Abstract

This paper deals with the solvability and uniqueness of the second-order three-point boundary value problems at resonance on a half-line

\[ x''(t) = f(t, x(t), x'(t)), \quad 0 < t < +\infty, \]

\[ x(0) = x(\eta), \quad \lim_{t \to +\infty} x'(t) = 0, \]

and

\[ x''(t) = f(t, x(t), x'(t)) + e(t), \quad 0 < t < +\infty, \]

\[ x(0) = x(\eta), \quad \lim_{t \to +\infty} x'(t) = 0, \]

where \( f : [0, +\infty) \times R^2 \to R, e : [0, +\infty) \to R \) are continuous and \( \eta \in (0, +\infty) \). By using the coincidence degree theory, we establish some existence and uniqueness criteria.

Keywords: Three-point boundary value problem; At resonance; Coincidence degree theory; Infinite intervals

1. Introduction

Second-order boundary value problems (BVPs) on infinite intervals, arising from the study of radially symmetric solutions of nonlinear elliptic equation and models of gas pressure in a semi-infinite porous medium [1], have received much attention, see [1–9] and the references therein.

Multi-point boundary value problems of second-order linear differential equations on a finite interval were initiated by V.A. Il’in and E.I. Moiseev [10,11] and three-point BVPs of nonlinear differential equations were studied by
C.P. Gupta. Since then, more general nonlinear multi-point BVPs on finite intervals have been discussed extensively [12–19]. The methods therein mainly depend on the Leray–Schauder continuation theorem, coincidence degree theory. However, few work is done for second-order multi-point BVPs on an infinite interval. In [19] the authors studied the following second-order three-point BVP on a half-line

\[ \begin{align*}
  x''(t) &= f(t, x(t), x'(t)), \quad 0 < t < +\infty, \\
  x(0) &= \alpha x(\eta), \\
  \lim_{t \to +\infty} x'(t) &= 0,
\end{align*} \]

(1.1)

where \( \alpha \neq 1, \eta \in [0, +\infty) \). With the help of the established Green function and the Leray–Schauder continuation theorem, suitable conditions imposed on \( f \) are presented for the existence of solutions.

Second-order three-point BVP (1.1)–(1.2) is at resonance when \( \alpha = 1 \), that is, the corresponding homogeneous BVP

\[ \begin{align*}
  x''(t) &= 0, \quad 0 < t < +\infty, \\
  x(0) &= x(\eta), \\
  \lim_{t \to +\infty} x'(t) &= 0
\end{align*} \]

(1.3)

has nontrivial solutions. In other words, see [12,14–18], the linear operator \( L \) defined by \( Lx = x'' \) is not invertible, even if boundary value condition (1.4) is added.

The methods used in [19] is not suitable to the resonance case. So in this paper, we intend to discuss the solvability of BVPs (1.1)–(1.4) and the BVP of

\[ \begin{align*}
  x''(t) &= f(t, x(t), x'(t)) + e(t), \quad 0 < t < +\infty,
\end{align*} \]

(1.5)

under boundary value condition (1.4) respectively. In what follows we always suppose that \( f : [0, +\infty) \times R^2 \to R \), \( e : [0, +\infty) \to R \) are continuous and \( \eta \in (0, +\infty) \).

The proof of our main results depends on the continuation theorem due to J. Mawhin [20]. After the preliminaries in Section 2, we present sufficient conditions for the existence of solutions to BVP (1.1)–(1.4) in Section 3. BVP (1.5)–(1.4) is investigated in Section 4 based on the results obtained for BVP (1.1)–(1.4). Since the integral mean-value theorem is not suitable to a function defined on an infinite interval, our method is different from those applied in finite case. The uniqueness of solutions to both systems are discussed in the last section.

2. Preliminaries

We present here some definitions and lemmas which are essential in the proof of our main results.

**Definition 2.1.** \( f : [0, +\infty) \times R^2 \to R \) is called an S-Carathéodory function if and only if

(i) for each \((u, v) \in R^2, t \mapsto f(t, u, v)\) is measurable on \([0, +\infty)\);  
(ii) for a.e. \( t \in [0, +\infty) \), \((u, v) \mapsto f(t, u, v)\) is continuous on \( R^2 \);  
(iii) for each \( r > 0 \), there exists \( \varphi_r \in L^1([0, +\infty)) \) satisfying \( \varphi_r(t) > 0, t \in (0, +\infty) \) and \( \int_0^{+\infty} s \varphi_r(s) ds < +\infty \) such that

\[
\max \{|u|, |v|\} \leq r \implies |f(t, u, v)| \leq \varphi_r(t), \text{ a.e. } t \in [0, +\infty).
\]

To obtain the existence results, we need the following theorem due to J. Mawhin [20].

**Theorem 2.1.** Let \( X, Y \) be Banach spaces. Let \( L : \text{dom} \ L \subset X \to Y \) be a Fredholm operator of index zero and \( N : X \to Y \) \( L \)-compact on \( \overline{\Omega} \) with \( \Omega \subset X \) open and bounded. Assume that the following conditions are satisfied:

1. \( Lx \neq \lambda Nx \) for every \((x, \lambda) \in [(\text{dom} \ L \setminus \text{Ker} \ L) \cap \partial \Omega] \times (0, 1)\);  
2. \( Nx \notin \text{Im} L \) for every \( x \in \text{Ker} \ L \cap \partial \Omega \);  
3. \( \deg(JQN\mid_{\overline{\Omega} \setminus \text{Ker} \ L}, \Omega \cap \text{Ker} \ L, 0) \neq 0 \), where \( Q : Y \to Y \) is a projection such that \( \text{Im} L = \text{Ker} Q, J : \text{Im} Q \to \text{Ker} L \) is an isomorphism.

Then the equation \( Lx = Nx \) has at least one solution in \( \text{dom} \ L \cap \overline{\Omega} \).
In this paper, we use the space $X$, $Y$ defined by
\begin{equation}
X = \left\{ x \in C^1[0, +\infty), \lim_{t \to +\infty} x(t) \text{ exists}, \lim_{t \to +\infty} x'(t) \text{ exists} \right\},
\end{equation}
\begin{equation}
Y = \left\{ y \in L^1[0, +\infty) \cap C[0, +\infty), \int_0^s |y(s)| ds < +\infty \right\},
\end{equation}
with the norm $\|x\|_X = \max\{\|x\|_\infty, \|x'\|_\infty\}$ and $\|y\|_Y = \max\{\|y\|_\infty, \|y\|_{L^1}, \|y\|_1\}$ respectively, where $\| \cdot \|_\infty$ is the supremum norm on $[0, +\infty)$ and $\|y\|_1 = \int_0^+ \|y(s)\| ds$. By the standard arguments, we can prove that $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ are Banach spaces.

Define the linear operator $L : \text{dom } L \subset X \to Y$ by
\begin{equation}
L : x \mapsto Lx = x'',
\end{equation}
where $\text{dom } L = \{ x \in X \cap C^2[0, +\infty), x(0) = x(\eta), \lim_{t \to +\infty} x'(t) = 0 \}$. Then by direct calculations, we obtain
\begin{align*}
\text{Ker } L &= \{ x \in X: x(t) = c \in R, t \in [0, +\infty) \}, \\
\text{Im } L &= \{ y \in Y: \int_0^+ G(s)y(s) ds = 0 \},
\end{align*}
where
\begin{equation*}
G(s) = \begin{cases} s, & 0 \leq s \leq \eta, \\
\eta, & \eta \leq s < +\infty. \end{cases}
\end{equation*}

Set $\omega(t) = e^{-t} / (1 - e^{-\eta})$ and define the continuous projection $Q : Y \to Y$ by
\begin{equation}
(Qy)(t) = \omega(t) \int_0^+ G(s)y(s) ds, \quad t \in [0, +\infty).
\end{equation}

It is easy to verify that the operator $Q$ is well defined. For any $y \in Y$, let $y_1 = y - Qy$. Then $y_1 \in Y$ and
\begin{equation*}
\int_0^+ G(s)y_1(s) ds = \left( 1 - \int_0^+ G(s)\omega(s) ds \right) \int_0^+ G(s)y(s) ds = 0,
\end{equation*}
which implies $Y = \text{Im } L \oplus \text{Im } Q$. So $\dim \text{Ker } L = \dim \text{Im } Q = 1$ and then $L$ is a Fredholm operator of index zero.

**Remark 2.1.** The definition of $Q$ is not unique. In fact, if we choose $\omega \in Y$ with $\omega > 0$ and $\int_0^+ G(s)\omega(s) ds = 1$, the conclusions are also hold. Here we note that different projections make no differences in the proof of Theorem 2.1, see [20]. So an explicit definition of $Q$ is enough. The same is for $P$ defined later.

Define the continuous projection $P : X \to \text{Ker } L$ by
\begin{equation}
(Px)(t) = x(0), \quad t \in [0, +\infty).
\end{equation}
Then $X = \text{Ker } L \oplus \text{Ker } P$. So for every $x \in X$, there is a unique decomposition $x(t) = \rho + x_1(t)$ such that $\rho \in R$ and $x_1 \in \text{Ker } P$. And the restriction of $L$ on $\text{dom } L \cap \text{Ker } P$, denoted by $L_p : \text{dom } L \cap \text{Ker } P \to \text{Im } L$, is invertible. Write $K_p = L_p^{-1}$ and we have
\begin{equation*}
(K_p y)(t) = - \int_0^+ G(t, s)y(s) ds, \quad t \in [0, +\infty)
\end{equation*}
for any $y \in \text{Im } L$, where
\begin{equation*}
G(t, s) = \begin{cases} s, & 0 \leq s \leq t, \\
t, & t \leq s < +\infty. \end{cases}
\end{equation*}
Let the nonlinear operator \( N : X \to Y \) be defined by
\[
(Nx)(t) = f(t, x(t), x'(t)), \quad t \in [0, +\infty).
\]
Then BVP (1.1)–(1.4) equal to
\[
\begin{align*}
    x &= Px + K_P(I - Q)Nx, \\
    JQNx &= 0,
\end{align*}
\]
where \( J : \text{Im} \, Q \to \text{Ker} \, L \) is an isomorphism.

In order to apply Theorem 2.1, we have to prove that \( N \) is \( L \)-compact, that is, the operators \( QN \) and \( K_P(I - Q)N \) are both completely continuous. Because the Arzelà–Ascoli theorem fails to the noncompact interval case, we will use the following criterion.

**Theorem 2.2.** (See [1].) Let \( M \subset C_\infty = \{ x \in C[0, +\infty), \ \lim_{t \to +\infty} x(t) \text{ exists} \} \). Then \( M \) is relatively compact if the following conditions hold:

(a) all functions from \( M \) are uniformly bounded;
(b) all functions from \( M \) are equicontinuous on any compact interval of \([0, +\infty)\);
(c) all functions from \( M \) are equiconvergent at infinity, that is, for any given \( \epsilon > 0 \), there exists a \( T = T(\epsilon) > 0 \) such that \( |f(t) - f(+\infty)| < \epsilon \), for all \( t > T \) and \( f \in M \).

**Lemma 2.3.** Let \( f \) be an \( S \)-Carathéodory function, then \( N \) is \( L \)-compact.

**Proof.** Obviously, \( QN \) and \( K_P(I - Q)N \) are continuous. So we just prove the compactness, i.e. \( QN \) and \( K_P(I - Q)N \) maps bounded sets into relatively compact ones.

Suppose \( U \subset X \) is a bounded set. Then there exists \( r > 0 \) such that \( \| x \|_X \leq r \), for all \( x \in U \). Because \( f \) is an \( S \)-Carathéodory function, there exists \( \varphi_r \in L^1[0, +\infty) \) satisfying \( \varphi_r(t) > 0, \ t \in (0, +\infty) \) and \( \int_0^{+\infty} s\varphi_r(s) \, ds < +\infty \) such that
\[
|f(t, x(t), x'(t))| \leq \varphi_r(t), \quad \text{a.e. } t \in [0, +\infty).
\]

Then for any \( x \in U \),
\[
\| QN x \|_Y = \max\{\| QN x \|_\infty, \ \| QN x \|_{L^1}, \ \| QN x \|_1 \} \leq \| \varphi_r \|_1 \cdot \| \omega \|_Y < +\infty.
\]
Noticing that \( \text{Im} \, Q \simeq R \), we have \( QN \) is compact.

Furthermore, denote \( K_{p, Q} = K_P(I - Q)N \) and for any \( x \in U \) we have
\[
\begin{align*}
    \left| (K_{p, Q} x)(t) \right| &\leq \int_0^t G(t, s) \left| f(s, x(s), x'(s)) - \omega(s) \int_0^s G(\tau) f(\tau, x(\tau), x'(\tau)) \, d\tau \right| \, ds \\
    &\leq \| \varphi_r \|_1 (1 + \| \omega \|_1) < +\infty
\end{align*}
\]
and
\[
\begin{align*}
    \left| (K_{p, Q} x)'(t) \right| &\leq \int_t^{+\infty} \left| f(s, x(s), x'(s)) - \omega(s) \int_0^s G(\tau) f(\tau, x(\tau), x'(\tau)) \, d\tau \right| \, ds \\
    &\leq \| \varphi_r \|_{L^1} + \| \omega \|_{L^1} \| \varphi_r \|_1 < +\infty,
\end{align*}
\]
that is, \( K_{p, Q} U \) is uniformly bounded. Meanwhile, for any \( t_1, t_2 \in [0, T) \) with \( T \) a positive constant
\[
\begin{align*}
    \left| (K_{p, Q} x)(t_1) - (K_{p, Q} x)(t_2) \right| &\leq \int_0^{+\infty} \left| G(t_1, s) - G(t_2, s) \right| \left| f(s, x(s), x'(s)) - \omega(s) \int_0^s G(\tau) f(\tau, x(\tau), x'(\tau)) \, d\tau \right| \, ds
\end{align*}
\]
\[\begin{align*}
&\leq \int_{0}^{+\infty} |G(t_1, s) - G(t_2, s)| \left( (\varphi_r(s) + \omega(s)) \|\varphi_r\|_1 \right) ds \\
&\to 0, \text{ as } t_1 \to t_2,
\end{align*}\]

and
\[\begin{align*}
&\left| (K_{P,Q} x)'(t_1) - (K_{P,Q} x)'(t_2) \right| \leq \int_{t_1}^{t_2} \left| f(s, x(s), x'(s)) - \omega(s) \int_{0}^{+\infty} G(\tau) f(\tau, x(\tau), x'(\tau)) d\tau \right| ds \\
&\quad \leq \int_{t_1}^{t_2} (\varphi_r(s) + \omega(s)) \|\varphi_r\|_1 ds \\
&\quad \to 0, \text{ as } t_1 \to t_2.
\end{align*}\]

So \(K_{P,Q} U\) is equicontinuous. From Theorem 2.2, we can see that if \(K_{P,Q} U\) and \((K_{P,Q})' U\) are equiconvergent at infinity, then \(K_{P,Q} U\) is relatively compact in \(X\). In fact,
\[\begin{align*}
&\left| (K_{P,Q} x)(t) - (K_{P,Q} x)(+\infty) \right| \leq \int_{0}^{+\infty} (s - G(t, s)) \left| f(s, x(s), x'(s)) - \omega(s) \int_{0}^{+\infty} G(\tau) f(\tau, x(\tau), x'(\tau)) d\tau \right| ds \\
&\quad \leq \int_{0}^{+\infty} (s - G(t, s)) (\varphi_r(s) + \omega(s)) \|\varphi_r\|_1 ds \\
&\quad \to 0, \text{ uniformly as } t \to +\infty.
\end{align*}\]

and
\[\begin{align*}
&\left| (K_{P,Q} x)'(t) - (K_{P,Q} x)'(+\infty) \right| \leq \int_{t}^{+\infty} \left| f(s, x(s), x'(s)) - \omega(s) \int_{0}^{+\infty} G(\tau) f(\tau, x(\tau), x'(\tau)) d\tau \right| ds \\
&\quad \leq \int_{t}^{+\infty} (\varphi_r(s) + \omega(s)) \|\varphi_r\|_1 ds \\
&\quad \to 0, \text{ uniformly as } t \to +\infty.
\end{align*}\]

So we can complete the proof. 

3. Existence result for (1.1)–(1.4)

In this section, we establish an existence result for BVP (1.1)–(1.4) by using Mawhin’s continuation theorem.

**Theorem 3.1.** Let \(f : [0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}\) be an S-Carathéodory function. Suppose that

(H1) there exist nonnegative functions \(p, q, r \in L^1[0, +\infty)\) satisfying
\[\int_{0}^{+\infty} sp(s) ds < +\infty, \quad \int_{0}^{+\infty} sq(s) ds < +\infty, \quad \int_{0}^{+\infty} sr(s) ds < +\infty\]

such that
\[|f(t, u, v)| \leq p(t)|u| + q(t)|v| + r(t) \quad \text{a.e. } t \in [0, +\infty) \text{ and all } (u, v) \in \mathbb{R}^2;\]
there exists $\beta > 0$ such that for all $u, v \in \mathbb{R}$, if $f(t, u, v)$ has zero points, then
\[
\inf_{t \in (0, +\infty)} \left\{ t : f(t, u, v) = 0 \right\} \leq \beta;
\]

there exist $B_1, l > 0, m, n \geq 0$ such that for all $u \in \mathbb{R}$ with $|u| > B_1$, it holds that
\[
|f(t, u, v)| \geq l|u| - m|v| - n \quad \text{for all } t \in [0, \beta] \text{ and } v \in \mathbb{R};
\]

there exists $B_2 > 0$ such that for all $u \in \mathbb{R}$ with $|u| > B_2$ either
\[
uf(t, u, 0) \leq 0 \quad \text{a.e. } t \in [0, +\infty),
\]
or else
\[
uf(t, u, 0) \geq 0 \quad \text{a.e. } t \in [0, +\infty).
\]

Then BVP (1.1)–(1.4) has at least one solution if
\[
\alpha := \left( \frac{m}{l} + \beta \right) \|p\|_{L^1} + \|p\|_1 + \|q\|_{L^1} < 1.
\]

\[\text{Proof.}\]
Let $X, Y, L, N, P, Q$ be defined as (2.1)–(2.6). We divide the proof into four steps.

**Step 1.** Let $\Omega_1 = \{ x \in \text{dom } L \setminus \text{Ker } L : Lx = \lambda Nx \text{ for some } \lambda \in [0, 1]. \}$ Then $\Omega_1$ is bounded.

It is easy to show that for $\lambda \in (0, 1]$ a function $x$ satisfies $Lx = \lambda Nx$ if and only if $x$ is a solution of
\[
x = Px + \lambda K_p(I - Q)Nx,
\]
\[
JQNx = 0.
\]

Suppose $x \in \Omega_1$, then $\lambda \neq 0$ and $QNx = 0$. So
\[
\omega(t) \int_0^{+\infty} G(s)f(s, x(s), x'(s)) \, ds = 0.
\]

For $\omega(t) > 0$, the continuity of $f$ implies that there exists $\xi \in [0, +\infty)$ such that
\[
f(\xi, x(\xi), x'(\xi)) = 0 \quad \text{and} \quad f(t, x(t), x'(t)) \neq 0, \quad t \in [0, \xi).
\]

Then from (H2) and (H3), we have $\xi \leq \beta$ and
\[
\left| x(\xi) \right| \leq \max \left\{ B_1, \frac{m}{l} \|x'\|_{\infty} + \frac{n}{l} \right\}.
\]

Therefore,
\[
\left| x(t) \right| \leq \left| x(\xi) \right| + \int_\xi^t x'(s) \, ds \leq \max \left\{ B_1, \frac{m}{l} \|x'\|_{\infty} + \frac{n}{l} \right\} + (\beta + t)\|x'\|_{\infty}.
\]

For $x \in \Omega_1$, from (3.1), (3.2) we have
\[
x(t) = x(0) - \lambda \int_0^{+\infty} G(t, s) f(s, x(s), x'(s)) \, ds, \quad t \in [0, +\infty),
\]
\[
x'(t) = -\lambda \int_t^{+\infty} f(s, x(s), x'(s)) \, ds, \quad t \in [0, +\infty).
\]

Then
\[ \|x'\|_{\infty} \leq \int_0^{+\infty} |f(s, x(s), x'(s))| \, ds \]
\[ \leq \int_0^{+\infty} \left\{ p(s) \left[ \max \left\{ B_1, \frac{m}{l} \|x'\|_{\infty} + \frac{n}{l} \right\} + (\beta + \tau) \|x'\|_{\infty} \right] + q(s) \|x'\|_{\infty} + r(s) \right\} \, ds \]
\[ \leq \alpha \|x'\|_{\infty} + \left( B_1 + \frac{n}{l} \right) \|p\|_{L^1} + \|r\|_{L^1} \]
implies
\[ \|x'\|_{\infty} \leq \frac{(B_1 + n/l) \|p\|_{L^1} + \|r\|_{L^1}}{1 - \alpha} := M_1. \]

It follows from
\[ \|x\|_{\infty} \leq |x(0)| + \int_0^{+\infty} |sf(s, x(s), x'(s))| \, ds \]
\[ \leq \max \left\{ B_1, \frac{m}{l} \|x'\|_{\infty} + \frac{n}{l} \right\} + \beta \|x'\|_{\infty} + \int_0^{+\infty} s \left( p(s)|x(s)| + q(s)|x'(s)| + r(s) \right) \, ds \]
\[ \leq \|p\|_1 \|x\|_{\infty} + \left( \frac{m}{l} + \beta + \|q\|_1 \right) \|x'\|_{\infty} + B_1 + \frac{n}{l} + \|r\|_1 \]
that
\[ \|x\|_{\infty} \leq \frac{(m/l + \beta + \|q\|_1) M_1 + B_1 + n/l + \|r\|_1}{1 - \|p\|_1} := M_2. \]

Therefore, \[ \|x\|_{X} \leq \min\{M_1, M_2\} := M. \] \( \Omega_1 \) is bounded.

**Step 2.** Let \( \Omega_2 = \{x \in \text{Ker} \, L, \, Nx \in \text{Im} \, L\} \). Then \( \Omega_2 \) is bounded.

Suppose \( x \in \Omega_2 \). Then \( x(t) = \rho \) for some \( \rho \in R \) and
\[ \int_0^{+\infty} G(s) f(s, \rho, 0) \, ds = 0. \]
Similarly, we can obtain that \( |\rho| \leq \max\{B_1, n/l\} \). Thus \( \|x\|_{X} = |\rho| \leq \max\{B_1, n/l\} \) and \( \Omega_2 \) is bounded.

**Step 3.** Let \( \Omega_3^{(1)} = \{x \in \text{Ker} \, L, \, (-1)^i \lambda x + (1 - \lambda) JQx = 0, \, \lambda \in [0, 1], \, i = 1, 2\} \), where \( J : \text{Im} \, Q \to \text{Ker} \, L \) is an isomorphism given by \( J(cw(t)) = c \) for each \( c \in R \). Then \( \Omega_3^{(1)} \) is bounded if the first part of the condition (H4) holds and \( \Omega_3^{(2)} \) is bounded if the second part of the condition (H4) holds.

If \( x \in \Omega_3^{(1)} \), then \( x(t) = \rho \) for some \( \rho \in R \) and
\[ \lambda \rho = (1 - \lambda) \int_0^{+\infty} G(s) f(s, \rho, 0) \, ds. \]
If \( \lambda = 0 \), then \( |\rho| \leq \max\{B_1, n/l\} \). And if \( \lambda \in (0, 1] \), one has \( |\rho| \leq B_2 \). Otherwise
\[ \lambda \rho^2 = (1 - \lambda) \int_0^{+\infty} G(s) \rho f(s, \rho, 0) \, ds \leq 0, \]
which is a contraction. So \( \Omega_3^{(1)} \) is bounded. Similarly, we can show that \( \Omega_3^{(2)} \) is bounded.
Step 4. Let $\Omega = \{x \in X, \|x\|_X < \max\{M, n/l, B_1, B_2\} + 1\}$. Then we can prove that (1.1)–(1.4) has at least one solution in $\text{dom } L \cap \overline{\Omega}$.

Clearly $\Omega_1 \cup \Omega_2 \cup \Omega_3^{(1)} (\cup \Omega_3^{(2)}) \subset \Omega$. So,

1. $Lx \neq \lambda Nx$ for every $(x, \lambda) \in ([\text{dom } L \setminus \text{Ker } L] \cap \partial \Omega) \times (0, 1)$,
2. $Nx \notin \text{Im } L$ for every $x \in \text{Ker } L \cap \partial \Omega$.

In order to complete the proof, it suffices to show that condition (3) in Theorem 2.1 holds, since $L$ is a Fredholm operator of index zero and $N$ is $L$-compact on $\overline{\Omega}$.

Consider the operator $H_i : \text{Ker } L \cap \overline{\Omega} \times [0, 1] \to X$ defined by

$$H_i(x, \lambda) = (-1)^i \lambda x + (1 - \lambda) J Q N x, \quad i = 1 \text{ or } 2.$$ 

By step 3, we can see that $H_i(x, \cdot) \neq 0$ for all $x \in \text{Ker } L \cap \partial \Omega$. Therefore,

$$\text{deg}(J Q N|_{\overline{\Omega} \cap \text{Ker } L}, \overline{\Omega} \cap \text{Ker } L, 0) = \text{deg}(H_i(\cdot, 0), \overline{\Omega} \cap \text{Ker } L, 0) = \text{deg}(H_i(\cdot, 1), \overline{\Omega} \cap \text{Ker } L, 0)$$

$$= \text{deg}((-1)^i I, \overline{\Omega} \cap \text{Ker } L, 0) \neq 0.$$ 

Theorem 2.1 yields that $Lx = Nx$ has at least one solution in $\text{dom } L \cap \overline{\Omega}$. So (1.1)–(1.4) is solvability. The proof is completed. □

4. Existence result of (1.5)–(1.4)

Since the integral mean-value theorem is ineffective to the half-line, the methods used in [14–18] for perturbed systems on finite intervals are not applicable now. In this section, we establish an existence result for the perturbed BVP (1.5)–(1.4) by applying the result obtained for (1.1)–(1.4).

Theorem 4.1. Let $e \in Y$ be such that $\int_0^{+\infty} G(s)e(s)ds = 0$. Suppose that the conditions in Theorem 3.1 hold with (H4) replaced with

(H5) there exists $B_3 > 0$ such that for all $(u, v) \in \mathbb{R}^2$ with $|u| > B_3$ and $|v| \leq \|e\|_1$, either

$$uf(t, u, v) \leq 0 \quad \text{a.e. } t \in [0, +\infty)$$

or else

$$uf(t, u, v) \geq 0 \quad \text{a.e. } t \in [0, +\infty).$$

Then (1.5)–(1.4) has at least one solution.

Proof. It is easy to verify that when $e \in Y$ with $\int_0^{+\infty} G(s)e(s)ds = 0$, the following linear BVP on a half-line

$$x''(t) = e(t), \quad 0 < t < +\infty,$$

$$x(0) = x(\eta), \quad \lim_{t \to +\infty} x'(t) = 0$$

has at least one solution. Choose one solution expressed by

$$E(t) = \int_0^{+\infty} G(t, s)e(s)ds, \quad t \in [0, +\infty).$$

(4.1)

Set $x(t) = z(t) + E(t)$. Then we have

$$z''(t) = f(t, z(t) + E(t), z'(t) + E'(t)), \quad 0 < t < +\infty,$$

$$z(0) = z(\eta), \quad \lim_{t \to +\infty} z'(t) = 0.$$ 

(4.2)

(4.3)
Next we show that (4.2)–(4.3) has at least one solution. Let $X$, $Y$, $L$, $P$, $Q$ be defined as in Theorem 3.1. Define the operator $N_1 : X \to Y$ by

$$(N_1 z)(t) = f(t, z(t) + E(t), z'(t) + E'(t)), \quad t \in [0, +\infty).$$

Then BVP (4.2)–(4.3) equal to

$$z = Pz + K_P(I - Q)N_1 z,$$
$$JQN_1 z = 0.$$

**Claim 1.** Let $\Omega_4 = \{z \in \text{dom } L \setminus \text{Ker } L, Lz = \lambda N_1 z \text{ for some } \lambda \in [0, 1]\}$ and $\Omega_5 = \{z \in \text{Ker } L, N_1 z \in \text{Im } L\}$. Then $\Omega_4$ and $\Omega_5$ are bounded.

In fact, set

$$n_1 = m \|E'\|_{\infty} + n + l \|E\|_{\infty},$$
$$r_1(t) = p(t) \|E\|_{\infty} + q(t) \|E'\|_{\infty} + r(t).$$

In a same way as in Theorem 3.1, we obtain

$$\|z'\|_{\infty} \leq \left( B_1 + \|E\|_{\infty} + n_1/l \right) p \|L\|_1 + \|r_1\|_1$$
$$\|z\|_{\infty} \leq \left( m/l + \beta + \|q\|_1 \right) M_3 + B_1 + \|E\|_{\infty} + n_1/l + \|r_1\|_1$$

for $z \in \Omega_4$. So $\|z\|_X \leq \max\{M_3, M_4\}$ and $\Omega_4$ is bounded. If $z \in \Omega_5$, then $z(t) = \rho$ for some $\rho \in R$ and

$$\int_0^{+\infty} G(s)f(s, \rho + E(s), E'(s)) \, ds = 0.$$ 

Similarly, $\|z\|_X = |\rho| \leq \max\{B_1 + \|E\|_{\infty}, n_1/l\}$. So $\Omega_5$ is bounded.

**Claim 2.** Set $\Omega_6^{(i)} = \{z \in \text{Ker } L, (-1)^i \lambda z + (1 - \lambda) JQN_1 z = 0, \lambda \in [0, 1]\}, i = 1, 2$, where $J$ is the same as in Theorem 3.1. Then $\Omega_6^{(1)}$ is bounded if the first part of (H5) holds and $\Omega_6^{(2)}$ is bounded if the second part of (H5) holds.

Assume that there exists $z_k(t) = c_k \in \Omega_6^{(1)} \setminus \{0\}$ such that $|c_k| \to +\infty$ as $k \to +\infty$. Then there exists $\lambda_k \in [0, 1]$ such that

$$\lambda_k c_k = (1 - \lambda_k) \int_0^{+\infty} G(s)f(s, c_k + E(s), E'(s)) \, ds.$$ 

Obviously, $\{\lambda_k\}$ has a convergent subsequence. Without loss of generality, we assume $\lambda_k \to \lambda_0$. Then $\lambda_0 \neq 1$. If $|c_k| > \max\{B_1 + \|E\|_{\infty}, B_3 + \|E\|_{\infty}\}$, then conditions (H2) and (H5) imply

$$\frac{f(t, c_k + E(t), E'(t))}{c_k} \leq -\frac{l}{2}, \quad t \in [0, +\infty)$$

for $k$ large enough. Then

$$0 \leq \frac{\lambda_0}{1 - \lambda_0} = \lim_{k \to +\infty} \int_0^{+\infty} G(s)f(s, c_k + E(s), E'(s)) \, ds \leq -\frac{l}{2} \int_0^{+\infty} G(s) \, ds = -\infty$$

which is a contraction. Thus $\Omega_6^{(1)}$ is bounded. So is $\Omega_6^{(2)}$. 

Choose $\Omega_0$, a bounded and open subset of $X$, such that $\Omega_4 \cup \Omega_5 \cup \Omega_6^{(1)} \cup \Omega_6^{(2)} \subset \Omega_0$. Then in the same way as in Theorem 3.1, we can obtain that $Lz = N_1z$ has at least one solution $z \in \text{dom } L \cap \overline{\Omega}$.

Obviously, $x(t) = z(t) + E(t)$ is a solution of (1.5)–(1.4). The proof is completed. \hfill $\Box$

5. Uniqueness results

Under the stronger conditions imposed on $f$, we can prove the uniqueness of solutions to the BVPs studied above.

**Theorem 5.1.** Suppose that $f$ is a S-Carathéodory function and (H2), (H4) in Theorem 3.1 hold. Suppose further that the following conditions hold.

(H6) there exist functions $p, q \in L^1[0, +\infty)$ satisfying $\int_0^{+\infty} sp(s) \, ds < +\infty$, $\int_0^{+\infty} s \cdot p(s) \, ds < +\infty$ such that
\[
|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq p(t)|u_1 - u_2| + q(t)|v_1 - v_2|
\]
for a.e. $t \in [0, +\infty)$ and all $(u_1, v_1), (u_2, v_2) \in \mathbb{R}^2$;

(H7) there exists $\beta_1 > 0$ such that for all $u_i, v_i \in \mathbb{R}$, $i = 1, 2$ if $f(t, u_1, v_1) - f(t, u_2, v_2)$ has zero points, then
\[
\inf_{t \in [0, +\infty)} \{t: f(t, u_1, v_1) - f(t, u_2, v_2) = 0\} \leq \beta_1;
\]

(H8) there exist $l > 0$, $m > 0$ such that
\[
|f(t, u_1, v_1) - f(t, u_2, v_2)| \geq l|u_1 - u_2| - m|v_1 - v_2|
\]
for all $t \in [0, \max\{\beta, \beta_1\}]$ and $(u_1, v_1), (u_2, v_2) \in \mathbb{R}^2$.

Then the BVP (1.1)–(1.4) has exactly one solution provided $\max\{\alpha, \alpha_1\} < 1$, where $\alpha$ is defined in Theorem 3.1 and
\[
\alpha_1 = \left(\frac{m}{l} + \beta_1\right)\|p\|_{L^1} + \|p\|_1 + \|q\|_{L^1}.
\]

**Proof.** Let $X$, $Y$, $L$, $Z$, $P$, $Q$ be defined as in Theorem 3.1. Since $f$ is an S-Carathéodory function, then $r(t) = |f(t, 0, 0)| \in Y$. It follows from Theorem 3.1 that BVP (1.1)–(1.4) has at least one solution.

Now suppose that (1.1)–(1.4) has two solutions $x_1, x_2 \in X$. Let $x = x_1 - x_2$. Then $x$ satisfies
\[
x(t) = x(0) + \int_0^{+\infty} G(t, s) \left[f(s, x_1(s), x_1'(s)) - f(t, x_2(s), x_2'(s))\right] \, ds, \tag{5.1}
\]
and
\[
\int_0^{+\infty} G(s) \left[f(s, x_1(s), x_1'(s)) - f(s, x_2(s), x_2'(s))\right] \, ds = 0. \tag{5.2}
\]
Equality (5.2) and condition (H7) imply that there exists $\xi \in [0, \beta_1)$ such that
\[
f(\xi, x_1(\xi), x_1'(\xi)) - f(\xi, x_2(\xi), x_2'(\xi)) = 0.
\]
From (H6), we can obtain that $x(\xi) \leq m\|x'\|_{\infty}/l$ and
\[
|x(t)| \leq \left(\frac{m}{l} + \beta_1 + t\right)\|x'\|_{\infty}, \quad t \in [0, +\infty).
\]
So
\[
\|x'\|_{\infty} \leq \int_0^{+\infty} \left|f(s, x_1(s), x_1'(s)) - f(s, x_2(s), x_2'(s))\right| \, ds
\]
\[
\begin{align*}
&\leq \int_0^{+\infty} \left( p(s) |x(s)| + q(s) |x'(s)| \right) ds \\
&\leq \alpha_1 \|x'\|_{\infty},
\end{align*}
\]
which means that \(\|x'\|_{\infty} = 0\). Meanwhile, (5.1) and (5.3) conclude that
\[
|x(t)| \leq |x(0)| + \int_0^{+\infty} G(t,s) \left| f(s, x_1(s), x_1'(s)) - f(s, x_2(s), x_2'(s)) \right| ds
\]
\[
\leq \int_0^{+\infty} s \left[ p(s) |x(s)| + q(s) |x'(s)| \right] ds
\]
\[
\leq \|p\|_1 \|x\|_{\infty}, \quad t \in [0, +\infty).
\]
No alternative but \(\|x\|_{\infty} = 0\). Therefore \(x_1 = x_2\). Then (1.1)–(1.4) has a unique solution. The proof is completed. 

Similarly, we can obtain a uniqueness result for the forced system (1.5)–(1.4).

**Theorem 5.2.** Let \(e \in Y\) be such that \(\int_0^{+\infty} G(s)e(s) ds = 0\). Suppose that the conditions in Theorem 5.1 hold with (H4) replaced by (H5). Then (1.5)–(1.4) has a unique solution.

**References**


