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An Asymptotic Property of Schachermayer's Space under Renorming

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Let X be a Banach space with closed unit ball B. Given $k \in \mathbb{N}$, X is said to be k-β, respectively, $(k+1)$-nearly uniformly convex ($(k+1)$-NUC), if for every $\varepsilon > 0$ there exists δ, $0 < \delta < 1$, so that for every $x \in B$ and every ε-separated sequence $(x_n) \subseteq B$ there are indices $(n_i)_{i=1}^{k}$, respectively, $(n_i)_{i=1}^{k+1}$, such that $(1/(k+1))\|x + \sum_{i=1}^{k} x_{n_i}\| \leq 1 - \delta$, respectively, $(1/(k+1))\|\sum_{i=1}^{k+1} x_{n_i}\| \leq 1 - \delta$. It is shown that a Banach space constructed by Schachermayer is 2-β, but is not isomorphic to any 2-NUC Banach space. Modifying this example, we also show that there is a 2-NUC Banach space which cannot be equivalently renormed to be 1-β. © 2000 Academic Press

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1. INTRODUCTION

In [4], Huff introduced the notion of nearly uniform convexity (NUC). A Banach space X with closed unit ball B is said to be NUC if for any $\varepsilon > 0$ there exists $\delta < 1$ such that for every ε-separated sequence in B, $\mathrm{co}((x_n)) \cap \delta B \neq \emptyset$. Here $\mathrm{co}(A)$ denotes the convex hull of a set A; a sequence (x_n)

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is ε-separated if $\inf\{\|x_n - x_m\| : m \neq n\} \geq \varepsilon$. Huff showed that a Banach space is NUC if and only if it is reflexive and has the uniform Kadec–Klee property (UKK). Recall that a Banach space X with closed unit ball B is said to be UKK if for any $\varepsilon > 0$ there exists $\delta < 1$ such that for every ε-separated sequence (x_n) in B which converges weakly to some $x \in X$ we have $\|x\| \leq \delta$. A recent result of Knaust *et al.* [5] gives an isomorphic characterization of spaces having NUC. They showed that a separable reflexive Banach space X is isomorphic to a UKK space if and only if X has a finite Szlenk index. More recent results concerning Szlenk indices and renormings are to be found in [2, 3].

Another property related to NUC is the property (β) introduced by Rolewicz [11]. In [6], building on the work of Prus [9, 10], the first author showed that a separable Banach space X is isomorphic to a space with (β) if and only if both X and X^* are isomorphic to NUC spaces. In [7], a sequence of properties lying in between (β) and NUC is defined. Let X be a Banach space with closed unit ball B. Given $k \in \mathbb{N}$, X is said to be k-β, respectively, $(k+1)$-nearly uniformly convex ($(k+1)$-NUC), if for every $\varepsilon > 0$ there exists δ, $0 < \delta < 1$, so that for every $x \in B$ and every ε-separated sequence $(x_n) \subseteq B$ there are indices $(n_i)_{i=1}^{k}$, respectively $(n_i)_{i=1}^{k+1}$, such that

$$\frac{1}{k+1}\left\| x + \sum_{i=1}^{k} x_{n_i} \right\| \leq 1 - \delta,$$

respectively

$$\frac{1}{k+1}\left\| \sum_{i=1}^{k+1} x_{n_i} \right\| \leq 1 - \delta.$$

It follows readily from the definitions that every k-β space is $(k+1)$-NUC, every $(k+1)$-NUC space is $(k+1)$-β, and that every k-β space (or $(k+1)$-NUC space) is NUC. It is proved in [7] that property 1-β is equivalent to the property (β) of Rolewicz. It is worth noting that the "non-uniform" version of property k-NUC has been well-studied. For $k \geq 2$, a Banach space X is said to have property (kR) if every sequence (x_n) in X which satisfies $\lim_{n_1} \cdots \lim_{n_k} \|x_{n_1} + \cdots + x_{n_k}\| = k \lim_n \|x_n\|$ is convergent [1]. It is clear that the property (kR) implies property $((k+1)R)$. It follows from James' characterization of reflexivity that every (kR) space is reflexive. A recent result of Odell and Schlumprecht [8] shows that a separable Banach space is reflexive if and only if it can be equivalently renormed to have property $(2R)$. Thus, all the properites (kR) are isomorphically equivalent. Similarly, "non-asymptotic" properties known as k-uniform rotundity have been studied [13]. These properites are also isomorphically equivalent to each other as they are all equivalent to superreflexivity. In this paper, we find that the situation is different for the properites k-NUC and k-β. To be precise, we use the space constructed by Schachermayer in [12] and a variant to distinguish the properties 1-β, 2-NUC, and 2-β isomorphically.

Let $T = \cup_{n=0}^{\infty}\{0, 1\}^n$ be the dyadic tree. If $\varphi = (\varepsilon_i)_{i=1}^m$ and $\psi = (\delta_i)_{i=1}^n$ are nodes in T, we say that $\varphi \leq \psi$ if $m \leq n$ and $\varepsilon_i = \delta_i$ for $1 \leq i \leq m$. Also, $\emptyset \leq \varphi$ for all $\varphi \in T$. Two nodes φ and ψ are said to be comparable if either $\varphi \leq \psi$ or $\psi \leq \varphi$; they are incomparable otherwise. Let $\varphi \in T$, denote by T_φ or $T(\varphi)$ the subtree rooted at φ, i.e., the subtree consisting of all nodes ψ such that $\varphi \leq \psi$. A node $\varphi \in T$ has length n if $\varphi \in \{0, 1\}^n$. The length of φ is denoted by $|\varphi|$. Given $\varphi = (\varepsilon_i)_{i=1}^n \in T$, let S_φ be the set consisting of all nodes $\psi = (\delta_i)_{i=1}^m$ such that $m \geq n$, $\delta_i = \varepsilon_i$ if $1 \leq i \leq n$, and $\delta_i = 0$ otherwise. Say that a subset A of T is admissible, respectively, acceptable, if there exists $n \in \mathbb{N} \cup \{0\}$ such that (a) $A \subseteq \cup_{|\varphi|=n} T_\varphi$ and (b) $|A \cap T_\varphi| \leq 1$ for all φ with $|\varphi| = n$, respectively, (a′) $A \subseteq \cup_{|\varphi|=n} S_\varphi$, and (b′) $|A \cap S_\varphi| \leq 1$ for all φ with $|\varphi| = n$. For subsets A and B of T, say that $A \ll B$ if $\max\{|\varphi| : \varphi \in A\} < \min\{|\varphi| : \varphi \in B\}$. Let $c_{00}(T)$ be the space of all finitely supported real-valued functions defined on T. For $x \in c_{00}(T)$, let

$$\|x\|_X = \sup\left(\sum_{i=1}^{k}\left(\sum_{\varphi\in A_i} |x(\varphi)|\right)^2\right)^{1/2},$$

where the sup is taken over all $k \in \mathbb{N}$ and all sequences of admissible subsets $A_1 \ll A_2 \ll \cdots \ll A_k$ of T. The norm $\|\cdot\|_Y$ is defined similarly except that the sup is taken over all sequences of acceptable subsets $A_1 \ll A_2 \ll \cdots \ll A_k$ of T. Schachermayer's space X is the completion of $c_{00}(T)$ with respect to the norm $\|\cdot\|_X$. The completion of $c_{00}(T)$ with respect to $\|\cdot\|_Y$ is denoted by Y.

Remark. The space X defined here differs from Schachermayer's original definition and is only isomorphic to the space defined in [12].

In [7], it was shown that X (with the norm given in [12]) is 8-NUC but is not isomorphic to any 1-β space. We first show that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are 2-β and 2-NUC respectively. We begin with a trivial lemma concerning the ℓ^2-norm $\|\cdot\|_2$.

LEMMA 1. *If α, β, and γ are vectors in the unit ball of ℓ^2, and $\|\alpha + \beta + \gamma\|_2/3 \geq 1 - \delta$, then* $\max\{\|\alpha - \beta\|_2, \|\alpha - \gamma\|_2, \|\beta - \gamma\|_2\} \leq \sqrt{18\delta}$.

PROPOSITION 2. *$(X, \|\cdot\|_X)$ is 2-β.*

Proof. Let x and x_n, $n \geq 1$, be elements in the unit ball of X such that (x_n) is ε-separated. Choose $\delta > 0$ such that

$$(1 - 3\delta)^2 + \left[(1 - 24\delta)^{1/2} - (1 - \varepsilon^2/9)^{1/2}\right]^2 > 1. \qquad (1)$$

Without loss of generality, we may assume that (x_n) converges pointwise (as a sequence of functions on T) to some $y_0 : T \to \mathbb{R}$. It is clear that if $y, z \in X$ and $\operatorname{supp} y \ll \operatorname{supp} z$, then $\|y+z\|_X^2 \geq \|y\|_X^2 + \|z\|_X^2$. It follows easily that $y_0 \in X$. Let $y_n = x_n - y_0$. It may be assumed that $(\|y_n\|_X)$ converges. As (x_n) is ε-separated, so is (y_n). We may thus further assume that $\|y_n\|_X > \varepsilon/3$ for all $n \in \mathbb{N}$. By going to a subsequence and perturbing the vectors x, y_0, and y_n, $n \geq 1$, by as little as we please, it may be further assumed that (a) they all belong to $c_{00}(T)$, (b) $\operatorname{supp} x \cup \operatorname{supp} y_0 \ll \operatorname{supp} y_1 \ll \operatorname{supp} y_2$, and (c) $\|y_1 \chi_{T\varphi}\|_\infty = \|y_2 \chi_{T\varphi}\|_\infty$ for all φ such that $|\varphi| \leq M$, where $\|\cdot\|_\infty$ is the sup norm and $M = \max\{|\psi| : \psi \in \operatorname{supp} x \cup \operatorname{supp} y_0\}$.

Claim. Let A be an admissible set such that $\min\{|\varphi| : \varphi \in A\} \leq M$. If $\sum_{\varphi \in A} |y_1(\varphi)| = c$, and $\sum_{\varphi \in A} |y_2(\varphi)| = d$, then there exists an admissble set B such that

$$\begin{aligned}\min\{|\varphi| : \varphi \in A\} &\leq \min\{|\varphi| : \varphi \in B\} \\ &\leq \max\{|\varphi| : \varphi \in B\} \leq \max\{|\varphi| : \varphi \in A\},\end{aligned}$$

$A \cap \operatorname{supp} y_0 \subseteq B$, and $\sum_{\varphi \in B} |y_1(\varphi)| \geq c + d$.

To prove the claim, let N be such that $A \subseteq \cup_{|\varphi|=N} T_\varphi$ and let $|A \cap T_\varphi| \leq 1$ for all φ with $|\varphi| = N$. Then $N \leq M$. Now, for each $\psi \in A \cap \operatorname{supp} y_2$, $\psi \in T_\varphi$ for some φ with $|\varphi| = N \leq M$. It follows that

$$\left\|y_1 \chi_{T_\varphi}\right\|_\infty = \left\|y_2 \chi_{T_\varphi}\right\|_\infty \geq |y_2(\psi)|.$$

Hence, there exists a $\psi' \in T_\varphi$ such that $|y_1(\psi')| \geq |y_2(\psi)|$. Now let

$$B = (A \cap (\operatorname{supp} y_0 \cup \operatorname{supp} y_1)) \cup \{\psi' : \psi \in A \cap \operatorname{supp} y_2\}.$$

It is easy to see that the set B satisfies the claim.

Suppose that $\|x + x_1 + x_2\|_X/3 \geq 1 - \delta$. Let $x + x_1 + x_2 = x + 2y_0 + y_1 + y_2$ be normed by a sequence of admissible sets $A_1 \ll A_2 \ll \cdots \ll A_k$. Denote by $\alpha = (a_i)_{i=1}^k$, $\beta = (b_i)_{i=1}^k$, $\gamma = (c_i)_{i=1}^k$, and $\eta = (d_i)_{i=1}^k$ respectively the sequences $(\sum_{\varphi \in A_i} |x(\varphi)|)_{i=1}^k$, $(\sum_{\varphi \in A_i} |y_0(\varphi)|)_{i=1}^k$, $(\sum_{\varphi \in A_i} |y_1(\varphi)|)_{i=1}^k$, and $(\sum_{\varphi \in A_i} |y_2(\varphi)|)_{i=1}^k$. Now

$$\|\alpha + (\beta + \gamma) + (\beta + \eta)\|_2/3 \geq \|x + x_1 + x_2\|_X/3 \geq 1 - \delta.$$

But $\|\alpha\|_2 \leq \|x\|_X \leq 1$. Similarly, $\|\beta + \gamma\|_2$, $\|\beta + \eta\|_2 \leq 1$. By Lemma 1, we obtain that $\|\alpha - \beta - \gamma\|_2$, $\|\alpha - \beta - \eta\|_2$, and $\|\gamma - \eta\|_2$ are all $\leq \sqrt{18\delta}$. Let j be the largest integer such that $a_j \neq 0$. Note that this implies supp $x \cap A_j \neq \emptyset$; hence $(\operatorname{supp} y_1 \cup \operatorname{supp} y_2) \cap A_i = \emptyset$ for all $i < j$. Thus, $c_i = d_i = 0$ for all $i < j$. Now

$$\left\|(b_{j+1} + d_{j+1}, \ldots, b_k + d_k)\right\|_2 \leq \left\|\alpha - \beta - \eta\right\|_2 \leq \sqrt{18\delta}. \tag{2}$$

Moreover,

$$1 \geq \|x_2\|_X^2 = \|y_0 + y_2\|_X^2 \geq \|y_0\|_X^2 + \|y_2\|_X^2 \geq \|\beta\|_2^2 + \|y_2\|_X^2$$
$$\Longrightarrow \qquad \|\beta\|_2^2 \leq 1 - \varepsilon^2/9. \tag{3}$$

Hence

$$\begin{aligned} 3(1-\delta) &\leq \|\alpha\|_2 + \|\beta+\gamma\|_2 + \|\beta+\eta\|_2 \leq 2 + \|\beta+\eta\|_2 \\ \Longrightarrow \quad (1-3\delta)^2 &\leq \|\beta+\eta\|_2^2 \\ &= \left\|\left(b_1,\ldots,b_{j-1},b_j+d_j\right)\right\|_2^2 \\ &\quad + \left\|\left(b_{j+1}+d_{j+1},\ldots,b_k+d_k\right)\right\|_2^2 \\ &\leq \left(\left\|\left(b_1,\ldots,b_{j-1},b_j\right)\right\|_2 + d_j\right)^2 + 18\delta \qquad \text{by (2)} \\ &\leq \left(\|\beta\|_2 + d_j\right)^2 + 18\delta \\ &\leq \left(\left(1-\varepsilon^2/9\right)^{1/2} + d_j\right)^2 + 18\delta. \qquad \text{by (3)} \end{aligned}$$

Therefore,

$$d_j \geq (1-24\delta)^{1/2} - \left(1-\varepsilon^2/9\right)^{1/2}. \tag{4}$$

Note that by the first part of the argument above we also obtain that

$$\|\beta+\gamma\|_2 \geq 1 - 3\delta. \tag{5}$$

Since $A_j \cap \operatorname{supp} x \neq \emptyset$, we may apply the claim to obtain an admissible set B. Using the sequence of admissible sets $A_1 \ll \cdots \ll A_{j-1} \ll B \ll A_{j+1} \ll \cdots \ll A_k$ to norm $x_1 = y_0 + y_1$ yields

$$\begin{aligned} 1 &\geq \|y_0+y_1\|_X^2 \geq \left\|\left(b_1,\ldots,b_{j-1},b_j+c_j+d_j,b_{j+1}+c_{j+1},\ldots,b_k+c_k\right)\right\|_2^2 \\ &\geq \left\|\left(b_1,\ldots,b_{j-1},b_j+c_j,b_{j+1}+c_{j+1},\ldots,b_k+c_k\right)\right\|_2^2 + d_j^2 \\ &= \|\beta+\gamma\|_2^2 + d_j^2 \\ &\geq (1-3\delta)^2 + \left[\left(1-24\delta\right)^{1/2} - \left(1-\varepsilon^2/9\right)^{1/2}\right]^2 \end{aligned}$$

by (5) and (4). As the last expression is >1 by (1), we have reached a contradiction. ∎

Remark. The same method can be used to show that X is 2-β with the norm given in [12].

PROPOSITION 3. *$(Y, \|\cdot\|_Y)$ is 2-NUC.*

Proof. Let (x_n) be an ε-separated sequence in the unit ball of Y. Choose $\delta > 0$ so that

$$\delta' = 12\delta + 2\sqrt{8\delta} \leq \varepsilon^2/18 \tag{6}$$

and

$$1 - 2\delta - (2+\sqrt{8})\sqrt{\delta} > \sqrt{1-(\varepsilon/3)^2}. \tag{7}$$

As in the proof of the previous proposition, it may be assumed that there exists a sequence $(y_n)_{n=0}^{\infty}$ in Y such that $x_n = y_0 + y_n$, supp $y_{n-1} \ll$ supp y_n for all $n \in \mathbb{N}$, and $\|y_j \chi_{S_\varphi}\|_\infty = \|y_k \chi_{S_\varphi}\|_\infty$ whenever $|\varphi| \leq M_i$ and $j, k > i$, where $M_i = \max\{|\psi| : \psi \in \text{supp } y_i\}$. We may also assume that $(\|y_n\|_Y)$ converges. Since $(y_n)_{n=1}^{\infty}$ is ε-separated, $\eta = \lim \|y_n\|_Y \geq \varepsilon/2$. The choice of δ' in (6) guarantees that $4(\eta^2 - \delta')^{1/2} > 7\eta/2 \geq 3\eta + \sqrt{\delta'}$. Hence there exist $\eta_+ > \eta > \eta_- > \varepsilon/3$ such that

$$4\theta \geq 3\eta_+ + \sqrt{(\eta_+)^2 - (\eta_-)^2 + \delta'}, \tag{8}$$

where $\theta = \sqrt{(\eta_-)^2 - \delta'}$. We may now further assume that $\eta_+ \geq \|y_n\|_Y \geq \eta_-$ for all $n \in \mathbb{N}$. Now suppose that $\|x_m + x_n\|_Y/2 > 1 - \delta$ for all $m, n \in \mathbb{N}$.

Claim. For all $m < n$ in $\mathbb{N}$, there exists an acceptable set A such that $\sum_{\varphi \in A} |y_i(\varphi)| > \theta$ for $i = m, n$.

First observe that there are acceptable sets $A_1 \ll A_2 \ll \cdots \ll A_k$ such that $\sum_{i=1}^{k}(\sum_{\varphi \in A_i} |(2y_0 + y_m + y_n)(\varphi)|)^2 > 4(1-\delta)^2$. Let $\alpha = (a_i)_{i=1}^k$, $\beta = (b_i)_{i=1}^k$, and $\gamma = (c_i)_{i=1}^k$ be the sequences $(\sum_{\varphi \in A_i} |y_j(\varphi)|)_{i=1}^k$ for $j = 0, m, n$, respectively. Then $\|2\alpha + \beta + \gamma\|_2 > 2(1-\delta)$ and $\|\alpha + \beta\|_2 \leq \|y_0 + y_m\|_Y = \|x_m\|_Y \leq 1$. Similarly, $\|\alpha + \gamma\|_2 \leq 1$. It follows from the parallelogram law that $\|\beta - \gamma\|_2 < 4 - 4(1-\delta)^2 \leq 8\delta$. Note also that $\|\alpha + \beta\|_2 \geq \|2\alpha + \beta + \gamma\|_2 - \|\alpha + \gamma\|_2 > 1 - 2\delta$. Similarly, $\|\alpha + \gamma\|_2 > 1 - 2\delta$. Let j_1, respectively j_2, be the largest j such that $a_j \neq 0$, respectively $b_j \neq 0$. Since supp $y_0 \cap A_{j_1} \neq \emptyset$, $b_1 = \cdots = b_{j_1 - 1} = 0$. Similarly, $c_1 = \cdots = c_{j_2 - 1} = 0$. Moreover, $j_1 \leq j_2$. Let us show that $j_1 < j_2$. For otherwise, $j_1 = j_2 = j$. Then

$$|b_j - c_j| \leq \|\beta - \gamma\|_2 < \sqrt{8\delta}. \tag{9}$$

Consider the set A_j. Choose $p \in \mathbb{N} \cup \{0\}$ such that $A_j \subseteq \cup_{|\varphi|=p} S_\varphi$ and $|A_j \cap S_\varphi| \leq 1$ for all φ with $|\varphi| = p$. Note that $p \leq M_0$. Let $G = \{\varphi : |\varphi| = p, A_j \cap S_\varphi \cap \text{supp } y_m \neq \emptyset\}$. If $\varphi \in G$, $\|y_n \chi_{S_\varphi}\|_\infty = \|y_m \chi_{S_\varphi}\|_\infty$. Hence there exists $\psi_\varphi \in S_\varphi \cap \text{supp } y_n$ such that $|y_n(\psi_\varphi)| = \|y_m \chi_{S_\varphi}\|_\infty$. It is easy to see that the set $B = \{\psi_\varphi : \varphi \in G\} \cup (A_j \cap \text{supp } y_0) \cup (A_j \cap \text{supp } y_n)$

is acceptable and that $\min\{|\varphi| : \varphi \in A_j\} \leq \min\{|\varphi| : \varphi \in B\}$. Hence $A_1 \ll \cdots \ll A_{j-1} \ll B$. Thus

$$\begin{aligned}
1 \geq \|x_n\|_Y^2 = \|y_0 + y_n\|_Y^2 &\geq \sum_{i=1}^{j-1} |a_i|^2 + \left(\sum_{\varphi\in B} |(y_0+y_n)(\varphi)|\right)^2 \\
&\geq \sum_{i=1}^{j-1} |a_i|^2 + \left(\sum_{\varphi\in A_j} |y_0(\varphi)| + \sum_{\varphi\in A_j} |y_n(\varphi)| + \sum_{\varphi\in G} |y_n(\psi_\varphi)|\right)^2 \\
&\geq \sum_{i=1}^{j-1} |a_i|^2 + \left(|a_j| + |c_j| + \sum_{\varphi\in G} \|y_m \chi_{S_\varphi}\|_\infty\right)^2 \\
&\geq \sum_{i=1}^{j-1} |a_i|^2 + \left(|a_j| + |c_j| + \sum_{\varphi\in A_j} |y_m(\varphi)|\right)^2 \\
&\geq \left\|(a_1, \ldots, a_{j-1}, a_j + b_j + c_j)\right\|_2^2 \\
&\geq \left\|(a_1, \ldots, a_{j-1}, a_j + b_j)\right\|_2^2 + |c_j|^2 \\
&\geq \|\alpha + \beta\|_2^2 + \left(|b_j| - \sqrt{8\delta}\right)^2
\end{aligned}$$

by (9),

$$> (1-2\delta)^2 + \left(\|\beta\|_2 - \sqrt{8\delta}\right)^2.$$

Therefore, $\|\beta\|_2 < (2+\sqrt{8})\sqrt{\delta}$. It follows that

$$\|\alpha\|_2 \geq \|\alpha+\beta\|_2 - \|\beta\|_2 > 1 - 2\delta - \left(2+\sqrt{8}\right)\sqrt{\delta}. \tag{10}$$

However,

$$\|\alpha\|_2^2 \leq \|y_0\|_Y^2 \leq \|x_m\|_Y^2 - \|y_m\|_Y^2 \leq 1 - (\eta_-)^2 < 1 - (\varepsilon/3)^2. \tag{11}$$

Combining (10) and (11) with the choice of δ in (7) yield a contradiction. This shows that $j_1 < j_2$. Applying the facts that $\|\alpha+\beta\|_2 > 1 - 2\delta$ and $\|(b_{j_1}, \ldots, b_{j_2-1})\|_2 \leq \|\beta - \gamma\|_2 < \sqrt{8\delta}$, we obtain that

$$\begin{aligned}
\left|b_{j_2}\right|^2 &> (1-2\delta)^2 - \left(\|\alpha\|_2 + \sqrt{8\delta}\right)^2 \\
&\geq (1-2\delta)^2 - \left(\sqrt{1-(\eta_-)^2} + \sqrt{8\delta}\right)^2 \geq \theta^2.
\end{aligned}$$

Similarly,

$$
\begin{aligned}
(1-2\delta)^2 &< \|\alpha+\gamma\|_2^2 = \|\alpha\|_2^2 + \left|c_{j_2}\right|^2 + \left\|\left(c_{j_2+1}, \ldots, c_k\right)\right\|_2^2 \\
&\le \|\alpha\|_2^2 + \left|c_{j_2}\right|^2 + \|\beta-\gamma\|_2^2 \\
&\le 1-\left(\eta_-\right)^2 + \left|c_{j_2}\right|^2 + 8\delta.
\end{aligned}
$$

Hence $\left|c_{j_2}\right| > \theta$. Thus the set $A = A_{j_2}$ satisfies the requirements of the claim.

Taking $m = 1$, $n = 2$, and $m = 2$, $n = 3$, respectively, we obtain acceptable sets A and A' from the claim. Since $A \cap \operatorname{supp} y_1 \neq \emptyset$, if $\varphi \in A \cap \operatorname{supp} y_2$, $\varphi \in S_{\varphi'}$ for some φ' such that $|\varphi'| \le M_1$. This implies that there exists $\psi_\varphi \in S_{\varphi'}$ such that $|y_3(\psi_\varphi)| = \left\|y_3\chi_{S_{\varphi'}}\right\|_\infty = \|y_2\chi_{S_{\varphi'}}\|_\infty \ge |y_2(\varphi)|$. Let $q = \min\{|\varphi| : \varphi \in \operatorname{supp} y_3\}$ and $\Phi = \{\sigma \in T : |\sigma| = q\}$. For $\sigma \in \Phi$, define $s(\sigma) = |y_3(\psi_\varphi)|$ if there exists $\varphi \in A \cap \operatorname{supp} y_2$ such that $\psi_\varphi \in S_\sigma$; otherwise, let $s(\sigma) = 0$. Also, let $t(\sigma) = |y_3(\varphi)|$ if there exists $\varphi \in A' \cap \operatorname{supp} y_3 \cap S_\sigma$; otherwise, let $t(\sigma) = 0$. Finally, let $r(\sigma) = \left\|y_3\chi_{S_\sigma}\right\|_\infty$ for all $\sigma \in \Phi$. Then $r(\sigma) \ge s(\sigma) \ge 0$ for all $\sigma \in \Phi$, $\sum_\sigma r(\sigma) \le \|y_3\|_Y < \eta_+$, and $\sum_\sigma s(\sigma) > \theta$. Hence $\sum_\sigma (r(\sigma) - s(\sigma)) < \eta_+ - \theta$. Similarly, $\sum_\sigma (r(\sigma) - t(\sigma)) < \eta_+ - \theta$. Therefore, $\sum_\sigma |t(\sigma) - s(\sigma)| < 2(\eta_+ - \theta)$. Let B be the set of all nodes in $A \cap \operatorname{supp} y_2$ that are comparable with some node in $A' \cap \operatorname{supp} y_3$. Then

$$
\sum_{\varphi \in A \setminus B} |y_2(\varphi)| \le \sum_{\varphi \in A \setminus B} \left|y_3(\psi_\varphi)\right| \le \sum_\sigma |t(\sigma) - s(\sigma)| < 2(\eta_+ - \theta).
$$

Hence $\sum_{\varphi \in B} |y_2(\varphi)| > \theta - 2(\eta_+ - \theta) = 3\theta - 2\eta_+$. Now let $l = \min\{|\varphi| : \varphi \in A' \cap \operatorname{supp} y_2\}$. Divide B into $B_1 = \{\varphi \in B : |\varphi| < l\}$ and $B_2 = \{\varphi \in B : |\varphi| \ge l\}$. Since B_1 and $A' \cap \operatorname{supp} y_2$ are acceptable sets such that $B_1 \ll A' \cap \operatorname{supp} y_2$,

$$
\begin{aligned}
\left(\eta_+\right)^2 > \left\|y_2\right\|_Y^2 &\ge \left(\sum_{\varphi \in B_1} \left|y_2(\varphi)\right|\right)^2 + \left(\sum_{\varphi \in A'} \left|y_2(\varphi)\right|\right)^2 \\
&> \left(\sum_{\varphi \in B_1} \left|y_2(\varphi)\right|\right)^2 + \theta^2.
\end{aligned}
$$

Thus

$$
\sum_{\varphi \in B_2} \left|y_2(\varphi)\right| > 3\theta - 2\eta_+ - \sqrt{(\eta_+)^2 - \theta^2}.
$$

Finally, since $B_2 \cup (A' \cap \operatorname{supp} y_2)$ is acceptable,

$$\eta_+ > \|y_2\|_Y \geq \sum_{\varphi \in B_2} |y_2(\varphi)| + \sum_{\varphi \in A' \cap \operatorname{supp} y_2} |y_2(\varphi)|$$

$$> 3\theta - 2\eta_+ - \sqrt{(\eta_+)^2 - \theta^2} + \theta.$$

This contradicts inequality (8). ∎

Before proceeding further, let us introduce some more notation. A branch in T is a maximal subset of T with respect to the partial order $\leq$. If γ is a branch in T and $n \in \mathbb{N} \cup \{0\}$, let φ_n^γ be the node of length n in γ. A collection of pairwise distinct branches is said to have separated at level L if for any pair of distinct branches γ and γ' in the collection the nodes of length L belonging to γ and γ' respectively are distinct. Finally, if $(\gamma_1, \ldots, \gamma_k)$ is a sequence of pairwise distinct branches which have separated at a certain level L, we say that a sequence of nodes $(\varphi_1, \ldots, \varphi_k) \in S(\gamma_1, \ldots, \gamma_k; L)$ if $\varphi_i \in T(\varphi_L^{\gamma_i})$, $1 \leq i \leq k$. Let us note that in this situation $\|\chi_{\{\varphi_i : 1 \leq i \leq k\}}\|_X = k$.

Suppose $|||\cdot|||$ is an equivalent norm on X which is 2-NUC. It may be assumed that there exists $\varepsilon > 0$ so that $\varepsilon\|x\|_X \leq |||x||| \leq \|x\|_X$ for all $x \in X$. Let $\delta = \delta(2\varepsilon) > 0$ be the number obtained from the definition of 2-NUC for the norm $|||\cdot|||$.

PROPOSITION 4. *Let $n \in \mathbb{N} \cup \{0\}$. Then there are pairwise incomparable nodes $\varphi_1, \ldots, \varphi_{2^n}$ such that whenever γ_i, γ_i' are distinct branches passing through φ_i, $1 \leq i \leq 2^n$, and $\{\gamma_i, \gamma_i' : 1 \leq i \leq 2^n\}$ have separated at level L, there is a sequence of nodes $(\psi_1, \ldots, \psi_{2^{n+1}}) \in S(\gamma_1, \gamma_1', \ldots, \gamma_{2^n}, \gamma_{2^n}'; L)$ satisfying $|||\chi_{\{\psi_i : 1 \leq i \leq 2^{n+1}\}}||| \leq (2(1-\delta))^{n+1}$.*

Proof. Assume that n is the first non-negative integer where the proposition fails. Let $\varphi_1, \ldots, \varphi_{2^{n-1}}$ be the nodes obtained by applying the proposition for the case $n-1$. (If $n = 0$, begin the argument with any node φ_1.) For each i, $1 \leq i \leq 2^{n-1}$, let $\psi_{2i-1,1}$ and $\psi_{2i,1}$ be a pair of incomparable nodes in T_{φ_i}. (If $n = 0$, let $\psi_{1,1}$ be any node in T_{φ_1}.) Since the proposition fails for the nodes $\psi_{1,1}, \ldots, \psi_{2^n,1}$, there are distinct branches $\gamma_{i,1}$, $\gamma_{i,1}'$ passing through $\psi_{i,1}$, $1 \leq i \leq 2^n$, and a number L_1 so that $\{\gamma_{i,1}, \gamma_{i,1}' : 1 \leq i \leq 2^n\}$ have separated at level L_1, but $|||\chi_{\{\xi_i : 1 \leq i \leq 2^{n+1}\}}||| > (2(1-\delta))^{n+1}$ for any sequence of nodes $(\xi_1, \ldots, \xi_{2^{n+1}}) \in S(\gamma_{1,1}, \gamma_{1,1}', \ldots, \gamma_{2^n,1}, \gamma_{2^n,1}'; L_1)$. However, since the proposition holds for the nodes $\varphi_1, \ldots, \varphi_{2^{n-1}}$, we obtain a sequence of nodes $(\xi_{1,1}, \ldots, \xi_{2^n,1}) \in S(\gamma_{1,1}', \ldots, \gamma_{2^n,1}'; L_1)$ such that

$$|||\chi_{\{\xi_{i,1} : 1 \leq i \leq 2^n\}}||| \leq (2(1-\delta))^n.$$

(Note that the preceding statement holds trivially if $n = 0$.) For each i, choose a node $\psi_{i,2}$ in $\gamma_{i,1}$ such that $|\psi_{i,2}| > L_1$. Then $\psi_{2i-1,2}$ and $\psi_{2i,2}$ are a pair of incomparable nodes in T_{φ_i}, and the argument may be repeated. (If $n = 0$, repeat the argument using the node $\psi_{1,2}$.) Inductively, we thus obtain sequences of branches $(\gamma_{1,r}, \gamma'_{1,r}, \ldots, \gamma_{2^n,r}, \gamma'_{2^n,r})_{r=1}^{\infty}$, a sequence of numbers $L_1 < L_2 < \cdots$, and sequences of nodes $(\xi_{1,r}, \ldots, \xi_{2^n,r})_{r=1}^{\infty}$ such that

1. the branches $\{\gamma_{i,r}, \gamma'_{i,r} : 1 \le i \le 2^n\}$ have separated at level L_r, $r \ge 1$,
2. $|||\chi_{\{\xi_i : 1 \le i \le 2^{n+1}\}}||| > (2(1-\delta))^{n+1}$ for any sequence of nodes
$$(\xi_1, \ldots, \xi_{2^{n+1}}) \in S(\gamma_{1,r}, \gamma'_{1,r}, \ldots, \gamma_{2^n,r}, \gamma'_{2^n,r}; L_r),$$
3. $(\xi_{1,r}, \ldots, \xi_{2^n,r}) \in S(\gamma'_{1,r}, \ldots, \gamma'_{2^n,r}; L_r)$, and
$$|||\chi_{\{\xi_{i,r} : 1 \le i \le 2^n\}}||| \le (2(1-\delta))^n, \qquad r \ge 1;$$
4. $\xi_{i,r} \in T(\varphi_{L_s}^{\lambda_{i,s}})$ whenever $r > s$, and $1 \le i \le 2^n$.

It follows that if $r > s$, then

$$(\xi_{1,r}, \xi_{1,s}, \ldots, \xi_{2^n,r}, \xi_{2^n,s}) \in S\Big(\gamma_{1,s}, \gamma'_{1,s}, \ldots, \gamma_{2^n,s}, \gamma'_{2^n,s}; L_s\Big). \tag{12}$$

Let $x_r = (2(1-\delta))^{-n} \chi_{\{\xi_{i,r} : 1 \le i \le 2^n\}}$, $r \ge 1$. By Item 3, $|||x_r||| \le 1$. Moreover, because of (12), if $r > s$, then

$$|||x_r - x_s||| \ge \varepsilon \|x_r - x_s\|_X = 2^{n+1}\varepsilon/(2(1-\delta))^n \ge 2\varepsilon.$$

Thus (x_r) is 2ε-separated in the norm $|||\cdot|||$. By the choice of δ, there are $r > s$ such that $|||x_r + x_s|||/2 \le 1 - \delta$. Therefore, $|||\chi_{\{\xi_{1,r}, \xi_{1,s}, \ldots, \xi_{2^n,r}, \xi_{2^n,s}\}}||| \le (2(1-\delta))^{n+1}$. But this contradicts Item 2 and the condition (12). ∎

THEOREM 5. *There is no equivalent* 2*-NUC norm on* X.

Proof. In the notation of the statement of Proposition 4, we obtain, for each n, nodes $\psi_1, \ldots, \psi_{2^{n+1}}$ such that $|||\chi_{\{\psi_i : 1 \le i \le 2^{n+1}\}}||| \le (2(1-\delta))^{n+1}$ and $\|\chi_{\{\psi_i : 1 \le i \le 2^{n+1}\}}\|_X = 2^{n+1}$. Hence $|||\cdot|||$ cannot be an equivalent norm on X. ∎

The proof that the space Y has no equivalent 1-β norm follows along similar lines. Suppose that $|||\cdot|||$ is an equivalent 1-β norm on Y. We may assume that $\varepsilon\|\cdot\|_Y \le |||\cdot||| \le \|\cdot\|_Y$ for some $\varepsilon > 0$. Let $\delta = \delta(\varepsilon)$ be the constant obtained from the definition of 1-β for the norm $|||\cdot|||$. Let $n \in \mathbb{N} \cup \{0\}$ and denote the set $\{\varphi \in T : |\varphi| = n\}$ by Φ.

PROPOSITION 6. *For any* m, $0 \le m \le n$, *any subset* Φ' *of* Φ *with* $|\Phi'| = 2^m$, *and any* $p \in \mathbb{N}$, *there exists an acceptable set* $A \subseteq \cup_{\varphi \in \Phi'} S_\varphi$ *such that* $|A| = 2^m$, $\min\{|\varphi| : \varphi \in A\} \ge p$, *and* $|||\chi_A||| \le 2^m(1-\delta)^m$.

Proof. The case $m = 0$ is trivial. Suppose the proposition holds for some m, $0 \leq m < n$. Let $\Phi' \subseteq \Phi$, $|\Phi'| = 2^{m+1}$, and let $p \in \mathbb{N}$. Divide Φ' into disjoint subsets Φ_1 and Φ_2 such that $|\Phi_1| = |\Phi_2| = 2^m$. By the inductive hypothesis, there exist acceptable sets B and C_j, $j \in \mathbb{N}$, such that $B \subseteq \cup_{\varphi \in \Phi_1} S_\varphi$, $|B| = 2^m$, $\min\{|\varphi| : \varphi \in B\} \geq p$, and $|||\chi_B||| \leq 2^m(1-\delta)^m$; and also $C_j \subseteq \cup_{\varphi \in \Phi_2} S_\varphi$, $|C_j| = 2^m$, $\min\{|\varphi| : \varphi \in C_1\} \geq p$, $C_j \ll C_{j+1}$, and $|||\chi_{C_j}||| \leq 2^m(1-\delta)^m$ for all $j \in \mathbb{N}$. It is easily verified that the sequence $\left(2^{-m}(1-\delta)^{-m}\chi_{C_j}\right)$ is ε-separated and has norm bounded by 1 with respect to $|||\cdot|||$. It follows that there exists j_0 such that $2^{-m}(1-\delta)^{-m}\left|\left|\left|\chi_B + \chi_{C_{j_0}}\right|\right|\right| \leq 2(1-\delta)$. The induction is completed by taking A to be $B \cup C_{j_0}$. ∎

Using the same argument as in Theorem 5, we obtain

THEOREM 7. *There is no equivalent* 1-β *norm on* Y.

We close with the obvious problem.

Problem. For $k \geq 3$, can every k-NUC Banach space, respectively, k-β Banach space, be equivalently renormed to be $(k-1)$-β, respectively, k-NUC?

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