $H = G$ the set of all bijections $Cl(G) \to Cl(G)$ is a group (under composition of maps). We call

$$\text{AutCT}(G) := \{ \tau : Cl(G) \to Cl(G) \mid \tau \text{ central Brauer class correspondence} \}$$

the group of character table automorphisms of $G$. (This coincides with the usual definition, see, for example, [4].) By Lemma 2.2, $\text{AutCT}(G)$ has a faithful permutation representation on $\text{Irr}(G)$ via the map $\chi \mapsto \chi \circ \tau^{-1}$ (where $\tau \in \text{AutCT}(G)$ and $\chi \in \text{Irr}(G)$).

Before we study finite Coxeter groups in general we consider the following quite typical application of the above techniques.

2.7. The Symmetric Groups. The character table automorphisms of $G = \mathfrak{S}_n$ have been determined by Peterson [31]: they are always trivial unless $n = 6$ in which case there is a unique non-trivial automorphism given by the group automorphism which maps the class of cycle type $(2)$ to the class of cycle type $(2, 2, 2)$.

Our methods will provide a new proof of this result. The first step in our argument is to consider the character $\chi$ of the natural permutation representation of $G$ and to show that if $\tau \in \text{AutCT}(G)$ fixes $\chi$ then $\tau$ is trivial. This is done as follows, using the above techniques.

The classes of $G$ are parametrized by partitions of $n$. Let $\alpha = (1 \leq \alpha_1 \leq \cdots \leq \alpha_k)$ be such a partition and $C_\alpha$ the corresponding class. Let $\tau \in \text{AutCT}(G)$ fix $\chi$. Then Corollary 2.5 implies that $\text{char.pol}(C_\alpha, \chi) = \text{char.pol}(C_\beta, \chi)$ where $\beta$ is the partition of $n$ such that $C_\beta = \tau(C_\alpha)$. Now note that $\text{char.pol}(C_\alpha, \chi) = f_\alpha$ where, for any partition $\alpha$ as above, we define the polynomial

$$f_\alpha := \prod_{i=1}^{k} (X^{\alpha_1} - 1).$$

It remains to observe the purely combinatorial fact that the partition $\alpha$ is uniquely determined by the polynomial $f_\alpha$. (Indeed, the smallest part $\alpha_1$ of $\alpha$ is determined by the fact that the lowest non-constant term of $f_\alpha$ is the monomial $(-1)^{k-1}mX^{\alpha_1}$, where $m$ is the number of parts of $\alpha$ equal to $\alpha_1$; so we can divide by $X^{\alpha_1} - 1$ and proceed by induction on the number of parts of $\alpha$.) Hence $\alpha = \beta$ and so $\tau(C_\alpha) = C_\alpha$. This holds for all $\alpha$, and so $\tau$ is trivial.

The character $\chi$ can be written in the form $\chi = 1_G + \rho$ where $\rho$ is the character of the reflection representation of $G$, regarded as the Coxeter group on the generators $((1, 2), \ldots, (n-1, n))$. We have just shown that the stabilizer of $\rho$ in $\text{AutCT}(G)$ is trivial, hence Theorem 1.2 holds for $G = \mathfrak{S}_n$. 
To complete the classification of all character table automorphisms of $G$ we have two alternative methods. One will be provided in Section 5 (cf. Corollary 5.4) where we show that every character table automorphism of a finite irreducible Coxeter group can be modified by a group automorphism so that it fixes the reflection character $\rho$. It then remains to use the known results on $\text{Aut}(G)$ (see [17, Satz II.5.5], for example). The other, more direct approach is based on the following observation (see [19, Theorem 2.4.10]):

(a) For $n \neq 1, 2, 3, 6$ there are only two irreducible characters of degree $n - 1$; these are the characters $\rho$ and $\rho \otimes \text{sgn}$.

(b) The group $\Xi_6$ has four irreducible characters of degree 5.

Now let $\tau \in \text{Aut}(G)$ be arbitrary. We want to show that $\tau$ is trivial if $n \neq 6$. We consider the character $\rho' := \tau^*(\rho)$. By what we proved before it is sufficient to show that $\rho' = \rho$. For $n = 1, 2, 3$ there is nothing to prove. Now let $n \geq 4$ and assume, if possible, that $\rho' = \rho \otimes \text{sgn}$. Let $C \in \text{Cl}(G)$ be the class containing the basic transposition $(1, 2)$. Then $\rho(C) = \rho'(C) = \rho \otimes \text{sgn}(C) = -\rho(C) = -(n - 2)$. But we have that $\chi = 1 \rho + \rho'$ is the character of a permutation representation of $G$. This implies, in particular, that $\chi(g) \geq 0$ for all $g \in G$ and hence $-(n - 2) = \rho(\tau(C)) \geq -1$, contradicting our assumption that $n \geq 4$. Hence this is impossible and we have $\rho' = \rho \otimes \text{sgn}$. Using the above property (a) this forces $\rho' = \rho$ if $n \neq 6$. The case $n = 6$ can be handled by checking explicitly the character table of $\Xi_6$ (see [19, p. 350]) and using the existence of the exceptional group automorphism described in [17, Satz II.5.5]. In any case, we have the following “uniform” result: every character table automorphism of $\Xi_n$ (for $n \geq 2$) which fixes the class containing the basic transposition $(1, 2)$ must be trivial.

3. CHARACTER TABLE AUTOMORPHISMS FOR EXCEPTIONAL TYPES

Throughout this and the following section we let $(W, S)$ be an irreducible finite Coxeter system, and $\rho$ the character of the standard reflection representation. Let $\text{Aut}(\rho(W))$ be the subgroup of all $\tau \in \text{Aut}(W)$ such that $\tau^*(\rho) = \rho$. The principal aim in this section is to establish some general results which will be useful in the proof of Theorem 1.2, and to check this theorem for Coxeter groups of exceptional types: for $E_6$, $E_7$, $E_8$ see Subsection 3.5, for $F_4$ see Subsection 3.6, for $I_2(m)$ see Subsection 3.7, for $H_3$, $H_4$ see Subsection 3.8. The classical types $B_n$ and $D_n$ will be considered in Section 4.
In the course of the proof we also produce additional information for each type to show that Theorem 1.4 holds.

**Lemma 3.1.** Let \( C \in \text{Cl}(G) \) such that \( C \cap S \neq \emptyset \) and \( \tau \in \text{Aut}_{\text{CT}}(W) \). Then \( \tau(C) \cap S \neq \emptyset \).

**Proof.** Let \( w \in \tau(C) \). We must show that \( w \) is conjugate to some element in \( S \). Since \( \tau \) preserves the powermaps, \( w \) is an element of order 2. Let \( W \subseteq \text{GL}(V) \) via the standard reflection representation. Then each \( s \in S \) is a reflection with root \( v_s \in V \) say. By a result of Richardson (see [16, Sect. 8.3]) we can find a subset \( J \subseteq S \) such that \(-I \in \text{GL}(V_j)\) lies in \( W_j \) and is conjugate to \( w \). Here, \( W_j \subseteq W \) denotes the subgroup generated by \( J \) and \( V_j \subseteq V \) the subspace spanned by the roots \( v_s, s \in J \); note that \( W_j \) leaves \( V_j \) invariant and we obtain an embedding \( W_j \subseteq \text{GL}(V_j) \).

It will therefore be sufficient to show that \( J \) consists of just one element of \( S \). Since \( w \) is conjugate to \(-I \in \text{GL}(V_j)\) the character value \( \rho(w) \) or, in other words, the trace of \( w \in \text{GL}(V) \) is given by \(|S| - 2|J|\). Since \( \tau \) fixes \( \rho \), this value must be equal to \(|S| - 2\) and so we conclude that \(|J| = 1\). 

**Lemma 3.2.** Every graph automorphism of \((W, S)\) fixes \( \rho \).

**Proof.** Let \( W \subseteq \text{GL}(V) \) such that \( \rho \) is given by the usual trace. There is a basis \( \{e_s \mid s \in S\} \) in \( V \) such that each \( s \in S \) is represented by the reflection with root \( e_s \), and the set \( \{e_s \mid s \in S\} \) is a system of fundamental roots (see [9, Theorem 64.28]).

Let \( \alpha : W \rightarrow W \) be a group automorphism such that \( \alpha(S) = S \). Then we can define a new action of \( W \) on \( V \) by composing the original action with \( \alpha \). Denote this new \( W \)-module by \( V^{\alpha} \); the corresponding character is \( \alpha^*(\rho) \). Since \( \alpha(S) = S \) the map \( V \rightarrow V^{\alpha}, e_s \mapsto e_{\alpha(s)} \), commutes with the action of \( W \) (see the explicit description of the action of \( W \) on \( V \) in [9, Theorem 64.28]). Hence \( V \) and \( V^{\alpha} \) are isomorphic as \( W \)-modules and so \( \alpha^*(\rho) = \rho \).

The following result shows that duality automorphisms provide good examples of table automorphisms which do not fix \( \rho \).

**Lemma 3.3.** Assume that \(-I \in W \) and let \( \mu : W \rightarrow \{\pm 1\} \) be a linear character such that \( \mu(-I) = 1 \). Then there exists a unique \( \delta \in \text{Aut}(W) \) such that \( \delta(s) = \mu(s)s \) for all \( s \in S \). Every duality automorphism in \( \text{Aut}_{\text{CT}}(W) \) is given by a group automorphism \( \delta \) arising in this way.

**Proof.** Assume first that \( \delta \in \text{Aut}(W) \) is such that \( \delta(s) = \mu_s s \) for all \( s \in S \) and some \( \mu_s = \pm 1 \). The defining relations for \((W, S)\) show that \( (\mu_s \mu_t)^{m(s, t)} = 1 \). Hence there exists a unique homomorphism \( \mu : W \rightarrow \{\pm 1\} \) such that \( \mu(s) = \mu_s \) for all \( s \in S \). Every group automorphism must fix \(-I \in W \). So we have \(-I = \delta(-I) = -\mu(-I)I \) and hence \( \mu(-I) = 1 \).
Conversely, if \( \mu : W \to \{ \pm 1 \} \) is a linear character we check using the defining relations that there is a unique group homomorphism \( \delta : W \to W \) such that \( \delta(s) = \mu(s)s \) for all \( s \in S \). We then have \( \delta(w) = \mu(w)w \) and so the kernel of \( \delta \) is contained in \( \{ \pm I \} \subseteq W \). This kernel is trivial if and only if \( \mu(-1) = 1 \).

There is one typical example of this construction. The unique linear character \( \text{sgn} : s \mapsto -1 \) (\( s \in S \)) is called the sign character of \( W \). We clearly have \( \text{sgn} = \det \) under the embedding \( W \subseteq \text{GL}(V) \). Suppose that \( -I \in W \). Then \( \text{sgn}(-I) = 1 \) if and only if \( |S| \) is even. Using furthermore the characterization of all types \( (W, S) \) for which \( -I \in W \) in [16, p. 59], we conclude that \( \text{sgn} \) defines a duality automorphism if and only if \( (W, S) \) is of type \( B_n (u \geq 2 \text{ even}), D_n (n \geq 4 \text{ even}), F_4, E_6, H_4, \) or \( I_2(m) (m \geq 4 \text{ even}) \).

### 3.4. Fake Degrees of Finite Coxeter Groups

Let \( (W, S) \) be arbitrary. It is possible to define a grading of the regular representation of \( W \) using the ring of co-invariants of \( W \) in its reflection representation (see [7, Sect. 11.1]). For \( \chi \in \text{Irr}(W) \) denote by \( n_i(\chi) \) the multiplicity of \( \chi \) in the \( i \)th piece of this grading and let

\[
P(\chi) := \sum_i n_i(\chi) X^i \in \mathbb{Z}[X].
\]

This polynomial is called the fake degree of \( \chi \), and the formula in [7, Proposition 11.1.1], shows that it can be computed from the character values of \( \chi \) and the characteristic polynomials of \( \rho \). Using Corollary 2.5 we can therefore conclude that

\[
P(\tau^*(\chi)) = P(\chi) \quad \text{for all } \chi \in \text{Irr}(W) \text{ and all } \tau \in \text{AutCT}_\rho(W).
\]

It follows for example that if every irreducible character of \( W \) is uniquely determined by its fake degree then \( \text{AutCT}_\rho(W) = 1 \). Now the fake degrees are known in all cases (or can be easily computed from the ordinary character table of \( W \)).

In the following paragraphs we use the above techniques to prove Theorem 1.2 and Theorem 1.4 for Coxeter groups of exceptional types.

### 3.5. Types \( E_6, E_7, E_8 \)

In these cases the tables in [7, Sect. 13.2] show that each \( \chi \in \text{Irr}(W) \) is determined by the pair \( (n, e) \) where \( n \) is the degree of \( \chi \) and \( e \) is the maximal power of \( X \) dividing the fake degree \( P(\chi) \). Hence Subsection 3.4 implies, without any further computation, that

(a) \( \text{AutCT}_\rho(W) = 1 \) for \( (W, S) \) of type \( E_6, E_7, E_8 \).

From the tables in [7, Sect. 13.2], we see that in each of these cases there are precisely 2 irreducible characters of degree \( |S| \). This and the previous
result already imply that $|\text{AutCT}(W)| \leq 2$. We now show that

(b) if $(W, S)$ is of type $E_6$, $E_7$ then $\text{AutCT}(W) = 1$, and

(c) if $(W, S)$ is of type $E_8$ then $\text{AutCT}(W)$ is cyclic of order 2 and generated by the non-trivial duality automorphism (given by $\text{sgn}$).

If $(W, S)$ is of type $E_6$ or $E_7$ then the conjugacy class of $S$ is the only class on which $\rho$ has value $+4$ or $+5$, respectively. Hence any $\tau \in \text{AutCT}(W)$ fixes $\rho$ and we are done by (a). (This property of the character values of $\rho$ can be checked, for example, using the characteristic polynomials in [6].)

If $(W, S)$ is of type $E_8$ then $-I \in W$ and $-I$ is in the kernel of $\text{sgn}$. Hence $\text{sgn}$ gives rise to a non-trivial duality automorphism (see the remarks following Lemma 3.3). This duality then interchanges the two characters $\rho$ and $\text{sgn} \otimes \rho$. Since these characters are the only characters of degree 8 we can compose any $\tau \in \text{AutCT}(W)$ with this duality if necessary, and then assume that $\tau \in \text{AutCT}_r(W)$. Again, we are done using (a).

### 3.6. Type $F_4$

Let $(W, S)$ be of type $F_4$. We show that

(a) every character table automorphism which fixes $\rho$ is given by a graph automorphism, and

(b) every character table automorphism can be modified by a duality automorphism so that it fixes $\rho$.

In particular, $\text{AutCT}(W)$ is isomorphic to the dihedral group of order 8.

To prove these statements we use the character table of $W$ which is printed in [7, p. 413]. Let $\tau \in \text{AutCT}(W)$. There are only four faithful irreducible characters of degree 4, and these are obtained from $\rho$ by tensoring with the four linear characters of $W$. Since the latter have $-I \in W$ in their kernel, they give rise to duality automorphisms by Lemma 3.3. This proves (b). Now let $\tau \in \text{AutCT}_r(W)$. Let $C$ and $C'$ denote the 12th and the 17th class in this table. These classes contain generators in $S$ and they are interchanged by the unique non-trivial graph automorphism of $W$. Lemma 3.1 shows that $\tau$ either fixes or interchanges these two classes. By composing $\tau$ with the non-trivial graph automorphism if necessary, we may therefore assume that $\tau(C) = C$ and $\tau(C') = C'$. But then we see in the table that every irreducible character of degree $\neq 6$ is uniquely determined by its values on 1, $C, C'$. Since the two characters of degree 6 are fixed anyway (they are the 2nd and 3rd exterior power of $\rho$), $\tau$ must be trivial. This proves (a).

### 3.7. Dihedral Groups

Let $m \geq 3$ and $W = \langle a, s \mid a^m = s^2 = 1, sas = a^{-1} \rangle$ the dihedral group of order $2m$. If we set $t := sa$ and $S := \{s, t\}$ then $(W, S)$ is a Coxeter system of type $I_2(m)$ (including types $A_2, B_2, G_2$). The irreducible characters of $W$ of degree 2 are afforded by the representa-
tions (cf. [17, Hilfssatz V.21.5])

\[
\varphi_j: a \mapsto \begin{pmatrix} e^j & 0 \\ 0 & e^{-j} \end{pmatrix}, \quad s \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

where \( e := \exp(2\pi i/m) \in \mathbb{C} \) is a primitive \( m \)th root of unity and \( 1 \leq j \leq (m - 2)/2 \) for \( m \) odd, \( 1 \leq j \leq (m - 2)/2 \) for \( m \) even, respectively. The standard reflection character corresponding to \( S \) is \( \rho = \varphi_1 \). We show that

(a) every \( \tau \in \text{Aut}_\rho(W) \) is given by a graph automorphism of \((W, S)\), and

(b) every \( \tau \in \text{Aut}_\rho(W) \) is the composition of an automorphism which fixes \( \rho \) and a field automorphism in \( \mathbb{Q}[e + e^{-1}] \) which is induced from a group automorphism of \( W \).

The elements of \( W \) are \( 1, a, \ldots, a^{m-1}, s, sa, \ldots, sa^{m-1} \). All elements of form \( sa^j \) for some \( j \) are involutions. For \( m \) odd there are \((m + 1)/2\) conjugacy classes with representatives \( 1, s, a, a^2, \ldots, a^{(m - 1)/2} \). For \( m \) even there are \((m + 6)/2\) conjugacy classes with representatives \( 1, s, t = sa, a, a^2, \ldots, a^{m/2} \).

Let \( \tau \in \text{Aut}_\rho(W) \).

Case 1. Assume first that \( \tau \) fixes the class of \( a \). Since \( \tau \) preserves the powermaps, it fixes all classes with a representative of the form \( a^i \) for some \( i \). If \( m \) is odd, all the remaining elements in \( W \) are conjugate to \( s \). Hence the class of \( s \) must also be fixed by \( \tau \). So \( \tau \) is trivial in this case. If \( m \) is even, there are two more classes with representatives \( s, t \), and these can be permuted by \( \tau \). Now there exists a group automorphism interchanging \( s \) and \( t \) and leaving all other classes fixed. Hence a non-trivial \( \tau \) is given by this automorphism.

Case 2. Now let \( C \in \text{Cl}(W) \). Using the above matrices we compute that

\[
\text{char.pol}(C, \rho) = \begin{cases} 
X^2 - (e^i + e^{-i})X + 1 & \text{if } a^i \in C \text{ for some } i, \\
X^2 - 1 & \text{if } sa^i \in C \text{ for some } i.
\end{cases}
\]

Now let \( \tau \in \text{Aut}_\rho(W) \). Then \( \text{char.pol}(C, \rho) = \text{char.pol}(\tau(C), \rho) \) for all \( C \in \text{Cl}(W) \), by Corollary 2.5. Comparing coefficients in the above polynomials it is easy to see that the class containing \( a \) is fixed by \( \tau \). So we are in Case 1 and conclude that we have verified Theorem 1.2 for \((W, S)\).

The General Case. Finally let \( \tau \) be arbitrary. Let \( C \) be the class of \( a \). Since \( \tau \) preserves the powermaps the image of \( C \) under \( \tau \) cannot be a class with a representative of the form \( sa^i \) (since all these elements are
involutions and since \( m \geq 3 \). Hence \( C \) is mapped to a class with a representative \( a^j \) for some \( j \geq 1 \) such that \( m \) and \( j \) are coprime. But then the map \( s \mapsto s, a \mapsto a^j \) extends to a group automorphism of \( W \) (check the defining relations). This group automorphism induces a field automorphism in \( \text{Aut CT}(W) \). By composing \( \tau \) with the inverse of this field automorphism if necessary we can therefore assume that the class \( C \) is fixed under \( \tau \). Again we are in Case 1.

3.8. Types \( H_3 \) and \( H_4 \). Let \((W, S)\) be given by one of the following diagrams:

\[
\begin{array}{cccccc}
H_3 & s_1 & s_2 & s_3 & s_4 & s_5 \\
H_4 & s_1 & s_2 & s_3 & s_4 & s_5
\end{array}
\]

In type \( H_3 \), the group \( W \) is a direct product of the alternating group \( \text{Alt}_5 \) and the cyclic group \( C_2 \) (see [16, Sect. 2.13; 5, Chap. VI, Sect. 4, Exercise 11]). In type \( H_4 \), representations affording the irreducible characters of \( W \), and the corresponding fake degrees, have been computed and listed in [1]. In both cases we see that the values of all irreducible characters lie in the field \( \mathbb{Q} [\sqrt{5}] \), and the value of \( \rho \) at \( s_3 s_2 \) is non-rational. Hence, in both cases, \( \text{Aut CT}(W) \) contains a unique non-trivial field automorphism. We show that

(a) \( \text{Aut CT}_r(W) = 1 \) for \((W, S)\) of type \( H_3, H_4 \),

(b) if \((W, S)\) is of type \( H_3 \) then \( \text{Aut CT}(W) \) is cyclic of order 2 generated by the unique non-trivial field automorphism, and

(c) if \((W, S)\) is of type \( H_4 \) then \( \text{Aut CT}(W) \) is elementary abelian of order 4 generated by the unique duality automorphism given by \( \text{sgn} \) and the unique non-trivial field automorphism.

Let \((W, S)\) be of type \( H_3 \). We have already mentioned above that then \( W \) is isomorphic to \( \text{Alt}_5 \times C_2 \). This easily implies (b) and that \( \text{Aut CT}_r(W) = 1 \).

Now let \((W, S)\) be of type \( H_4 \). We see from the list in [1] that each irreducible character of \( W \) is uniquely determined by its fake degree. Hence \( \text{Aut CT}_r(W) = 1 \) by Subsection 3.4. There are 4 faithful irreducible characters of degree 4. This and the previous result already imply that \( |\text{Aut CT}(W)| \leq 4 \). Now we also have \(-I \in W\) and \(-I\) lies in the kernel of the sign character (see the remarks following Lemma 3.3). Hence there is a non-trivial duality automorphism \( \delta \in \text{Aut}(W) \) (arising from \( \text{sgn} \)). This automorphism fixes the class of \( s_3 s_2 \) but does not fix the class of \( s_1 \), while algebraic conjugation fixes the latter and does not fix the former. We conclude that \( |\text{Aut CT}(W)| = 4 \), and (c) follows.

The question remains whether the non-trivial field automorphism is also induced by a group automorphism of \( W \). This is indeed so.
If \((W, S)\) is of type \(H_3\) then \(W \cong \text{Alt}_5 \times C_2 \subset S_5 \times C_2\) and the unique non-trivial field automorphism of \(W\) can be realised by conjugation with an element in \(S_5 \setminus \text{Alt}_5\).

If \((W, S)\) is of type \(H_4\), we can explicitly check the existence of a group automorphism inducing the non-trivial field automorphism using GAP [34] and CHEVIE [13]: there exist precisely two elements \(c \in W\) which are conjugate to \(s_3\) and such that the map \(s_2 \mapsto s_1\), \(s_2 \mapsto s_1 s_2 s_1\), \(s_3 \mapsto c\), \(s_4 \mapsto s_1\) extends to a group automorphism of \(W\). Reduced expressions for the two “good” choices of \(c\) are given by 3213214321321432132123 and 34321321234321213212343, respectively (where we write \(32 \cdots\) instead of \(s_3 s_2 \cdots\)). Once these elements are known, it can also be checked by hand that the above map extends to a group automorphism of \(W\).

If one is willing to accept computer proofs for the exceptional types, one can alternatively use the GAP function \text{TableAutomorphisms}\) (see [34]) to compute the group of all character table automorphisms of \(W\). The character tables for all types of \((W, S)\) are contained in the CHEVIE system [13].

4. CHARACTER TABLE AUTOMORPHISMS FOR CLASSICAL TYPES

We keep the notation from the previous section and consider now Coxeter groups of type \(B_n\) and \(D_n\) (see Propositions 4.3, 4.4, and Subsection 4.5). Our strategy will be very roughly the same as for the symmetric groups \(S_n\) in Subsection 2.7. We show that there are only very few (faithful) irreducible characters of degree \(n\) and that all these are permuted among themselves by duality automorphisms. Then we prove using the characteristic polynomials of the elements of the group that every automorphism which fixes the reflection character \(\rho\) must be a graph automorphism. In particular, we show that Theorem 1.2 and Theorem 1.4 hold for types \(B_n\) and \(D_n\).

At the end of this section we can then also complete the proof of Theorem 1.1 but note that, once Theorem 1.2 is established, the surjectivity of the map \(\text{Aut}(W) \to \text{AutCT}(W)\) will also follow from a general argument in Section 5.

4.1. Coxeter Groups of Types \(B_n\) and \(D_n\). Let \(n \geq 2\) and let \(W_n \subset \text{GL}(n, \mathbb{R})\) be the group of all matrices which have exactly one non-zero entry in each row and column, and where this non-zero entry is \(\pm 1\). Let \(\rho\) be the character of \(W_n\) given by the trace.

Let \(t \in W_n\) be the diagonal matrix with entries \((-1, 1, \ldots, 1)\) and \(s_i \in W_n^*\) (for \(1 \leq i \leq n - 1\)) the matrix obtained from the identity matrix
by interchanging the $i$th and the $(i + 1)$st row. Then $S = \{t, s_1, \ldots, s_{n-1}\}$ generates $W_n$ and $(W_n, S)$ is a Coxeter system of type $B_n$, and $\rho$ is the reflection character of $(W_n, S)$.

Let $u := t s_1 t \in W_n$ and $S' := \{u, s_1, \ldots, s_{n-1}\}$. We define $W'_n \subseteq W_n$ to be the subgroup generated by $S'$. Then $(W'_n, S')$ is a Coxeter system of type $D_n$ and the reflection character is given by the restriction of $\rho$ to $W'_n$.

The relations among the generators are given by the following diagrams.

$$B_n$$

$$D_n$$

In each case the generators $\{s_1, \ldots, s_{n-1}\}$ form a subsystem of type $A_{n-1}$. Let $H \cong \mathbb{Z}_n$ be the subgroup generated by them.

In type $B_n$ we define $t_0 := t$ and $t_i := s_i t_{i-1} s_i$ for $1 \leq i \leq n - 1$. Then $N := \langle t_0, \ldots, t_{n-1} \rangle$ is an elementary abelian normal subgroup of order $2^n$ in $W_n$, and we have a semidirect product decomposition $W_n = N.H$.

In type $D_n$ we define $u_1 := s_1 u$ and $u_i := s_i u_{i-1} s_i$ for $2 \leq i \leq n - 1$. (Under the above embedding we have $u_i = t t_i$ for all $i$, where we let $u_0 = 1$ by convention.) Then $N' := \langle u_1, \ldots, u_{n-1} \rangle$ is an elementary abelian normal subgroup of order $2^{n-1}$ in $W'_n$ and we have a semidirect product decomposition $W'_n = N'.H$.

The classes of $W_n$ are parametrized by pairs of partitions $(\alpha, \beta)$ such that the total sum of the parts of $\alpha$ and $\beta$ equals $n$ (see [6, Proposition 24]). This corresponds to the signed cycle type of elements, and we use the convention that $\alpha$ corresponds to the negative cycles and $\beta$ to the positive cycles. There is a similar parametrization of the classes of $W'_n$ by pairs $(\alpha, \beta)$ as above where the number of parts of $\alpha$ must be even and where there are two classes if $\alpha$ is empty and all parts of $\beta$ are even (see [6, Proposition 25]). If $\alpha = (1 \leq \alpha_1 \leq \cdots \leq \alpha_r)$ and $\beta = (\beta_1 \leq \cdots \leq \beta_s)$ and $C(\alpha, \beta)$ denotes the corresponding class in $W_n$ or $W'_n$, we have $\text{char.pol}(C(\alpha, \beta)) = f_{\alpha, \beta}$ where, for any pair of partitions as above, we define the polynomial

$$f_{\alpha, \beta} := \prod_{i=1}^{r} (X^{\alpha_i} + 1) \prod_{j=1}^{s} (X^{\beta_j} - 1);$$

see [6, Table 3].
Lemma 4.2. Let \( n \geq 3 \).

(a) If \( \chi \in \text{Irr}(W_n) \) has degree \( n \) and satisfies \( \chi(t) \neq \pm n \) then \( \chi = \mu \otimes \rho \) for some linear character \( \mu: W_n \to \{ \pm 1 \} \).

(b) If \( \chi \in \text{Irr}(W'_n) \) has degree \( n \) and satisfies \( \chi(s_1u) \neq n \) then \( \chi \) can be extended to a character of \( W_n \) as in (a).

Proof. In order to prove (a) and (b) let us recall the construction of the irreducible characters of \( W_n \) (see, for example, [36, Theorem 4.1]). We apply Clifford theory with respect to the normal subgroup \( N \). Let \( \chi \in \text{Irr}(W_n) \) and \( \eta \in \text{Irr}(N) \) be a constituent of the restriction of \( \chi \) to \( N \). Let \( a \geq 0 \) be the number of factors \( t_i \) such that \( \eta(t_i) = 1 \). The stabilizer of \( \eta \) in \( W_n \) is a subgroup of the form \( N.H_a \), where \( \mathbb{Z}_n \times \mathbb{Z}_{n-a} \cong H_a \subseteq H \), and the character \( \eta \) can be trivially extended to a character \( \eta \) of \( N.H_a \). Then \( \chi \) can be obtained by induction from a character of the form \( \eta \otimes \psi \) where \( \psi \in \text{Irr}(H_a) \). In particular, this implies that \( \chi(1) = \psi(1)[W_n: N.H_a] = \psi(1)\frac{a}{1}/(a(n-a)) \). Concerning estimates for this degree, note the following property of binomial coefficients:

\[
 b(n, a) \geq \frac{1}{2}n(n-1) \quad \text{if } 1 < a < n - 1.
\]

(a) Assume that \( \chi \in \text{Irr}(W_n) \) has degree \( n \) and satisfies \( \chi(t) \neq \pm n \). The above inequality implies that \( a = 0, 1, n - 1, n \). Assume that \( a = 0 \) or \( a = n \). Then the restriction of \( \chi \) is just a multiple of \( \eta \) hence \( \chi(t) = \pm n \). This is impossible and so we have \( a = 1 \) or \( n - 1 \). In this case, \( \chi \) is induced from a linear character of \( N.H_a \). Hence there are only four possibilities for \( \chi \): we can choose \( a = 1, n - 1 \) and we can choose one of the two linear characters of \( H_a \cong \mathbb{Z}_{n-1} \). On the other hand, there are four linear characters of \( W_n \) given by \( t, s_1 \mapsto \pm 1 \). If we tensor \( \rho \) with these four linear characters we obtain four distinct characters with value \( \neq \pm n \) on \( t \) (since \( n \geq 3 \)). So we are done.

(b) Assume that \( \chi \in \text{Irr}(W'_n) \) is of degree \( n \) such that \( \chi(u_1) \neq n \). Since \( W'_n \) is a normal subgroup in \( W_n \) of index \( 2 \), there are two possibilities: either \( \chi \) can be extended to \( W_n \), or the induction of \( \chi \) to \( W_n \) is an irreducible character of \( W_n \). Assume that the first is the case. If the extension of \( \chi \) to \( W_n \) had value \( \pm n \) on \( t \) then the discussion in the above proof of (a) shows that the restriction of that extension to \( N \) would have to be a multiple of a linear character \( \eta \) of \( N \) such that either \( \eta(t_i) = 1 \) for all \( i \) or \( \eta(t_i) = -1 \) for all \( i \). But then \( \chi(u_1) = \chi(t_1) = n \), contrary to our assumption. So the extension is a character of \( W_n \) as in (a), and hence we conclude that \( \chi = \mu \otimes \rho \) for some linear character \( \mu \) of \( W_n \).
Now assume that the induction of $\chi$ is irreducible. This can only happen if $n \geq 4$ is even (since $W_n = W'_n \times \langle -I \rangle$ for odd $n$). Then this induced character of $W'_n$ has degree $2n$. With the above notation, it is induced from a character $\tilde{\eta} \otimes \psi$ with $\psi \in \text{Irr}(H_n)$. The above inequality for binomial coefficients and a similar argument as in (a) shows that the only possibilities are $a = 1, n - 1$ (for $n \geq 6$) or $a = 0, n$ (for $n = 4$). In the first case, this forces that $\psi(1) = 2$. But since $\psi \in \text{Irr}(H_n)$ and $H_n \cong \mathbb{Z}_{n-1}$, this is impossible. Hence we are in the second case $n = 4$. The restriction of our character of degree $8 = 2n$ to $N$ is a multiple of a linear character of $N$ invariant under $W'_n$. Hence a similar statement would hold for the restriction of $\chi$ to $N'$ which, again, is impossible.

**Proposition 4.3.** Let $n \geq 3$. Then $\text{AutCT}(W'_n)$ is elementary abelian of order 4 if $n$ is even, and is cyclic of order 2 if $n$ is odd. In both cases all character table automorphisms are given by duality automorphisms, and $\text{AutCT}_p(W'_n) = 1$.

**Proof.** Consider any $\tau \in \text{AutCT}(W'_n)$. Then $\rho := \tau^*(\rho)$ is a faithful irreducible character of degree $n$; hence Lemma 4.2(a) implies that $\rho' = \rho \otimes \mu$ for some linear character $\mu$ of $W'_n$. We show that $\mu$ defines a duality automorphism as in Lemma 3.3. First note that $-I \in W'_n$. We have $\rho(-I) = -n$ and $\rho'(-I) = -\mu(-I)n$. The fact that $\rho'$ is faithful therefore implies that $\mu(-I) = 1$, and so $\mu$ indeed defines a duality automorphism $\delta \in \text{AutCT}(W'_n)$. (Note that for odd $n$ there are only two linear characters $\mu$ such that $\mu(-I) = 1$ while for even $n$ all 4 linear characters have this property.) Then $\tau \circ \delta$ fixes $\rho$. Hence it will be sufficient to show that $\text{AutCT}_p(W'_n) = 1$.

Assume that $\tau \in \text{AutCT}_p(W'_n)$. We must show that all classes are fixed by $\tau$. This will be done in three steps.

**Step 1.** Let $C$ be the class of $t$ and $C'$ the class of $s_1$. We start by showing that these two classes are fixed.

Indeed, since $\tau \in \text{AutCT}_p(W'_n)$ we conclude, using Lemma 3.1, that $\tau(C) = \{C, C'\}$. Now $C$ just consists of $t, t_1, \ldots, t_{n-1}$, and hence $|C| = n$. On the other hand, $s_1$ has $n(n - 1)/2$ conjugates in $\mathbb{Z}_n$ and $t$ does not commute with $s_1$. Hence $|C'| > n(n - 1)/2$. So, if we had $\tau(C) = C'$ then this would imply $n = |C| = |C'| > n(n - 1)/2$, i.e., $n < 3$, contrary to our assumption. Thus, $\tau$ fixes both $C$ and $C'$.

**Step 2.** Let $\pi: W'_n \to H$ be the canonical map with kernel $N$. As a normal subgroup, $N$ is generated by $t$. Hence $\tau(N)$ (notation of Corollary 2.3(b)) is generated by the class $\tau(C)$. Using Step 1 we conclude that $\tau(N) = N$. This invariance of $N$ under $\tau$ implies that the characters of $W'_n$ with $N$ in their kernel are permuted by $\tau^*$. Then we obtain an induced
character and powermap preserving bijection

$$\tau : \text{Cl}(H) \to \text{Cl}(H)$$

such that $$\tau \circ \pi = \pi \circ \tau.$$  

Lemma 2.2 shows that $$\tau$$ is a central Brauer class correspondence. By Peterson [31] (see also our remarks in Subsection 2.7), $$\tau$$ must be trivial. This is clear for $$n \neq 6$$; for $$n = 6$$ note that the class of $$s_1$$ (the basic transpositions in $$H$$) is fixed by $$\tau$$.

**Step 3**. Now let $$(\alpha, \beta)$$ be any pair of partitions as in (4.1). If we take the collection of all parts of $$\alpha$$ and $$\beta$$ and put them in increasing order, we obtain a partition of $$n$$ which we shall denote by $$\gamma := \Gamma(\alpha, \beta)$$.

Assume that $$\tau(C(\alpha, \beta)) = C(\alpha', \beta')$$, and let $$\gamma' = \Gamma(\alpha', \beta')$$. Then the images of these classes in $$H$$ via $$\pi$$ are the classes of $$H \cong \mathbb{S}_n$$ parametrized by the partitions $$\gamma$$ and $$\gamma'$$, respectively. (We just forget the signs of the cycles.) Using Step 2 we conclude that $$\gamma = \gamma'$$. Since $$\tau$$ fixes $$\rho$$ we have $$\text{char.pol}(C(\alpha, \beta)) = \text{char.pol}(C(\alpha', \beta'))$$. It remains to observe the purely combinatorial fact that the pair $$(\alpha, \beta)$$ is uniquely determined by the two polynomials $$f_\gamma$$ (see Subsection 2.7) and $$f_{\alpha, \beta}$$. Indeed, the quotient of $$f_\gamma$$ and $$f_{\alpha, \beta}$$ is $$\Pi(X^{\alpha_i} - 1)/\Pi(X^{\alpha'} + 1)$$. Hence the condition that $$f_\gamma = f_{\alpha, \beta}$$ and $$f_{\alpha, \beta} = f_{\alpha', \beta'}$$ implies that $$f_{\alpha, \alpha'} = f_{\alpha', \alpha}$$. Assume, if possible, that the smallest parts $$\alpha_1$$ and $$\alpha'_1$$ of $$\alpha$$ and $$\alpha'$$, respectively, are different. We choose notation so that $$\alpha_1 < \alpha'_1$$. The lowest non-constant term of $$f_{\alpha, \alpha'}$$ is the monomial $$m(-1)^{r'}X^{\alpha_1}$$ where $$r'$$ is the number of parts of $$\alpha'$$ and $$m$$ is the number of parts of $$\alpha$$ equal to $$\alpha_1$$. Similarly, the lowest non-constant term of $$f_{\alpha', \alpha}$$ is $$m(-1)^{-1}X^{\alpha_1}$$ where $$r$$ is the number of parts of $$\alpha$$. This implies, in particular, that $$r - r'$$ is odd. On the other hand, the constant term of $$f_{\alpha, \alpha'}$$ is $$(-1)^r$$ and that of $$f_{\alpha', \alpha}$$ is $$(-1)^{r'}$$. This implies that $$r - r'$$ is even, a contradiction. So the assumption was wrong, and the smallest parts of $$\alpha, \alpha'$$ are equal. We can proceed by an induction on the parts of $$\alpha$$ to conclude that $$\alpha = \alpha'$$. Then $$f_\gamma = f_{\gamma'}$$ also implies that $$\beta = \beta'$$. This completes the proof. □

**Proposition 4.4.** If $$n \geq 6$$ is even then $$\text{Aut}_{\text{CT}}(W_n')$$ is elementary abelian of order 4 and generated by the unique non-trivial graph automorphism and the unique non-trivial duality automorphism (given by sgn). Moreover, $$\text{Aut}_{\text{CT}}(W_n')$$ is cyclic of order 2 generated by the unique non-trivial graph automorphism. If $$n \geq 5$$ is odd then $$\text{Aut}_{\text{CT}}(W_n') = 1$$.

**Proof.** If $$n \geq 5$$ is odd then we have a direct product decomposition $$W_n = \langle -I \rangle \times W_n'$$, and the result in this case follows easily from the result for $$W_n'$$. 
Let us assume from now on that \( n \) is even. Let \( \tau \in \text{Aut}(W'_n) \). The strategy is roughly the same as in the proof of Proposition 4.3.

The composition of \( \rho \) with \( \tau \) is a faithful irreducible character of \( W'_n \) of degree \( n \). Using Lemma 4.2 we conclude that \( \tau^* (\rho) = \mu \otimes \rho \) for some linear character \( \mu: W'_n \to (\pm 1) \). As in the proof of Proposition 4.3 we can check that \( \mu \) defines a duality automorphism of \( W'_n \).

Hence we can assume that \( \tau \in \text{Aut}(W'_n) \) and it remains to show that it is given by a graph automorphism.

**Step 1.** At first we show that the class of \( s_1 \) is fixed. This follows immediately from Lemma 3.1 since all generators in \( S \) are conjugate.

**Step 2.** Let \( \pi': W'_n \to H \) be the canonical map with kernel \( N' \). Since \( \tau \) induces an isomorphism of the normal subgroup lattices by Corollary 2.3(b) and since \( n \geq 6 \), \( N' \) is invariant under \( \tau \). As in Step 2 of the proof of Proposition 4.3 this implies that \( \pi' \circ \tau = \pi' \).

**Step 3.** Now let \( C \in \text{Cl}(W'_n) \) be labelled by a pair of partitions \( (\alpha, \beta) \) as in (4.1). The same argument as in Step 3 of the proof of Proposition 4.3 shows that \( \tau(C) \) is also labelled by the pair \( (\alpha, \beta) \).

Let us summarize what we have achieved so far. Let \( (\alpha, \beta) \) be a pair of partitions as above (where \( \alpha \) has an even number of parts). There are two cases:

**Case I.** If \( \alpha \) is empty and all parts of \( \beta \) are even there are two classes in \( W'_n \) labelled by \((-\beta, \beta)\) (cf. the remarks in Subsection 4.1). We denote them by \( C(\gamma) \) and \( C(\gamma)' \), respectively, where \( \gamma \) is the partition of \( n/2 \) obtained by dividing each part of \( \beta \) by \( 2 \). Note that we have such a pair of classes for all partitions \( \gamma \) of \( n/2 \). We can fix the notation so that \( C(\gamma) \) has a representative in \( \langle s_1, s_2, \ldots, s_{n-1} \rangle \) and \( C(\gamma)' \) has a representative in \( \langle u, s_2, \ldots, s_{n-1} \rangle \). By Step 3 we know that \( \tau \) fixes or interchanges the classes \( C(\gamma), C(\gamma)' \).

**Case II.** If \( (\alpha, \beta) \) is not of the above form then there is a unique class of \( W'_n \) labelled by this pair. Step 3 now shows that \( \tau \) fixes such a class.

Assume that \( \text{Aut}(W'_n) \) is non-trivial. Then there exists a class \( C(\gamma) \) as in Case I such that \( \tau(C(\gamma)) = C(\gamma)' \). The unique non-trivial graph automorphism of \( W'_n \) given by \( u \mapsto s_1, s_i \mapsto s_i \) (\( i \geq 2 \)) interchanges the two classes \( C(\gamma) \) and \( C(\gamma)' \). Hence by composing \( \tau \) with this graph automorphism if necessary we can assume that \( \tau \) fixes them. So it will be sufficient to show that

\[ \text{if } \tau \text{ is non-trivial then } \tau(C(\gamma)) = C(\gamma)' \text{ for all partitions } \gamma \text{ of } n/2. \]
In order to prove this statement it is clearly sufficient to find some \( \chi \in \text{Irr}(W'_n) \) such that \( \chi \) and \( \tau^*(\chi) \) have different values on \( C(\gamma) \) for all partitions \( \gamma \) of \( n/2 \).

For the following description of the irreducible characters of \( W'_n \) see [36]. First recall that the irreducible characters of \( W_n \) are labelled by pairs of partitions \((\lambda, \mu)\) such that the sum of the parts of \( \lambda \) and \( \mu \) equals \( n \). The characters with labels \((\lambda, \mu)\) and \((\mu, \lambda)\) have the same restriction to \( W'_n \) and this restriction is irreducible if and only if \( \lambda \neq \mu \). If \( \lambda = \mu \) then the restriction has two irreducible constituents which we denote by \( \chi_\lambda \) and \( \chi'_\lambda \), respectively. Note that we obtain such a pair of characters for any partition \( \lambda \) of \( n/2 \). By [36, Theorem 7.5], the notation can be arranged so that

\[
\chi_\lambda(C(\gamma)) - \chi'_\lambda(C(\gamma)) = 2^{k(\gamma)[n/2]}(\lambda)(\gamma),
\]

where \([\lambda]\gamma)\) denotes the value of the irreducible character of \( S_{n/2} \) labelled by \( \lambda \) on the class with elements of cycle type \( \gamma \) and \( k(\gamma) \geq 0 \) is the number of non-zero parts of \( \gamma \). Hence, if we choose \( \lambda = (n/2) \) then the corresponding character of \( S_{n/2} \) is the trivial character, and the above difference is non-zero for all partitions \( \gamma \) of \( n/2 \). This completes the proof.

4.5. Type \( D_4 \). We show that

(a) every \( \tau \in \text{Aut}_\text{CT}(W'_4) \) can be modified by a duality automorphism so that it fixes \( \rho \), and

(b) the group \( \text{Aut}_\text{CT}(W'_4) \) contains 6 elements which are all given by graph automorphisms.

In particular, \( \text{Aut}_\text{CT}(W'_4) \) is isomorphic to the dihedral group of order 12.

There are only two characters of degree 4 (see [7, p. 499], for example). We have \(-I \in W'_4 \) and this element lies in the kernel of \( \text{sgn} \) hence we can modify any \( \tau \in \text{Aut}_\text{CT}(W'_4) \) by a duality automorphism so that it fixes \( \rho \). Assume now that this is the case. From [6] we can write down representatives for the conjugacy classes of \( W \): \( I, 01, 01201320123, 0, 012, 013, 03, 13, 020123, 02, 0123, 023, 123 \) (where we simply write, for example, 12 instead of \( s_1s_2 \) and represent \( u \) by 0). We can also construct explicitly the reflection representation and compute the character values of \( \rho \) on these representatives. The following table contains the orders of the above representatives, the 2-powermap, and the values of \( \rho \).
This table shows that all those classes are fixed on which $\rho$ has a non-zero value. Looking at the order of elements we see that only the classes 2, 7, 8 respectively nos. 5, 12, 13, can be permuted among themselves. Taking also into account the 2-powermap we see that the permutation of the classes 2, 7, 8 is determined by the permutation of the classes 5, 12, 13. These arguments imply that $|\text{Aut}_{CT}(W'_\rho)| \leq 6$. On the other hand, there are six different graph automorphisms, and using the representatives it is easy to check that they generate $\text{Aut}_{CT}(W'_\rho)$.

4.6. Proof of Theorem 1.1. At first we check that $\text{Aut}(W') \to \text{Aut}_{CT}(W')$ is surjective. We have already seen that every $\tau \in \text{Aut}_{CT}(W')$ can be written as a product of a graph, a duality, and a field automorphism. The first two types of automorphisms are induced by group automorphisms (by their definition). In Subsections 3.7, 3.8 we have also verified that the field automorphisms for the types $I_2(m), H_3, H_4$ are induced by group automorphisms. (A somewhat different argument in the case of crystallographic groups will be given in Corollary 5.4.)

It remains to show that the kernel of $\text{Aut}(W') \to \text{Aut}_{CT}(W')$ consists precisely of $\text{Inn}(W')$. For this purpose we have to consider a group automorphism $\alpha: W \to W$ such that $\alpha(C) = C$ for all $C \in C(W)$ and we must prove that $\alpha$ is given by conjugation with some element of $W$.

As in the proof of Lemma 3.2, let $W' \subset GL(V)$ such that $\rho$ is given by the usual trace and let $\Pi := \{ e_s \mid s \in S \} \subset V$ be a basis such that each $s \in S$ is represented by the reflection with root $e_s$. The set $\Phi := \{ w(e_s) \mid w \in W, s \in S \}$ is the root system of $W \subset GL(V)$ and $\Pi \subset \Phi$ is a system of fundamental roots (see [9, Theorem 64.28]).

Since $\alpha$ fixes all classes we have $\alpha^*(\rho) = \rho$ and hence there exists some $g \in GL(V)$ such that $\alpha(w) = gwg^{-1}$ for all $w \in W$. Thus, $\alpha$ is realized as conjugation with $g$ inside $GL(V)$.

Let $r \in \Phi$ and $w_r \in W$ the reflection with root $r$. Since $\alpha$ fixes all classes we have $\alpha(w_r) = ww_rw_r^{-1}$ for some $w \in W$. Hence $\alpha(w_r)$ is the reflection with root $w(r)$. On the other hand, we have $\alpha(w_r)(g(r)) = gw_rg^{-1}(g(r)) = gw_r(r) = g(-r) = -g(r)$. This means that $g(r)$ is an eigenvector of $\alpha(w_r)$ with eigenvalue $-1$. Since $\alpha(w_r)$ is a reflection, the eigenspace to the eigenvalue $-1$ has dimension 1. So there exists a sign $\varepsilon_r$. 
and a positive real number $\lambda$, such that $g(r) = e_r \cdot \lambda w(r)$. Let

$$
\Pi' := \{ \lambda^{-1} g(r) \mid r \in \Pi \} \subset \Phi.
$$

Then every root in $\Phi$ can be expressed as a linear combination in $\Pi'$ where the coefficients are either all $\geq 0$ or all $\leq 0$. Hence $\Pi'$ is a system of fundamental roots in $\Phi$ (see [16, Sect. 1.3]). But any two systems of fundamental roots are conjugate in $W$ (see [16, Sect. 1.4]), so there exists some $w \in W$ such that $w(\Pi) = \Pi'$. We conclude that $g' := w^{-1} g$ maps each $r \in \Pi$ to a positive scalar multiple of some fundamental root in $\Pi$. It follows that $g'Sg'^{-1} = S$ and so $g'$ defines a graph automorphism of $(W, S)$.

Finally, it remains to observe that non-trivial graph automorphisms exist only for types $A_n$, $D_n$, $E_6$, $F_4$, and $I_2(m)$. In types $A_n$, $D_n$ ($n$ odd), $E_6$, and $I_2(m)$ ($m$ odd) such a non-trivial graph automorphism is given by conjugation with the longest element in $W$ hence it is inner. In the remaining cases, it is easily checked using the results in this and the previous section that every non-trivial graph-automorphism acts non-trivially on the classes of $W$. Hence our graph automorphism defined by the element $g'$ (which still fixes all classes of $W$) must be inner and we are done.

4.7. The Orders of $\text{Aut}CT(W)$. For each type of $(W, S)$, except $A_3$ and $I_2(m)$, the order of $\text{Aut}CT(W)$ can be written as a product of three factors $g$, $d$, $f$ which are the number of graph, duality, and field automorphisms, respectively. For the crystallographic types we omit the third factor which is always 1 in these cases. The types $A_2$, $B_2$, $G_2$ are included in $I_2(m)$. For type $I_2(m)$, $m \geq 4$ even, the duality automorphism defined by $\text{sgn}$ is the composition of the non-trivial graph automorphism and a field automorphism. Since $I_2(m)$, $m \geq 3$ odd, has no duality automorphisms, we can omit the factor $d$ in these cases. The results are given in Table I. ($\varphi(m)$ denotes the Euler function.)

5. THE ZASSENHAUS Z2-CONJECTURE

As before, let $(W, S)$ be a finite irreducible Coxeter system and $\rho \in \text{Irr}(W)$ be the character of the standard reflection representation. In this section we show that $\rho$ essentially determines $W$ up to isomorphism. This gives a positive answer for the isomorphism problem of integral group rings of finite Coxeter groups. On the other hand it may happen that $W$ possesses non-equivalent faithful reflection representations (see types $I_2(m)$, $H_3$, and $H_4$). We will show that such representations can always be
TABLE 1
Orders of $\text{AutCT}(W')$

| Type of $(W, S)$ | $|\text{AutCT}(W')|$ | Condition on $n$ |
|-----------------|------------------------|------------------|
| $A_n$           | $1 = 1 \cdot 1$       | $n \neq 5$       |
|                 | $2$                    | $n = 5$          |
| $B_n$           | $2 = 1 \cdot 2$       | $n \geq 3$ odd  |
|                 | $4 = 1 \cdot 4$       | $n \geq 4$ even |
| $D_n$           | $1 = 1 \cdot 1$       | $n \geq 5$ odd  |
|                 | $4 = 2 \cdot 2$       | $n \geq 6$ even |
|                 | $12 = 6 \cdot 2$      | $n = 4$          |

| Type | $|\text{AutCT}(W')|$ | Type | $|\text{AutCT}(W')|$ |
|------|------------------------|------|------------------------|
| $E_t$| $8 = 2 \cdot 4$        | $H_2$| $2 = 1 \cdot 1 \cdot 2$|
| $E_4$| $1 = 1 \cdot 1$        | $H_6$| $4 = 1 \cdot 2 \cdot 2$|
| $E_7$| $1 = 1 \cdot 1$        | $I_5(m)$| $1 \cdot \varphi(m)/2$ |
| $E_6$| $2 = 1 \cdot 2$        |      | $2 \cdot \varphi(m)$   |

obtained from each other by group automorphisms. These results together enable us to complete the proof of Theorem 1.3, that is, the Zassenhaus $Z_2$-Conjecture will be proved for finite Coxeter groups.

**Lemma 5.1.** Let $H$ be a finite group and $\tau : \text{Cl}(H) \to \text{Cl}(W)$ a central Brauer class correspondence. Then $\tau^*(\rho)$ is the character of a faithful real reflection representation of $H$.

**Proof.** Let $\rho' := \rho^*(\rho) \in \text{Irr}(H)$. It is clear that $\rho'$ is real valued. We first check that $\rho'$ is even afforded by a real representation. For this purpose we consider the Frobenius–Schur indicator; recall that if $G$ is any finite group and $\chi \in \text{Irr}(G)$ is a real valued character then $\nu_2(\chi) = (\sum_{g \in G} \chi(g^2))/|G| \in \{\pm 1\}$, and we have $\nu_2(\chi) = 1$ if and only if $\chi$ is afforded by a real representation (see [18, Theorem 4.19]). Since $\tau$ preserves powermaps we conclude that $\nu_2(\rho') = \nu_2(\rho) = 1$. Hence, indeed, $\rho'$ is afforded by a real representation.

Since $\rho$ is the character of a faithful representation, the same holds for $\rho'$. Thus, we obtain an embedding $H \subset \text{GL}(V)$ where $V$ is a real vectorspace of dimension $|S|$, and $\rho'$ is the character given by the usual trace. We can also choose an $H$-invariant scalar product on $V$ so that $H$ lies in the orthogonal group of $V$ with respect to this product.

It remains to check that $H$ is generated by reflections in $\text{GL}(V)$, that is, by elements of order 2 with precisely one eigenvalue $-1$. This is done as follows. Let $C_1, \ldots, C_k \in \text{Cl}(H)$ such that $\tau(C_1) \cup \cdots \cup \tau(C_k) = \{wsw^{-1} | w \in W, s \in S\}$. (We always have $k = 1$ or $k = 2$.) Since $\tau$ preserves powermaps, every $C_i$ is a class of elements of order 2. Hence every
element \( h \in C_i \) has eigenvalues \( \pm 1 \). The value of \( \rho' \) at \( h \) is the same as the value of \( \rho \) at some element in \( \tau(C_i) \). Since \( \tau(C_i) \) consists of conjugates of elements in \( S \), the latter value equals \( |S| - 2 \). We conclude that \( h \) has precisely one eigenvalue \( -1 \). Finally, consider the normal subgroup of \( H \) generated by the elements in \( C_1 \cup \cdots \cup C_k \). This normal subgroup equals \( H \) since \( S \) generates \( W \) and \( \tau \) preserves the latter of normal subgroups (see Corollary 2.3(b)). This completes the proof.

**Corollary 5.2.** The isomorphism problem has a positive solution for \( W \). This means that if \( H \) is a finite group and \( \sigma : \bar{Z}H \rightarrow \bar{Z}W \) is a ring isomorphism, then \( H \) and \( W \) are isomorphic groups.

**Proof.** By [18, (3.17); 30, Proposition 2, Sect. 3], the isomorphism \( \sigma \) induces a central Brauer class correspondence between \( H \) and \( W \). Lemma 5.1 implies that \( \sigma \ast (\rho) \) is the character of a faithful reflection representation of \( H \) on some real vectorspace \( V \) with \( \dim V = \sigma \ast (\rho)(1) = \rho(1) = |S| \). There is a corresponding root system in \( V \) in which we choose a system of fundamental roots (see the remarks following [9, Definitions 64.4 and Theorem 64.6]). Let \( S' \subset H \) be the reflections corresponding to the fundamental roots. Then \((H, S')\) is a finite irreducible Coxeter system, see [9, Theorem 64.14].

We have \( |H| = |W| \) and \( |S'| = \dim V = |S| \). Using the formulae for the orders of finite irreducible Coxeter groups (see the table in [16, sect. 2.11]) it is a straightforward matter to check that, given \( n \geq 1 \), the finite irreducible Coxeter systems \((W, S)\) such that \( |S| = n \) are distinguished by the order of \( W \). We conclude that there exists a group isomorphism \( \alpha : W \rightarrow H \) such that \( \alpha(S) = S' \).

**Proposition 5.3.** Let \( H \) be a finite group. Assume that \( \rho_1, \rho_2 \in \text{Irr}(H) \) are the characters of two faithful real reflection representations of \( H \) of the same degree. Then there exists a group automorphism \( \alpha : H \rightarrow H \) such that \( \rho_2 = \rho_1 \circ \alpha \).

**Proof.** Let \( n \geq 1 \) be an integer and consider an \( n \)-dimensional real vectorspace \( V \) such that \( H \subset \text{GL}(V) \) is an irreducible subgroup generated by reflections. Let \( \Phi \) be the corresponding irreducible root system in \( V \) and \( \Pi \subset \Phi \) a system of fundamental roots. We choose an \( H \)-invariant scalar product on \( V \) and assume that all roots have norm \( 1 \). Let \( S \subset H \) denote the set of reflections with roots in \( \Pi \). Then \((H, S)\) is a finite irreducible Coxeter system. We have \( |S| = |\Pi| = n \).

Let \( M := (m_{st})_{s,t \in S} \) be the corresponding Coxeter matrix, where \( m_{st} \) denotes the order of \( st \in H \). We have already seen in the proof of Corollary 5.2 that \( M \) is determined by the pair \((|H|, n)\). On the other hand,
the equivalence class of the representation $H \subset \text{GL}(V)$ is determined by the Cartan matrix $A := (a_{a, \beta})_{a, \beta \in \Pi}$, where $a_{a, \beta}$ denotes the scalar product between $a$ and $\beta$.

In general, the matrix $A$ always determines the matrix $M$. If $\Phi$ is crystallographic then the converse also holds and $M$ determines $A$, see [5, Chap. VI, Sect. 4, Théorème 3]. Suppose this is the case. Then we deduce that, given $S$, there is only one irreducible character of degree $n$ which is the character of a faithful reflection representation of $H$. A different choice of the subset $S$ leads to a Coxeter system of the same type, and hence there exists a group automorphism which transforms them into each other.

It remains to consider the non-crystallographic types. The problem then is that there are several (algebraically conjugate) Cartan matrices $A$ corresponding to the same Coxeter matrix $M$. It is easily checked using the results in Subsections 3.7 and 3.8 that any two irreducible characters corresponding to faithful reflection representations of $H$ can be obtained from each other by a field automorphism. But we have also checked in Subsections 3.7 and 3.8 that every such field automorphism is induced from a group automorphism. This completes the proof.

Corollary 5.4. If $\tau \in \text{Aut}_{\text{CT}}(W)$ then there exists a group isomorphism $\alpha \in \text{Aut}(W)$ such that $\tau \circ \alpha$ fixes $\rho$.

Proof. By Lemma 5.1 (applied with $H := W$) the character $\tau^* (\rho)$ is the character of some faithful real reflection representation of $W$. Since both $\rho$ and $\tau^* (\rho)$ are irreducible and have the same degree, Proposition 5.3 implies that there exists a group automorphism $\alpha : W \to W$ such that $\alpha^* (\tau^* (\rho)) = \rho$. So $\tau \circ \alpha$ fixes $\rho$.

5.5. Proof of Theorem 1.3. Corollary 5.4 and Theorem 1.2 together provide a proof for the surjectivity of the map $\text{Aut}(W) \to \text{Aut}_{\text{CT}}(W)$. It is known (see [4, Corollary 3.5]) that this implies that each normalized automorphism of $\mathbb{Z}W$ admits a Zassenhaus decomposition, that is, it can be written as the composition of a group automorphism which is induced by a group automorphism of $W$ followed by a central automorphism. Since a positive solution to the isomorphism problem for $W$ has already been established in Corollary 5.2, this completes the proof of Theorem 1.3.

Remark 5.6. A proof that the Zassenhaus Z2-Conjecture is valid for finite irreducible Coxeter groups can also be extracted case by case from known general results on the topic. We point out that some of these results depend on classification results concerning finite simple groups (for example, [22]) or their proof is up to now not completely published, see [33, 35, 32].
We will now give a brief summary of these results. We use the abbreviation (IP) for the isomorphism problem and (AUT) for the property that normalized ring automorphisms admit a Zassenhaus decomposition.

(a) Problem (IP) has a positive solution for a given group if this group is determined up to isomorphism by its character table (without powermap). This applies to all finite irreducible Coxeter groups except for the dihedral groups \( I_2(m) \) with \( 4|m \) and the Weyl groups of type \( D_n \) with \( 4|n \), see [29] and the references there. Some cases (for example, types \( E_6, E_7, H_3 \)) can also be settled using [22], where it is shown that (IP) has a positive solution for finite simple groups and their automorphism groups. Note that in the above excluded cases there do exist non-isomorphic groups with the same character table. Nevertheless, since all dihedral groups are metabelian, (IP) has a positive answer in these cases by [37].

For the Weyl groups of type \( D_n \) see c below.

(b) Property (AUT) has been shown explicitly for type \( A_n \) in [31], for wreath products \( C \wr S_g \) (hence, in particular, type \( B_n \)) in [15], and for metacyclic groups (hence, in particular, type \( I_2(m) \)) in [10]. Property (AUT) can also be checked explicitly for each group \( W \) of exceptional type by computing \( \text{Aut}(W) \) and verifying that the map \( \text{Aut}(W) \to \text{Aut}(W) \) is surjective.

(c) By a theorem of Roggenkamp and Scott, see [33] or [35], \( Z_2 \) holds for groups \( G \) which have as generalized Fitting subgroup \( F^*(G) \) a \( p \)-group. This theorem applies with \( p = 2 \) to the Weyl groups of type \( B_n, D_n, \) and \( F_4 \) and establishes (IP) and (AUT) for these groups at the same time.

6. AUTOMORPHISMS OF IWAHORI–HECKE ALGEBRAS

Let \((W, S)\) be a finite irreducible Coxeter system and \( H \) the generic Iwahori–Hecke algebra as introduced in Section 1. We shall use specialization techniques to obtain restrictions on the possible automorphisms of \( H \) and prove Theorem 1.6.

For simplicity we assume that \( W \) is crystallographic (see Remark 6.9 for the other cases). Let \( K = \mathbb{Q}(\nu) \) be the field of fractions of \( A \) and \( KH \) the corresponding algebra over \( K \). By [27] the algebra \( KH \) is then split semisimple, and we have \( \chi(T_w) \in A \) for all irreducible characters \( \chi \) of \( KH \) and all \( w \in W \).

6.1. Specializations and Decomposition Maps. For any integral domain \( R \) and any homomorphism \( f: A \to R \), we write \( RH := R \otimes_A H \), and denote the standard basis elements \( 1 \otimes T_w \) of \( RH \) again by \( T_w \), for
Thus, the multiplication in $RH$ is given by
\[ T_w T_{w'} = T_{ww'}, \quad \text{if } l(ww') = l(w) + l(w'), \]
\[ T_s^2 = f(u)T_1 + (f(u) - 1)T_s \quad \text{for } s \in S. \]

Let $F$ be the field of fractions of $R$ and $FH$ the algebra obtained by extending scalars from $R$ to $F$. We will assume throughout that $F$ has characteristic 0. Then, by [11, Proposition 5.4], the algebra $FH$ also is split (but not necessarily semisimple). Let $\text{Irr}(FH)$ be the set of irreducible characters of $FH$ and $\mathbb{Z}\text{Irr}(FH)$ the free abelian group with basis $\text{Irr}(FH)$. We denote by $\mathbb{N}_0 \text{Irr}(FH)$ the subset of all linear combinations of irreducible characters with non-negative coefficients.

Following [14], we define the decomposition map
\[ d_f : \mathbb{N}_0 \text{Irr}(KH) \to \mathbb{N}_0 \text{Irr}(FH) \]
associated with $f : A \to R$ as follows. Let $\text{Maps}(KH, K[X])$ be the set of all maps from $KH$ to $K[X]$, where $X$ is an indeterminate and where we regard this set as a semigroup with pointwise multiplication of maps. We have a semigroup homomorphism $\rho_K : \mathbb{N}_0 \text{Irr}(KH) \to \text{Maps}(KH, K[X])$ given by sending a character $\chi$ of $KH$ to the map which assigns to each element $h \in KH$ the characteristic polynomial of $h$ in a representation affording $\chi$. Since $A$ is integrally closed in $K$ and all character values at basis elements $T_w$ lie in $A$, we can restrict $\rho_K$ to a map $\rho_A : \mathbb{N}_0 \text{Irr}(KH) \to \text{Maps}(H, A[X])$ (cf. [14, Lemma 2.10]). Similarly, we have a map $\rho_F : \mathbb{N}_0 \text{Irr}(FH) \to \text{Maps}(FH, F[X])$. The homomorphism $f$ induces a canonical map $H \to FH$. Using composition with this map, we obtain a new map $\rho_f : \mathbb{N}_0 \text{Irr}(FH) \to \text{Maps}(H, F[X])$. There is a canonical reduction map $\text{Maps}(H, A[X]) \to \text{Maps}(H, F[X])$, given by applying $f$ to the coefficients of polynomials in $A[X]$. Then $d_f$ is the unique map such that the following diagram is commutative:

\[
\begin{array}{ccc}
\mathbb{N}_0 \text{Irr}(KH) & \xrightarrow{\rho_A} & \text{Maps}(H, A[X]) \\
\downarrow d_f & & \downarrow f \\
\mathbb{N}_0 \text{Irr}(FH) & \xrightarrow{\rho_f} & \text{Maps}(H, F[X]).
\end{array}
\]

(The necessity of replacing $\rho_K$ by the map $\rho_f$ with values in $\text{Maps}(H, F[X])$ was not taken into account in [14], and the statement of [14, Proposition 2.11] has to be corrected accordingly.) The advantage of this definition of $d_f$ is that it shows very easily compatibility properties with algebra automorphisms; see below.
6.2. Automorphisms and Specializations. We consider an \( A \)-automorphism \( \sigma: H \rightarrow H \) and a specialization \( f: A \rightarrow R \) as above. Then \( \sigma \) induces an \( R \)-algebra automorphism of \( RH \) which we denote by \( \sigma_f \) and call the specialized automorphism of \( RH \). (If \( \sigma \) is given by the matrix \( (a_{w,w'})_{w,w' \in W} \) with respect to the basis \( (T_w \mid w \in W) \) of \( H \) then the induced automorphism of \( RH \) is given by the matrix \( (f(a_{w,w'}))_{w,w' \in W} \).)

If \( \chi \) is an irreducible character of \( KH \) then \( \chi_f: KH \rightarrow K, T_w \mapsto \chi(\sigma(T_w)) \), is again an irreducible character. Hence \( \sigma \) also induces a map from \( \mathbb{N}_0 \text{Irr}(KH) \) into itself which we denote by the same symbol. Similarly, we obtain a map from \( \mathbb{N}_0 \text{Irr}(FH) \) into itself which is induced from \( \sigma_f \). On the other hand, the actions of \( \sigma \) on \( H \) and \( \sigma_f \) on \( FH \) are compatible with taking characteristic polynomials and with the canonical maps \( H \rightarrow FH \) and \( \text{Maps}(H, A[X]) \rightarrow \text{Maps}(H, F[X]) \). The definition of \( d_f \) in terms of the commutative diagram in Subsection 6.1 then immediately implies that the diagram

\[
\begin{array}{ccc}
\mathbb{N}_0 \text{Irr}(KH) & \xrightarrow{d_f} & \mathbb{N}_0 \text{Irr}(FH) \\
\downarrow \sigma & & \downarrow \sigma_f \\
\mathbb{N}_0 \text{Irr}(KH) & \xrightarrow{d_f} & \mathbb{N}_0 \text{Irr}(FH)
\end{array}
\]

is also commutative. Thus, we see that the decomposition map commutes with automorphisms of \( H \). A somewhat different argument to prove this commutativity is given in [4, Sect. 2].

We define the \( f \)-modular Brauer graph as follows. It has vertices labelled by the irreducible characters of \( KH \) and an edge joining the characters \( \chi, \chi' \in \text{Irr}(KH) \) if there exists some \( \psi \in \text{Irr}(FH) \) such that \( \psi \) appears with non-zero multiplicity in the decomposition of \( d_f(\chi) \) and \( d_f(\chi') \). The connected components of this graph define a partition of \( \text{Irr}(KH) \) into "blocks." The above commutative diagram implies that each \( A \)-automorphism \( \sigma: H \rightarrow H \) permutes the blocks of \( KH \) and induces a symmetry of the Brauer graph.

Remark 6.3. Let \( R = \mathbb{Z} \) and \( f: A \rightarrow \mathbb{Z}, v \mapsto 1 \), so that \( RH \) is the group algebra of \( W \) over \( \mathbb{Z} \).

Since \( QW \) is semisimple, Tits' Deformation Theorem (see [9, Sect. 68A; 12]) implies that the decomposition map \( d_f \) is a bijection between \( \text{Irr}(KH) \) and \( \text{Irr}(W) \).

Let \( \sigma: H \rightarrow H \) be an \( A \)-algebra automorphism and \( \sigma_f \) the corresponding automorphism of \( ZW \). The commutative diagram in Subsection 6.2 shows that the bijection \( \text{Irr}(KH) \leftrightarrow \text{Irr}(W) \) is compatible with the induced actions of \( \sigma \) on \( \text{Irr}(KH) \) and of \( \sigma_f \) on \( \text{Irr}(W) \).
In particular we see that $\sigma$ acts trivially on $\text{Irr}(KH)$ if and only if $\sigma_1$ acts trivially on $\text{Irr}(W)$.

6.4. Specializations associated with the Coxeter Number. If we choose an ordering of the generators in $S$ and multiply them together, we obtain an element $w_\sigma \in W$. Its order, $h \geq 1$ say, is called the Coxeter number of $W$. These numbers are given in the following table (see [16, p. 80])

<table>
<thead>
<tr>
<th>Type of $(W, S)$</th>
<th>$A_{n-1}$</th>
<th>$B_n$</th>
<th>$D_n$</th>
<th>$G_2$</th>
<th>$F_4$</th>
<th>$E_6$</th>
<th>$E_7$</th>
<th>$E_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coxeter no.</td>
<td>$n$</td>
<td>$2n$</td>
<td>$2n - 2$</td>
<td>6</td>
<td>12</td>
<td>12</td>
<td>18</td>
<td>30</td>
</tr>
</tbody>
</table>

Let $\Phi_{2h} \in \mathbb{Z}[v]$ be the $2h$th cyclotomic polynomial, $\xi_{2h} \in \mathbb{C}$ a primitive $2h$th root of unity, and $R = \mathbb{Z}[\xi_{2h}]$ with quotient field $F = \mathbb{Q}(\xi_{2h})$. As in [11, Sect. 5], we consider the specialization $f_h: A \to R, v \mapsto \xi_{2h}$, and call the associated decomposition map $d_h := d_{j_h}: \mathbb{N}_0 \text{Irr}(KH) \to \mathbb{N}_0 \text{Irr}(FH)$ the $\Phi_h$-modular decomposition map of $H$.

Let $P := \sum_{w \in W} v^{2h(w)}$ be the Poincaré polynomial of $W$. From the list in [7, p. 75], we see that $\Phi_{2h}$ has multiplicity 1 as an irreducible factor of $P$. On the other hand, with every irreducible character $x$ of $KH$, there is associated a polynomial $D_x \in \mathbb{Q}[v]$ called the generic degree of $x$ (see, for example, [9, Sect. 68C]). These polynomials divide $P$ in $\mathbb{Q}(v, v^{-1})$. Hence we conclude that $\Phi_{2h}$ has multiplicity either 0 or 1 as an irreducible factor of $P/D_x$. Accordingly, we say that $x$ has $F$-defect 0, respectively 1.

By [11, Proposition 9.3], this implies that the connected components of the $\Phi_h$-modular Brauer graph are trees (in fact, just straight lines). It is also known (cf. [11, Sect. 8]) that if $x \in \text{Irr}(KH)$ has $\Phi_h$-defect 0 then $(x)$ is a block of $KH$. Therefore, in order to find non-trivial blocks we have to consider the characters of $\Phi_h$-defect 1. It turns out that these are precisely the characters which correspond to the exterior powers of the character $\rho$ of the standard reflection of $W$ under the bijection in Remark 6.3.

**Lemma 6.5.** Let $0 \leq k \leq |S|$ and $\rho^\wedge k$ be the $k$th exterior power of $\rho$. Then $\rho^\wedge k$ is irreducible and the character of $KH$ corresponding to it under the bijection in Remark 6.3 has $\Phi_h$-defect 1. All irreducible characters of $KH$ with $\Phi_h$-defect 1 are obtained in this way.

**Proof.** For the irreducibility of $\rho^\wedge k$ see [8, Theorem 9.13]. The statement about $\Phi_h$-defects could be checked case-by-case using the known tables for generic degrees. A general argument using algebraic groups goes as follows.

Let $G$ be a simple algebraic group defined over $\mathbb{F}_q$, with corresponding Frobenius map $F$. We assume that $G$ contains an $F$-stable maximal torus $T_0$ which is split, and that the Weyl group of $T_0$ is isomorphic to $W$. Let $B \subseteq G$ be an $F$-stable Borel subgroup of $G$ containing $T_0$. Then the
irreducible characters of $G^F$ (over $\mathbb{C}$) appearing in the permutation representation of $G^F$ on the cosets of $B^F$ are in 1–1 correspondence with the irreducible characters of $KH$. Moreover, the degree of such an irreducible character of $G^F$ is given by evaluating the generic degree of the corresponding irreducible character of $KH$ at $q$.

To fix notation, let $\chi$ be an irreducible character of $KH$ and $[\chi]$ be the corresponding irreducible character of $G^F$. Then $\deg([\chi]) = D_\chi(q)$. Let $T \subseteq G$ be an $F$-stable maximal torus obtained from $T_0$ by twisting with our Coxeter element $w_0 \in W$, and $R_{T,1}^G$ the corresponding Deligne–Lusztig generalized character. By [26, Lemma 3.30], the multiplicity of $[\chi]$ in $R_{T,1}^G$ is given by

$$( [\chi], R_{T,1}^G ) = D_\chi( \zeta_h ),$$

where $\zeta_h$ is a primitive $h$th root of unity. (Note that $F$ acts trivially on $W$ by assumption and, also, that we can assume $q$ to be sufficiently large, if necessary.) Thus, we can conclude that $\chi$ has $h$-defect 1 if and only if $([\chi], R_{T,1}^G) = D_\chi( \zeta_h ) \neq 0$, that is, if and only if $[\chi]$ appears with non-zero multiplicity in $R_{T,1}^G$. Finally, by [25, (7.8)], this is the case if and only if $\chi$ corresponds to an exterior power of $\rho$ under the bijection $\text{Irr}(KH) \leftrightarrow \text{Irr}(W)$. This completes the proof.

**Theorem 6.6.** Let $h$ be the Coxeter number of $W$. Then the irreducible characters of $KH$ corresponding to the exterior powers of $\rho$ under the bijection in Remark 6.3 form a single $\Phi_h$-block, and this is the only block which contains more than one irreducible character. Its Brauer tree is given by

$$
\begin{array}{c}
\rho^0 \\
\rho^1 \\
\vdots \\
\rho^{|\mathcal{S}|}
\end{array}
$$

**Proof.** We first introduce some notation. The decomposition matrix of $H$ is the matrix of the map $d_h$ with respect to the canonical bases of $\mathbb{N}_0 \text{Irr}(KH)$ and $\mathbb{N}_0 \text{Irr}(FH)$ consisting of the irreducible characters. Consider a character of $KH$, given as a linear combination of irreducible characters of $KH$ with non-negative coefficients. We say that this character is projective if the vector of coefficients is a sum of the columns of the decomposition matrix of $H$. Now let $S' \subseteq S$ be a subset, $W' = \langle S' \rangle \subseteq W$ the corresponding parabolic subgroup, and $H' = \langle T_\mathcal{S} w \in W' \rangle \subseteq H$ the corresponding parabolic subalgebra. Denote by $\text{Ind}^H_{H'}$ the operation of induction of representations; note that this work on the level of $W$ and on
the level of $KH$, and that we have a commutative diagram (cf. [12], (1.5)):

\[
\begin{array}{ccc}
\mathbb{N}_0 \text{Irr}(W) & \hookrightarrow & \mathbb{N}_0 \text{Irr}(KH) \\
\uparrow \text{Ind}_S^W & & \uparrow \text{Ind}_S^{KH} \\
\mathbb{N}_0 \text{Irr}(W') & \hookrightarrow & \mathbb{N}_0 \text{Irr}(KH')
\end{array}
\]

Moreover, the operation $\text{Ind}_S^{KH}$ takes projectives characters of $KH'$ to projective characters of $KH$ (see [11, (12.2)]).

Now let $n := |S|$ and $\mathcal{B} = \{ \rho^{\wedge k} \mid 0 \leq k \leq n \}$. By Lemma 6.5 we know that $\mathcal{B}$ is a union of non-trivial blocks of $KH$. We say that a character belongs to $\mathcal{B}$ if it can be written as a linear combination of the irreducible characters in $\mathcal{B}$. Let $D$ be the part of the decomposition matrix of $H$ with rows and columns labelled by the irreducible characters of $KH$ and $FH$, respectively, which belong to the union of blocks $\mathcal{B}$.

We will use the above construction to produce a set of ''good'' projective characters in $\mathcal{B}$. For this purpose we note that if $S' \subset S$ is a proper subset then the Poincaré polynomial of $W'$ is not divisible by $\Phi_{2n}$. (This can be checked easily from the list in [7, p. 75]; a more conceptual proof would be given by using one of the known characterizations of the Coxeter number, for example, that in terms of the height of the highest root of the underlying root system, see [16, Theorem in Section 3.20]). This implies that every irreducible character of $KH'$ is projective (cf. [11, Proposition 8.2]). Hence the induction takes irreducible characters from $KH'$ to projective characters of $KH$.

Let us take any chain of subsets $S = S_0 \supset S_1 \supset \cdots \supset S_n = \emptyset$ such that $|S_i| = n - i$ for all $i$. Let $W_i = \langle S_i \rangle \subseteq W$ the corresponding parabolic subgroup and $H_i = \langle T_w \mid w \in W_i \rangle \subseteq H$ the corresponding parabolic subalgebra. Let $\text{ind}_{S_i} : KH_i \to K$ be the 1-dimensional representation given by $\text{ind}_{S_i} : T_s \mapsto v$ for all $s \in S_i$. Under the bijection $\text{Irr}(KH_i) \leftrightarrow \text{Irr}(W_i)$, this corresponds to the trivial character of $W_i$.

For $0 \leq k, i \leq n$ denote by $m(k, i)$ the multiplicity of $\rho^{\wedge k}$ in $\text{Ind}_S^{KH}(\text{ind}_{S_i})$. By [8, Theorem 9.15], we have $m(k, i) = \binom{i}{k}$. In particular, if we let $M := (m_{rs}) \in \mathbb{N}^{(n+1)\times n}$ where $m_{rs} = m(n - r, n - s)$ for $0 \leq r \leq n$ and $0 \leq s \leq n - 1$, then

\[
m_{rs} = \begin{cases} 
1 & \text{if } r = s \\
0 & \text{if } s > r,
\end{cases}
\]

and so this is a lower triangular matrix with 1's along the diagonal. The above commutative diagram shows that we obtain the same multiplicities on the level of $KH$. The above remarks concerning induction and projec-
tive characters imply that the columns of $M$ are sums of the columns of $D$ (note that we omitted the column corresponding to the subset $S_0 = S$), and so we have a factorization $M = DP$ where $P$ is a matrix with non-negative integer coefficients.

We can now prove that $B$ is just a single block. Indeed, each non-trivial block in $B$ yields exactly one non-trivial linear relation among the columns of $D$ (since its Brauer graph is a straight line). But the columns of $M$ are obviously linearly independent. So the equation $M = DP$ implies that $D$ has exactly rank $n$ and hence $B$ is a single block. This also implies that $P$ is a square matrix of size $n$ such that $\det P = \pm 1$.

It remains to prove that the characters in $B$ are arranged on the tree in the correct order. The following are standard arguments concerning triangular decomposition matrices. Since $B$ is a single block and its Brauer graph is just a straight line, every column of $D$ contains precisely two non-zero entries (which are 1). Hence, if we take ind, say, as the first character there is a unique ordering of the rows of $D$ so that $D$ has a lower unipotent shape (and this ordering gives the correct labelling of the vertices of the Brauer tree). It is easy to check that, under these conditions, the factorization problem $M = DP$ has a unique solution, and that this solution just says that the rows of $D$ have to be ordered in the same way as the rows of $M$ so as to give a lower triangular shape. This completes the proof. (For $(W, S)$ of type $E_6, E_7, E_8$ the result has already been obtained by explicit computation in [11, Theorems 12.5, 12.6, 12.7], respectively.)

Now we return to $A$-automorphisms of $H$ and consider their effect on the two representations ind: $T_s \mapsto u$ and sgn: $T_s \mapsto -1$ ($s \in S$).

**Corollary 6.7.** Let $\sigma: H \to H$ be an $A$-algebra automorphism and consider the induced bijection on $\Irr KH$. Then either $\sigma$ fixes ind or interchanges ind and sgn. If it fixes ind then it also fixes the character of $KH$ corresponding to $\rho \in \Irr W$.

**Proof.** By Subsection 6.2, the automorphism $\sigma$ induces a symmetry of the $\Phi_n$-modular Brauer graph of $H$. By Theorem 6.6, this graph consists of isolated vertices and a straight line with vertices labelled by the exterior powers of $\rho$. Hence, the latter component must be left invariant. So either $\sigma$ interchanges the two end points of this straight line or fixes all vertices of it. Since these two end points are labelled by ind and sgn, and since $\rho$ labels one of the other vertices, we are done.

6.8. **Proof of Theorem 1.6.** Let $\sigma: H \to H$ be an $A$-algebra automorphism. Corollary 6.7 shows that by composing $\sigma$ with the Alvis–Curtis
duality if necessary, we can always assume that \( \sigma \) is normalized (in the sense of Definition 1.5). We must show there exists a graph automorphism of \( H \) so that its composition with \( \sigma \) fixes every element in the center of \( H \).

Since \( \sigma \) is assumed to be normalized, it fixes ind hence it also fixes the irreducible character of \( KH \) corresponding to \( \rho \in \text{Irr}(W) \), by Corollary 6.7.

Consider the specialization \( v \mapsto 1 \) and denote by \( \sigma_1 \) the induced (normalized) automorphism of \( \mathbb{Z}W \). Since the bijection \( \text{Irr}(KH) \leftrightarrow \text{Irr}(W) \) is compatible with the induced actions of \( \sigma \) on \( \text{Irr}(KH) \) and of \( \sigma_1 \) on \( \text{Irr}(W) \) (see Remark 6.3) we conclude that \( \sigma_1 \) fixes the character \( \rho \).

By [18, (3.17); 30, Proposition 2, Sect. 3], the automorphism \( \sigma_1 \) actually induces a character table automorphism of \( W \). Since it fixes \( \rho \), this character table automorphism is given by a graph automorphism of \( W \), by Theorem 1.2. Such a graph automorphism always exists also on the level of \( H \) (check the defining relations for \( H \)). Using once more the compatibility in Remark 6.3 we conclude that if we compose \( \sigma \) with the inverse of this graph automorphism on \( H \) we obtain a normalized automorphism which fixes all irreducible characters of \( KH \).

An automorphism of \( H \) which fixes all irreducible characters of \( KH \) leaves each Wedderburn component of \( KH \) invariant, hence it fixes every element in the center of \( H \).

It remains to show that \( \text{Aut}_c(H) \) is the desired semidirect product. The central \( \mathbb{A} \)-algebra automorphisms of \( H \) form certainly a normal subgroup of \( \text{Aut}_c(H) \). By looking at the defining relations of \( H \) we see that every graph automorphism of \( H \) commutes with the Alvis–Curtis duality. Since \( \Gamma \) is the group of outer graph automorphisms, we get the semidirect product as stated. The proof is complete.

**Remark 6.9.** Let \( W \) be of non-crystallographic type \( H_3, H_4 \) or \( I_3(m) \) \((m \geq 5, m \neq 6)\). The Coxeter numbers are given by 10, 30, and \( m \), respectively. In these cases, \( K \) no longer is a splitting field. Nevertheless, one can still set up an analogous theory of specializations, decomposition maps, and Brauer trees as above, by working in a suitable enlargement of \( K \). We again have a commutative diagram as in Subsection 6.2 for specialized algebras, and J. Mueller has shown in [28] that the statement of Theorem 6.6 also remains true in this case. Consequently, the above arguments go through in these cases as well, and hence Theorem 1.6 holds in general.

### 6.10. Unequal Parameters

We close with a corresponding result for Iwahori–Hecke algebras with different parameters. Let \( \mathbb{A} := \mathbb{Z}[v, v^{-1}] \) where the \( v \) are indeterminates such that \( v_s = v_t \) whenever \( s, t \in S \) are
conjugate in \( W \). Let \( H \) be the generic Iwahori–Hecke algebra over \( A \). This is defined similarly as before but now we have quadratic relations of the form

\[
T_s^2 = u_s T_1 + (u_s - 1) T_s
\]

for all \( s \in S \), where \( u_s := v_s^2 \). Again, we have 1-dimensional representations \( \text{ind}: H \to A \), \( T_s \to u_s \), and \( \text{sgn}: H \to A \), \( T_s \to -1 \). The Alvis–Curtis duality \( *: H \to H \) is now defined by \( T_s \to -u_s T_s^{-1} \). We say that an \( A \)-algebra automorphism of \( H \) is normalized if it fixes \( \text{ind} \).

We have a corresponding theory of specializations and compatibility properties as above. (These things can be treated in a rather general framework, see [14]). We can now state:

**Proposition 6.11.** Let \( \sigma: H \to H \) be an \( A \)-algebra automorphism.

1. Either \( \sigma \) fixes \( \text{ind} \) or interchanges \( \text{ind} \) and \( \text{sgn} \). Hence, by composing with the Alvis–Curtis duality if necessary, we can assume that \( \sigma \) is normalized.
2. If \((W,S)\) is not of type \( D_{2n} \) (for some \( n \geq 2 \)) then every normalized \( \sigma \) is central.
3. If \((W,S)\) is of type \( D_{2n} \), then every normalized \( \sigma \) is the composition of a central and a graph automorphism.

The group of \( A \)-algebra automorphisms of \( H \) is a semidirect product

\[
\text{Aut}_A(H) \cong \text{Aut}_{\text{cent}}(H) \rtimes (\Gamma \times \langle * \rangle),
\]

where \( \text{Aut}_{\text{cent}}(H) \) denotes the group of central \( A \)-algebra automorphisms of \( H \), \( \Gamma \) is the group of outer graph automorphisms, and \( * \) is the Alvis–Curtis duality. Here \( \Gamma \neq 1 \) only if \( H \) is of type \( D_{2n} \).

**Proof.** Consider the specialization \( f: \mathbb{A} \to A, v_s \to v \) for all \( s \in S \). Let \( K \) be the field of fractions of \( A \) and \( K H \) the corresponding algebra over \( K \). By specialization, we obtain an \( A \)-algebra automorphism \( \sigma_f: H \to H \). It is known (see [12]) that \( K H \) is split semisimple and that \( \chi(T_s) \in \mathbb{Z}[v_s | s \in S] \) for all \( \chi \in \text{Irr}(K H) \) and all \( w \in W \) if \( W \) is of crystallographic type. If \( W \) is non-crystallographic then we have to work in a suitable enlargement of \( K \) as in Remark 6.9. Hence the relation between \( H \) and \( H \) is completely analogous to the relation between \( H \) and \( \mathbb{Z} W \) in Remark 6.9. Since also the Alvis–Curtis duality on \( H \) is the specialization of the Alvis–Curtis duality on \( H \) we immediately deduce (a) from the corresponding result for \( H \) in Corollary 6.7.
So let us now assume that \( \sigma \) is normalized. Then \( \sigma_f \) is also normalized and Theorem 1.6 implies that \( \sigma_f \) is the composition of a graph and a central automorphism of \( H \).

Recall that non-central graph automorphisms exist only for \( (W,S) \) of type \( D_{2n} \) (for \( n \geq 2 \), \( F_4 \), and \( I_2(2m) \) (for \( m \geq 2 \)). In type \( D_{2n} \), \( all \ s \in S \) are conjugate, and so there is nothing to prove since \( A = A; H = H \).

Now let \( (W,S) \) be of type \( F_4 \) or \( I_2(2m) \) (for some \( m \geq 2 \)). Then there exists a unique non-trivial graph automorphism \( \alpha \) and there exist \( s,t \in S \) which are not conjugate in \( W \) and such that \( \alpha(s) = t \). Hence the induced action of \( \sigma_f \) on \( \text{Irr}(KH) \) is either trivial or given by this graph automorphism. Assume, if possible, that the latter is the case.

The algebra \( H \) has four 1-dimensional representations, namely ind, sgn, a representation \( \epsilon_s \) such that \( \epsilon_s(T_u) = u_s \) and \( \epsilon_s(T_t) = -1 \), and a representation \( \epsilon_t \) such that \( \epsilon_t(T_u) = -1 \) and \( \epsilon_t(T_t) = u_t \). Using the bijection \( \text{Irr}(KH) \leftrightarrow \text{Irr}(KH) \) and its compatibility with the automorphisms \( \sigma \) and \( \sigma_f \) we conclude that

\[
\sigma^*(\text{ind}) = \text{ind} \quad \text{and} \quad \sigma^*(\text{sgn}) = \text{sgn} \quad \text{and} \quad \sigma^*(\epsilon_s) = \epsilon_t. \quad (1)
\]

Now forget the specialization \( f \) and consider instead another specialization \( f' : A \to A \) such that \( f'(u_s) = u \) and \( f'(u_t) = -1 \); denote the corresponding algebra over \( A \) by \( AH' \) and the corresponding algebra over \( K \) by \( KH' \). We have a specialized automorphism \( \sigma_f : AH' \to AH' \). As in Subsection 6.1 the map \( f' \) induces a decomposition map \( d_f \) and we see that

\[
d_f(\text{ind}) = d_f(\epsilon_s) \quad \text{and} \quad d_f(\text{sgn}) = d_f(\epsilon_t). \quad (2)
\]

By a similar argument as in Subsection 6.2 this decomposition map is compatible with the actions of \( \sigma \) on \( \text{Irr}(KH) \) and of \( \sigma_f \) on \( \text{Irr}(KH') \). Hence we conclude that

\[
d_f(\text{ind}) = d_f(\sigma^*(\text{ind})) \quad \text{by (1)}
\]

\[
= \sigma_f^*(d_f(\text{ind})) \quad \text{by compatibility}
\]

\[
= \sigma_f^*(d_f(\epsilon_s)) \quad \text{by (2)}
\]

\[
= d_f(\sigma^*(\epsilon_s)) \quad \text{by compatibility}
\]

\[
= d_f(\epsilon_t) \quad \text{by (1)}
\]

\[
= d_f(\text{sgn}) \quad \text{by (2)},
\]

which is absurd.
Hence $\sigma_i$ is trivial. This implies that $\sigma$ also acts trivially on $\text{Irr}(KH)$ and so $\sigma$ is central.

The semidirect product structure of $\text{Aut}_A(H)$ is shown similarly as in the proof of Theorem 1.6. This completes the proof.

REFERENCES


