JOURNAL OF COMBINATORIAL THEORY 5, 386–396 (1968)

Some Remarks on the van der Waerden Conjecture

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Communicated by H. J. Ryser

ABSTRACT

Elementary proofs of the van der Waerden conjecture are given for the cases $n = 3$ and $n = 4$. Some partial results are found for the case $n = 5$, and the conjecture is shown true for a special class of matrices.

1. INTRODUCTION

Let $\mathbf{A} = (a_{ij})$ be an $n \times n$ matrix. The permanent of $\mathbf{A}$ is defined by:

$$\text{per}\,\mathbf{A} = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where S_n is the symmetric group of order n. Van der Waerden conjectured [1] that if $\mathbf{A}$ is doubly stochastic then per $\mathbf{A} \geqslant n!/n^n$, with equality holding if and only if $\mathbf{A} = (1/n)\mathbf{J}$, where J is the matrix having all entries equal to one. If $n = 2$ it is trivial to prove the conjecture true; it is known to hold for $n = 3$ [2]. If $\mathbf{A}$ satisfies certain additional hypotheses, positive semidefinite symmetric for example, the conjecture holds. For a survey of the present state of the conjecture and related problems see [3].

In this paper we present elementary proofs for $n = 3$ and $n = 4$ and some partial results, especially for $n = 5$. In our proofs we make extensive use of the following theorem due to Julian Keilson [4]:

THEOREM 1. *Let $\phi(x_1, x_2, \ldots, x_n)$ be a real symmetric function in the variables $x_1, x_2, \ldots, x_n$, defined on the set*

$$C = \left\{(x_1, \ldots, x_n) \mid 0 \leqslant x_i \leqslant 1, \quad i = 1, 2, \ldots, n, \quad \sum_{i=1}^{n} x_i = \alpha\right\}$$

* The research of the second author was supported in part by National Science Foundation Grant GP 5801.

and linear in each variable taken separately. Then the maximum and minimum of ϕ over C are assumed either at the symmetric point or among the symmetric points of the boundary, i.e., points of the type

$$\left(1, 1,..., 1, \frac{\alpha - r}{k}, \frac{\alpha - r}{k},..., \frac{\alpha - r}{k}, 0, 0,..., 0\right)$$

having r ones, $r \leqslant \alpha$, k terms $(\alpha - r)/k$, and $(n - r - k)$ zeros,

$$0 \leqslant \alpha - k \leqslant r \leqslant \alpha \leqslant k + r \leqslant n.$$

The relevance of Keilson's theorem to the permanent function is based on the following expansion due to Ryser [3]. Let $\mathbf{a}^1, \mathbf{a}^2, \cdots, \mathbf{a}^n$ denote the columns of $\mathbf{A}$ and $e_r(\mathbf{x}) = e_r(x_1, x_2,..., x_n)$ denote the r-th elementary function of $(x_1, x_2,..., x_n)$ $r = 1, 2,..., n$; let $T_r(\mathbf{A})$ denote the set of all $\binom{n}{r}$ sums of the columns of $\mathbf{A}$ taken r at a time; i.e., let

$$T_r(\mathbf{A}) = \{\mathbf{x} = \mathbf{a}^{i_1} + \mathbf{a}^{i_2} + \cdots + \mathbf{a}^{i_r} \mid (i_1, i_2,..., i_r) \text{ is a } r\text{-subset of } (1, 2,..., n)\}.$$

Then

$$\operatorname{per} \mathbf{A} = \sum_{T_n(A)} e_n(\mathbf{x}) - \sum_{T_{n-1}(\mathbf{A})} e_n(\mathbf{x}) + \cdots + (-1)^{n-1} \sum_{T_1(\mathbf{A})} e_n(\mathbf{x}). \quad (1.1)$$

When $\mathbf{A}$ is doubly stochastic we have

$$\sum_{T_n(\mathbf{A})} e_n(\mathbf{x}) = 1.$$

In Section 2 we present two different proofs of the conjecture when $n = 3$. One of these is a direct proof based on the above expansion (1.1), the other is a simple application of Theorem 1. In Section 3 we prove identities connecting

$$\sum_{T_r(\mathbf{A})} e_k(\mathbf{x})$$

with

$$\sum_{T_1(\mathbf{A})} e_s(\mathbf{x}), \qquad k = 2, 3,$$

which are needed in the application of Keilson's theorem to the cases $n = 4, 5$ of the conjecture. In Section 4, the van der Waerden conjecture is proved for $n = 4$. In Section 5 we state some partial results for the case

$n = 5$. In the last section we show that the van der Waerden bound holds when m columns of the $n \times n$ doubly stochastic matrix $\mathbf{A}$ are equal and so are the remaining $(n - m)$ columns.

2. The Case $n = 3$.

In this section we present two elementary proofs of the conjecture for $n = 3$. (An earlier proof occurs in [2].) The first argument illustrates a characteristic aspect of the behavior of the permanent of a doubly stochastic matrix, namely, it is the sum of two, or more, vector functions, some of which take on a maximum and some of which assume a minimum at the point of symmetry; the second proof is an example of the application of Theorem 1.

Theorem 2. *If* $\mathbf{A} = (a_{ij})$ *is a* 3×3 *doubly stochastic matrix then* per $\mathbf{A} \geqslant 3!/3^3 = 2/9$ *with equality if and only if* $\mathbf{A} = \frac{1}{3}\mathbf{J}$.

Proof: From (1.1), for $n = 3$ we have

$$\text{per}\,\mathbf{A} = 1 - \sum_{T_2(\mathbf{A})} e_3(\mathbf{x}) + \sum_{T_1(\mathbf{A})} e_3(\mathbf{x}) = 1 + \sum_{T_1(\mathbf{A})} (-e_2 + 2e_3)(\mathbf{x}),$$

that is,

$$\text{per}\,\mathbf{A} = 1 + \sum_{i=1}^{3} (-e_2 + 2e_3)(\mathbf{a}^i), \tag{2.1}$$

where the stochastic vector $\mathbf{a}^i$ is the i-th column of $\mathbf{A}$. The theorem follows directly from the following:

Lemma. *If* $\mathbf{x} = (x_1, x_2, x_3)$ *is a stochastic vector* $(0 \leqslant x_i \leqslant 1, \sum x_i = 1)$ *then* $f(\mathbf{x}) = e_2(\mathbf{x}) - 2e_3(\mathbf{x}) = x_1x_2 + x_1x_3 + x_2x_3 - 2x_1x_2x_3$ *assumes its maximum at* $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Proof 1. $f(\mathbf{x})$ is a continuous symmetric function, and assumes its global maximum on the closed compact set

$$\left\{\mathbf{x} \mid 0 \leqslant x_i \leqslant 1, \sum x_i = 1\right\} \text{ at, say, } \mathbf{a} = (a_1, a_2, a_3).$$

If $\mathbf{a} \neq (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ then at least one of the $a_i < \frac{1}{3}$. Suppose $a_3 < \frac{1}{3}$. Then

$$f(a_1, a_2, a_3) - f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3\right) = (a_1 - a_2)^2\left[\frac{1}{2}\left(-\frac{1}{2} + a_3\right)\right] \leqslant 0,$$

with equality if and only if $a_1 = a_2$ and $a_3 = 1 - 2a_1$, since $a_3 < \frac{1}{3} < \frac{1}{2}$. Also for $\frac{1}{3} < a_1 \leqslant \frac{1}{2}$,

$$f(a_1, a_1, 1 - 2a_1) = 4a_1^3 - 5a_1^2 + 2a_1, f' = 2(3a_1 - 1)(2a_1 - 1),$$

and $f'' = 24a_1 - 10$. Hence

$$f(a_1, a_1, 1 - 2a_1) < \tfrac{7}{27} \qquad \text{for} \quad \tfrac{1}{2} \geqslant a_1 > \tfrac{1}{3},$$

which proves the lemma.

PROOF 2: $f(\mathbf{x})$ satisfies the conditions of Theorem 1, and therefore attains its extrema on $\{\mathbf{x} \mid 0 \leqslant x_i \leqslant 1, \sum x_i = 1\}$ among the points $(1, 0, 0)$, $(\frac{1}{2}, \frac{1}{2}, 0)$, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Now $f(1, 0, 0) = 0$, $f(\frac{1}{2}, \frac{1}{2}, 0) = \frac{1}{4}$, and $f(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = 7/27$. So $f(\mathbf{x})$ assumes its maximum uniquely at $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

3. TWO IDENTITIES

For $n > 3$ the expansion for per $\mathbf{A}$ does not reduce to a single sum over $T_1(\mathbf{A})$. For example, for $n = 4$ we have

$$\text{per}\,\mathbf{A} = 1 + \sum_{T_1(\mathbf{A})} (-e_2 + e_3 - 2e_4)(\mathbf{x}) + \sum_{T_2(\mathbf{A})} e_4(\mathbf{x}),$$

where the $\sum_{T_1(\mathbf{A})}$ assumes a minimum and the $\sum_{T_2(\mathbf{A})}$ assumes a maximum at $(\frac{1}{4} \cdots \frac{1}{4})$. Our method of proof consists in expressing per $\mathbf{A}$ as a sum of sums over $T_r(\mathbf{A})$ of symmetric vector functions such that each one of these assumes its minimum at the symmetric point. To find suitable functions we require identities relating $\sum_{T_r(\mathbf{A})} e_k(\mathbf{x})$ to sums of $e_s(\mathbf{x})$ over $T_1(\mathbf{A})$, $s \leqslant k$. We prove such identities for $k = 2$ and 3. For $k > 3$ we were unable to find appropriate relationships; the reason for this, as one may see from the proof below, is that for $k > 3$ the Newton identities are non-linear. For our purposes we need only the case $r = 2$, but we include the proof for general r.

THEOREM 3. *If $\mathbf{A} = (a_{ij})$ is an $n \times n$ doubly stochastic matrix then*

$$\sum_{T_r(\mathbf{A})} e_2(\mathbf{x}) = \binom{n-2}{r-1} \sum_{T_1(\mathbf{A})} e_2(\mathbf{x}) + \binom{n}{r}\binom{r}{2}, \tag{3.1}$$

and

$$\sum_{T_r(\mathbf{A})} e_3(\mathbf{x}) = \left[\frac{(n-2r)}{(r-1)}\binom{n-3}{r-2}\right] \sum_{T_1(\mathbf{A})} e_3(\mathbf{x})$$

$$+ (r-1)\binom{n-2}{r-1} \sum_{T_1(\mathbf{A})} e_2(\mathbf{x}) + \binom{r}{3}\binom{n}{r}. \tag{3.2}$$

PROOF: For a vector $\mathbf{x} = (x_1, x_2, \dots, x_n)$ let

$$S_r(\mathbf{x}) = \sum_{i=1}^{n} x_i^r.$$

For $\mathbf{x} \in T_r(\mathbf{A})$ we have by Newton's identities

$$\begin{aligned} S_1(\mathbf{x}) &= e_1(\mathbf{x}) = r, \\ S_2(\mathbf{x}) &= r^2 - 2e_2(\mathbf{x}), \\ S_3(\mathbf{x}) &= r^3 - 3re_2(\mathbf{x}) + 3e_3(\mathbf{x}). \end{aligned} \tag{3.3}$$

Any $\mathbf{x} \in T_r(\mathbf{A})$ has coordinates $a_{ij_1} + a_{ij_2} + \cdots + a_{ij_r}$, $i = 1, 2, \dots, n$, and

$$\begin{aligned} \sum_{T_r(\mathbf{A})} S_2(\mathbf{x}) &= \sum_{T_r(\mathbf{A})} \sum_{i=1}^{n} (a_{ij_1} + \cdots + a_{ij_r})^2 \\ &= \sum_{T_r(\mathbf{A})} \sum_{i=1}^{n} \Big[\sum_{k=1}^{r} a_{ij_k}^2 + \sum_{k \neq l} a_{ij_k} a_{ij_l} \Big] \\ &= \binom{n-1}{r-1} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 + \binom{n-2}{r-2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(1 - a_{ij}) \\ &= \Big[\binom{n-1}{r-1} - \binom{n-2}{r-2} \Big] \sum_{T_1(\mathbf{A})} S_2(\mathbf{x}) + n \binom{n-2}{r-2}. \end{aligned}$$

Substituting for $S_2(\mathbf{x})$ given in (3.3) and simplifying we find

$$\sum_{T_r(\mathbf{A})} (r^2 - 2e_2(\mathbf{x})) = \binom{n-2}{r-1} \sum_{T_1(\mathbf{A})} (1 - 2e_2(\mathbf{x})) + n \binom{n-2}{r-2},$$

and

$$\sum_{T_r(\mathbf{A})} e_2(\mathbf{x}) = \binom{n-2}{r-1} \sum_{T_1(\mathbf{A})} e_2(\mathbf{x}) + \binom{n}{r}\binom{r}{2}. \tag{3.1}$$

Next to prove (3.2) we proceed similarly:

$$\begin{aligned} \sum_{T_r(\mathbf{A})} S_3(\mathbf{x}) &= \sum_{T_r(\mathbf{A})} \sum_{i=1}^{n} (a_{ij_1} + \cdots + a_{ij_r})^3 \\ &= \sum_{T_r(\mathbf{A})} \sum_{i=1}^{n} \Big[\sum_{k=1}^{r} a_{ij_k}^3 + 3 \sum_{k \neq l} a_{ij_k}^2 a_{ij_l} + \sum_{k \neq l \neq m} a_{ij_k} a_{ij_l} a_{ij_m} \Big]. \end{aligned}$$

$$\sum_{T_r(\mathbf{A})} S_3(\mathbf{x}) = \binom{n-1}{r-1} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^3 + 3 \binom{n-2}{r-2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2(1 - a_{ij})$$

$$+ \binom{n-3}{r-3} \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} - 3a_{ij}^2 + 2a_{ij}^3)$$

$$= \left[\binom{n-1}{r-1} - 3\binom{n-2}{r-2} + 2\binom{n-3}{r-3}\right] \sum_{T_1(\mathbf{A})} S_3(\mathbf{x})$$

$$+ 3\left[\binom{n-2}{r-2} - \binom{n-3}{r-3}\right] \sum_{T_1(\mathbf{A})} S_2(\mathbf{x}) + n\binom{n-3}{r-3}.$$

Substituting for $S_2(\mathbf{x})$ and $S_3(\mathbf{x})$ and simplifying we obtain (3.2).

In this paper we use only the particular case $r = 2$:

$$\sum_{T_2(\mathbf{A})} e_2(\mathbf{x}) = (n-2) \sum_{T_1(\mathbf{A})} e_2(\mathbf{x}) + \binom{n}{2}, \tag{3.4}$$

$$\sum_{T_2(\mathbf{A})} e_3(\mathbf{x}) = (n-4) \sum_{T_1(\mathbf{A})} e_3(\mathbf{x}) + (n-2) \sum_{T_1(\mathbf{A})} e_2(\mathbf{x}). \tag{3.5}$$

4. The Case $n = 4$

If $\mathbf{A} = (a_{ij})$ is a 4×4 doubly stochastic matrix, (1.1) becomes

$$\text{per } \mathbf{A} = 1 - \sum_{T_3(\mathbf{A})} e_4(\mathbf{x}) + \sum_{T_2(\mathbf{A})} e_4(\mathbf{x}) - \sum_{T_1(\mathbf{A})} e_4(\mathbf{x})$$

$$= 1 + \sum_{T_1(\mathbf{A})} (-e_2 + e_3 - 2e_4)(\mathbf{x})$$

$$+ \frac{1}{2} \sum_{T_2(\mathbf{A})} (+e_2 - e_3 + 2e_4)(\mathbf{x}) - \frac{1}{2} \cdot 6.$$

Using (3.4) and (3.5) for $n = 4$, we write

$$\text{per } \mathbf{A} = c + \sum_{T_1(\mathbf{A})} f(\mathbf{x}) + \sum_{T_2(\mathbf{A})} g(\mathbf{x}),$$

where $c = -3 + 3\alpha$ and

$$f(\mathbf{x}) = [-(1-\alpha)\, e_2 + (1-\beta)\, e_3 - 2e_4](\mathbf{x}) = -g(\mathbf{x}).$$

We seek values of (α, β) for which $f(\mathbf{x})$ assumes its global minimum over $\{\mathbf{x} \mid 0 \leqslant x_i \leqslant 1, \sum x_i = 1\}$ at $(\frac{1}{4}\,\frac{1}{4}\,\frac{1}{4}\,\frac{1}{4})$ and $g(\mathbf{x})$ assumes its minimum over $\{\mathbf{x} \mid 0 \leqslant x_i \leqslant 1, \sum x_i = 2\}$ at $(\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2})$. Theorem 1 requires that

(α, β) satisfy a number of inequalities. If the set of (α, β) satisfying these inequalities is non-empty, which it is, then the proof of the conjecture for $n = 4$ is complete. One particular decomposition with the desired properties is:

$$\operatorname{per} \mathbf{A} = -\frac{1}{3} + \frac{1}{9} \sum_{T_1(\mathbf{A})} (-4e_2 + 9e_3 - 18e_4)(\mathbf{x})$$
$$+ \frac{1}{18} \sum_{T_2(\mathbf{A})} (4e_2 - 9e_3 + 18e_4)(\mathbf{x}).$$

Now, we have the following values for the summands:

$\mathbf{x}$	$(-4e_2 + 9e_3 - 18e_4)(\mathbf{x})$	$\mathbf{x}$	$(4e_2 - 9e_3 + 18e_4)(\mathbf{x})$
$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	$-129/128$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$2\frac{5}{8}$
$(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	-1	$(1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$2\frac{2}{3}$
$(0, 0, \frac{1}{2}, \frac{1}{2})$	-1	$(0, \frac{2}{3}, \frac{2}{3}, \frac{2}{3})$	$2\frac{2}{3}$
$(0, 0, 0, 1)$	0	$(0, 0, 1, 1)$	4
		$(1, 0, \frac{1}{2}, \frac{1}{2})$	$2\frac{3}{4}$,

which, in light of Theorem 1, establish:

THEOREM 4. *If* $\mathbf{A} = (a_{ij})$ *is a* 4×4 *doubly stochastic matrix then* $\operatorname{per} \mathbf{A} \geqslant 4!/4^4 = 3/32$, *with equality if and only if* $\mathbf{A} = \frac{1}{4}\mathbf{J}$.

5. THE CASE $n = 5$

For this case we have not been able to obtain a suitable decomposition of per **A** as the sum of two functions, each of which assumes its minimum at the symmetric point, as for the case $n = 4$, and have no proof of the conjecture for $n = 5$. However, partial results, for example those in the following two Theorems, may be obtained by use of the same techniques.

THEOREM 5. *Let* $\mathbf{A} = (a_{ij})$ *be a* 5×5 *doubly stochastic matrix such that any one of the following conditions hold*

(i) $\max_{i,j} a_{ij} \leqslant \frac{1}{3}$;
(ii) *some row (or column)* $= (\frac{1}{5}\ \frac{1}{5}\ \frac{1}{5}\ \frac{1}{5}\ \frac{1}{5})$;
(iii) *the elements of the sum of any two columns (or rows) are* $\leqslant \frac{1}{2}$;
(iv) *any row (or column)* $= (\frac{1}{2}\ \frac{1}{2}\ 0\ 0\ 0)$.

Then

$$\text{per } \mathbf{A} \geqslant \frac{5!}{5^5} = \frac{24}{625}.$$

THEOREM 6. *If* $\mathbf{A} = (a_{ij})$ *is a* 5×5 *doubly stochastic matrix then*

$$\text{per } \mathbf{A} \geqslant \frac{22.4609375}{625}.$$

(Note the van der Waerden bound is 24/625.)

We prove case (i) of Theorem 5 for illustration, and omit the remaining proofs because they are tedious and introduce no new technique.

PROOF OF 5(i): We may write

$$\text{per } \mathbf{A} = 1 + \sum_{T_1(\mathbf{A})} (-e_2 + e_3 - e_4 + 2e_5)(\mathbf{x}) + \sum_{T_2(\mathbf{A})} (e_2 - e_3 + e_4 - 2e_5)(\mathbf{x}) - 10.$$

Then, using (3.1) and (3.2) for $n = 5$, we have

$$\text{per } \mathbf{A} = -.84 + \sum_{T_1(\mathbf{A})} f(\mathbf{x}) + \sum_{T_2(\mathbf{A})} g(\mathbf{x}),$$

where

$$f(\mathbf{x}) = (-.184e_2 + .456e_3 - e_4 + 2e_5)(\mathbf{x}),$$
$$g(\mathbf{x}) = (.184e_2 - .456e_3 + e_4 - 2e_5)(\mathbf{x}).$$

By Theorem 1, $g(\mathbf{x})$ assumes its minimum on $\{0 \leqslant x_i \leqslant 1, \sum x_i = 2\}$ uniquely at $\mathbf{x} = (\frac{2}{5}, \frac{2}{5}, \ldots, \frac{2}{5})$. Now consider $f(\mathbf{x})$, where

$$\mathbf{x} \in \left\{0 \leqslant x_i \leqslant \frac{1}{3}, \sum x_i = 1\right\};$$

or, equivalently, consider

$$\bar{f}(\mathbf{x}) = \left(-.184\frac{e_2}{9} + .456\frac{e_3}{27} - \frac{e_4}{81} + \frac{2e_5}{243}\right)(\mathbf{x}) \text{ over } \{0 \leqslant x_i \leqslant 1, \sum x_i = 3\}.$$

Again by Theorem 1, $\bar{f}(\mathbf{x})$ assumes its minimum for $\mathbf{x} \in \{0 \leqslant x_i \leqslant 1, \sum x_i = 3\}$ at the symmetry point. Hence when (i) holds the theorem follows.

6. A Special Case

A special class of doubly stochastic matrices defined in Theorem 7 is easily handled by our technique.

THEOREM 7. *Let* $\mathbf{A} = (a_{ij})$ *be an* $n \times n$ *doubly stochastic matrix having* $m \leqslant n$ *identical columns,* $= \mathbf{a}/m$, *and the remaining* $n - m$ *columns identical,* $= \mathbf{b}/n - m$. *Then*

$$\text{per } \mathbf{A} \geqslant n!/n^n, \textit{ with equality if and only if } \mathbf{A} = \frac{1}{n}\mathbf{J}.$$

PROOF: We have

$$A = (\mathbf{a}/m, \mathbf{a}/m, \ldots, \mathbf{a}/m, \mathbf{b}/n - m, \mathbf{b}/n - m, \ldots, \mathbf{b}/n - m)$$

where

$$\sum_{i=1}^{n} a_i = m \text{ and } \sum_{i=1}^{n} b_i = n - m, \qquad a_i + b_i = 1, \qquad i = 1, 2, \ldots, n.$$

It may be verified that (1.1) reduces to

$$\text{per } \mathbf{A} = \frac{(n-m)!}{(n-m)^{n-m}} \frac{m!}{m^m} \left[\left(e_m - \binom{m+1}{m} e_{m+1} + \binom{m+2}{m} e_{m+2} + \cdots + (-1)^{n-m} \binom{n}{m} e_n\right)(\mathbf{a})\right]. \tag{6.1}$$

We denote the function in the bracket by $\phi(\mathbf{a})$. By Theorem 1, $\phi(\mathbf{a})$ assumes its minimum among the points

$$\mathbf{x}(r, k) = \left(1, 1, \ldots, 1, \frac{m-r}{k}, \ldots, \frac{m-r}{k}, 0, 0, \ldots, 0\right)$$

where there are r ones, k of the $(m-r)/k$ components, and $(n-k-r)$ zeros. We have $0 \leqslant m - k \leqslant r \leqslant m \leqslant k + r \leqslant n$. From (6.1) we see

$$\phi(\mathbf{x}(r, k)) = \sum_{p=m}^{k+r} (-1)^{p-m} \binom{p}{m} e_p(\mathbf{x}). \tag{6.2}$$

We wish to evaluate ϕ at the critical points $\mathbf{x}(r, k)$. If $r = m$, or if $m - r = k$ then all components are one, and $e_m = 1$ is the only non-vanishing e_p for $m \leqslant p \leqslant k + r$. In this case $\phi = 1$. We assume that $r < m$, and $m - k < r$, and $0 \leqslant m - k < r < m \leqslant k + r \leqslant m$. To compute the $e_p(\mathbf{x})$, we may choose σ of the ones and $p - \sigma$ of the components $(m - r)/k$ for $\sigma = 0, 1,..., r$. Hence

$$e_p(\mathbf{x}(r, k)) = \sum_{\sigma=0}^{r} \binom{r}{\sigma}\binom{k}{p-\sigma}\left(\frac{m-r}{k}\right)^{p-\sigma}, \qquad m \leqslant p \leqslant k + r. \tag{6.3}$$

Note that $p - \sigma \geqslant m - r \geqslant 0$ but that $\binom{k}{p-\sigma}$ will vanish for $\sigma < p - k$. Substituting (6.3) into (6.2) we obtain

$$\phi(\mathbf{x}(r, k)) = \sum_{p=m}^{k+r} (-1)^{p-m} \binom{p}{m} \sum_{\sigma=0}^{r} \binom{r}{\sigma}\binom{k}{p-\sigma}\left(\frac{m-r}{k}\right)^{p-\sigma}.$$

We change the summation index to $\tau = p - \sigma$, and

$$\phi(\mathbf{x}(r, k)) = \sum_{\sigma=0}^{r} \sum_{\tau=m-r}^{k} (-1)^{-m+\tau+\sigma} \binom{\tau+\sigma}{m}\binom{r}{\sigma}\binom{k}{\tau}\left(\frac{m-r}{k}\right)^{\tau}.$$

We note that the extra terms introduced for $\tau < m$, and omitted for $\tau > k$, are all zero. But

$$\sum_{\sigma=0}^{r} (-1)^{\sigma} \binom{\tau+\sigma}{m}\binom{r}{\sigma} = (-1)^{r} \binom{\tau}{m-r}$$

and

$$\begin{aligned} \phi(\mathbf{x}(r, k)) &= \sum_{\tau=m-r}^{k} (-1)^{-m+\tau+r} \binom{\tau}{m-r}\binom{k}{\tau}\left(\frac{m-r}{k}\right)^{\tau} \\ &= \sum_{\tau=m-r}^{k} (-1)^{-m+\tau+r} \binom{k}{m-r}\binom{k-m+r}{\tau-m+r}\left(\frac{m-r}{k}\right)^{\tau} \\ &= \binom{k}{m-r}\left(\frac{m-r}{k}\right)^{m-r} \sum_{\tau=m-r}^{k} (-1)^{\tau-m+r} \\ &\quad \times \binom{k-m+r}{\tau-m+r}\left(\frac{m-r}{k}\right)^{\tau-m+r} \\ &= \binom{k}{m-r}\left(\frac{m-r}{k}\right)^{m-r}\left(1 - \frac{m-r}{k}\right)^{k-m+r}, \end{aligned}$$

and finally

$$\phi(\mathbf{x}(r, k)) = \frac{k!}{k^k} \frac{(k-m+r)^{k-m+r}}{(k-m+r)!} \frac{(m-r)^{m-r}}{(m-r)!}.$$

Recall we have $k > m - r > 0$. For a fixed r, this function is decreasing for increasing k. Hence

$$\phi(\mathbf{x}(r, k)) \geqslant \phi(\mathbf{x}(r, n - r)).$$

But we also have $\phi(\mathbf{x}(r, n - r)) \geqslant \phi(\mathbf{x}(0, n))$ and Theorem 7 follows.

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