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Properties of parallelotopes equivalent to Voronoi's conjecture

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Abstract

A parallelotope is a polytope whose translation copies fill space without gaps and intersections by interior points. Voronoi conjectured that each parallelotope is an affine image of the Dirichlet domain of a lattice, that is to say a Voronoi polytope. We give several properties of a parallelotope and prove that each of them is equivalent to it being an affine image of a Voronoi polytope.

1. Introduction

Let us have a decomposition of $\mathbf{R}^n$ into equal convex polytopes (*tiles*) such that the decomposition is simultaneously a covering and a packing and the intersection of any two polytopes is empty or a common face of each. Such a partition is called *tiling*. Let this tiling be invariant under a group $\mathcal{T}$ of translations, and the group $\mathcal{T}$ is transitive on polytopes of the tiling. Then each tile of such a partition is called *parallelotope*. Here the prefix *parallelo* emphasizes that each tile is a parallel translation of a prototile. (Following [6] we use the word *parallelotope* rather than *parallelohedron* which was used by Voronoi in [12]. Recall also that a *polyhedron* is a three-dimensional *polytope*.)

Voronoi in Section 8 of [12] defines a parallelohedron as follows. A polytope P with a group of translations $\mathcal{T}$ is called a parallelohedron if the space $\mathbf{R}^n$ can be filled by non-overlapping congruent copies of P using translations taken from $\mathcal{T}$.

Each parallelotope necessarily satisfies the following three conditions:

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(i) a parallelotope is centrally symmetric;
(ii) each facet (=($n-1$)-dimensional face) of a parallelotope is centrally symmetric;
(iii) for $n > 1$, the projection of a parallelotope along any ($n-2$)-dimensional face is either a parallelogram or a centrally symmetric hexagon.

The edges of the polygon of item (iii) above are projections of four or six facets of the projected parallelotope P. These facets form a *belt* of the parallelotope P. Hence the property (iii) of a parallelotope P has another formulation. Namely,

(iii)′ for $n > 1$, each belt of a parallelotope contains four or six facets.

Venkov [11] (and independently McMullen [8]) proved that the above three conditions are sufficient for a polytope to be a parallelotope. Aleksandrov [1], knowing Venkov's result, simplified the proof of Venkov.

There is a special well known case of a parallelotope, namely, the Voronoi polytope related to a point of a lattice L. The Voronoi polytope $P_V(t_0)$ related to a point $t_0 \in L$ is the set of points of $\mathbf{R}^n$ which are at least as close to t_0 as to any other point of L.

The main conjecture of Voronoi is that any parallelotope P can be mapped into a Voronoi polytope P_V under an affine transformation $x \to Ax$ of the space containing P. Here A is a non-degenerate $n \times n$ matrix, where n is the dimension of the space.

Call a k-face (=k-dimensional face) of a parallelotope *primitive* if it belongs to the minimal possible number $n-k+1$ of parallelotopes of its tiling. Obviously any facet of a parallelotope is primitive. According to Zhitomirskii [13], a parallelotope is called *k-primitive* if each of its k-face are primitive. Besides, the k-primitivity implies the $(k+1)$-primitivity. A 0-primitive parallelotope is simply called *primitive*. Obviously, any parallelotope is $(n-1)$-primitive.

Voronoi proved his conjecture for primitive parallelotopes. If a parallelotope is primitive, then each its belts contain six facets, but not conversely. On the other hand, if each belt consists of six facets, then the parallelotope is $(n-2)$-primitive. This implies that each $(n-2)$-face belongs to three parallelotopes. Zhitomirskii [13] extend the result of Voronoi over $(n-2)$-primitive parallelotopes.

McMullen [7] proved that a parallelotope which is a zonotope is combinatorially equivalent to a Voronoi polytope. Later Erdahl [3] completed the result of McMullen proving that a zonotopal parallelotope is affinely equivalent to a Voronoi polytope. Delaunay [5] proved Voronoi's conjecture in complete generality for the dimensions $n \le 4$.

We give here several conditions on a parallelotope P each of which is equivalent to: the Voronoi's conjecture is true for P.

2. Parallelotopes

Now we consider a description of a parallelotope $P = P(0)$ with the center in origin by a system of linear inequalities. We denote by A^{T} the transpose of a matrix A and by $x^{\mathrm{T}}y = y^{\mathrm{T}}x$ the scalar product of two column vectors $x, y \in \mathbf{R}^n$, and set $x^{\mathrm{T}}x = x^2$.

Being a convex polytope a parallelotope is described by a system of linear inequalities $\{q_i^{\mathrm{T}}x \le \alpha_i\}$. Since, by (i) of the Introduction, P is centrally symmetric, each facet F_i of P has the opposite facet $-F_i$. If F_i lies in the affine hyperplane given by the equality $q_i^{\mathrm{T}}x = \alpha_i$, then the opposite facet $-F_i$ lies in the affine hyperplane $q_i^{\mathrm{T}}x = -\alpha_i$. By the facet F_i, the parallelotope P is adjacent to a parallelotope $P(t_i)$ which is a parallel shift of P by the translation vector $t_i \in \mathcal{T}$.

Let $\mathcal{I}_P$ be the set of indices of pairs of opposite facets of P. Then the set $\{t_i : i \in \mathcal{I}_P\}$ of translation vectors generates the translation group $\mathcal{T}$ and a lattice L. The points of L, i.e. the centers of parallelotopes of the tiling, are the end-points of lattice vectors. We can identify each lattice vector with an element $t \in \mathcal{T}$. By this identification the origin is the zero point and simultaneously the zero lattice vector 0 of L.

Obviously, the point $\frac{1}{2}t_i$ is the center of the facet F_i. Hence $\frac{1}{2}q_i^{\mathrm{T}}t_i = \alpha_i$. We have

$$P(0) = \{x \in \mathbf{R}^n : -\tfrac{1}{2}q_i^{\mathrm{T}}t_i \le q_i^{\mathrm{T}}x \le \tfrac{1}{2}q_i^{\mathrm{T}}t_i,\ i \in \mathcal{I}_P\}. \qquad (1)$$

Here the *facet vectors* q_i are determined only up to a non-zero multiple β_i. We say that the facet vectors q_i and the lattice vectors t_i, $i \in \mathcal{I}_P$, giving the description (1) of a parallelotope P, are *associated*.

The parallelotope $P(t)$ with the center in the point $t \in L$ is a translation of $P(0)$ by the vector t, and therefore it is described as follows:

$$P(t) = \{x \in \mathbf{R}^n : -\tfrac{1}{2}q_i^{\mathrm{T}}t_i \le q_i^{\mathrm{T}}(x-t) \le \tfrac{1}{2}q_i^{\mathrm{T}}t_i,\ i \in \mathcal{I}_P\}. \qquad (2)$$

Note that, by definition of a Voronoi polytope, a facet F_i of the Voronoi polytope $P_V(0)$ is orthogonal to the lattice vector t_i and bisects it. The lattice vector t_i connects the centers of $P_V(0)$ and $P_V(t_i)$, where $P_V(t_i)$ is the Voronoi polytope adjacent to $P_V(0)$ by the facet F_i. In other words, we can set $q_i = t_i$ in the descriptions (1) and (2) of parallelotopes $P(0)$ and $P(t)$ in the case they are Voronoi polytopes.

3. Linear transforms of parallelotopes

Note that the usual Euclidean norm x^2 is used in the definition of the Voronoi polytope $P_V(0)$. But we can use an arbitrary positive quadratic form $f(x) = x^{\mathrm{T}}Dx$ as a norm of x. Here D is a symmetric positive definite $n \times n$ matrix. Then the above definition gives a parallelotope P_f. Call such a parallelotope the *Voronoi polytope with respect to the quadratic form* $f(x)$. Such a parallelotope relates to a lattice L (or to a translation group $\mathcal{T}$). Consider the Voronoi polytope P_f with respect to the quadratic form $x^{\mathrm{T}}Dx$ in detail. By definition, we have

$$P_f(0) = \{x \in \mathbf{R}^n : x^{\mathrm{T}}Dx \le (x-t)^{\mathrm{T}}D(x-t) \text{ for all } t \in \mathcal{T}\}.$$

It is well known, that a finite set $\{\pm t_i : i \in \mathcal{I}_f\}$ of vectors $t_i \in \mathcal{T}$ is sufficient for the description of $P_f(0)$.

Using the symmetry of D and joining the inequalities for t_i and $-t_i$, we simplify this as follows:

$$P_f(0) = \{x \in \mathbf{R}^n : -\tfrac{1}{2}t_i^{\mathrm{T}}Dt_i \le t_i^{\mathrm{T}}Dx \le \tfrac{1}{2}t_i^{\mathrm{T}}Dt_i,\ i \in \mathcal{I}_f\}. \qquad (3)$$

For $D = I_n$, where I_n is the identity matrix, we have $f(x) = x^2$ and $P_f(0) = P_V(0)$.

Lemma 1. *Let P be a parallelotope given by* (1). *Let A be an $n \times n$ non-degenerate matrix, and $D = A^{\mathrm{T}}A$. The following assertions are equivalent:*

(i) *the affine transformation* $x \to Ax$ *transforms* P *into a Voronoi polytope;*
(ii) P *is the Voronoi polytope with respect to the quadratic form* $f(x) = x^{\mathrm{T}}Dx$*;*
(iii) *the facet vectors* q_i *satisfy the equality* $q_i = Dt_i$, $i \in \mathcal{I}_P$.

Proof. (i) $\Rightarrow$ (iii). Consider the affine transformation $x \to Ax$. The new facet vector has the form $(A^{\mathrm{T}})^{-1}q$. In fact, for $x \in F$, the point Ax belongs to a facet of AP. Hence

$$((A^{\mathrm{T}})^{-1}q)^{\mathrm{T}}Ax = q^{\mathrm{T}}((A^{\mathrm{T}})^{-1})^{\mathrm{T}}Ax = q^{\mathrm{T}}A^{-1}Ax = q^{\mathrm{T}}x = 0.$$

The new center of the transformed facet is $\frac{1}{2}At_i$. For AP to be a Voronoi polytope, we have to have

$$(A^{\mathrm{T}})^{-1}q_i = At_i, \qquad \text{i.e. } q_i = A^{\mathrm{T}}At_i, \qquad \text{i.e. } q_i = Dt_i.$$

(ii) $\Rightarrow$ (i) The positive definite matrix D can be represented as the product $D = A^{\mathrm{T}}A$, where the matrix A is non-degenerate. Hence the form $x^{\mathrm{T}}Dx = x^{\mathrm{T}}A^{\mathrm{T}}Ax = (Ax)^{\mathrm{T}}(Ax)$ is the quadratic form $(Ax)^2$ in the transformed space, i.e. $P = P_f = P_{(Ax)^2}$. Let $y = Ax$. Then $x = A^{-1}y$ and $P = P_{(Ax)^2} = A^{-1}P_{y^2} = A^{-1}P_V$. Hence $AP = P_V$.

(iii) $\Rightarrow$ (ii) If we set in (1) $q_i = Dt_i$, we obtain description (3) of a parallelotope. Hence the Voronoi polytope with respect to a quadratic form is also a special case of a parallelotope, when $q_i = Dt_i$. In other words, in this case, the parallelotope $P(0)$ of (1) is $P_f(0)$ for $f(x) = x^{\mathrm{T}}Dx$. $\square$

4. A canonical representation of a parallelotope

Consider a vertex v of a facet F_i. Let v be the intersection of facets F_j, $j \in \mathcal{I}(v)$. Then $i \in \mathcal{I}(v)$ and

$$q_j^{\mathrm{T}}v = \tfrac{1}{2}q_j^{\mathrm{T}}t_j, \qquad j \in \mathcal{I}(v).$$

Since each facet of P is centrally symmetric, there is a symmetric to v vertex $v^s \in F_i$. We have

$$q_k^{\mathrm{T}}v^s = \tfrac{1}{2}q_k^{\mathrm{T}}t_k, \qquad k \in \mathcal{I}(v^s).$$

Note that the point $\frac{1}{2}(v + v^s)$ is the center $\frac{1}{2}t_i$ of the facet F_i. Hence

$$v + v^s = t_i, \qquad \text{i.e. } v^s = t_i - v.$$

Recall that there are the following two types of belts in the parallelotope P:

(1) 3-belts containing six facets $\pm F_i, \pm F_j, \pm F_k$;
(2) 2-belts containing four facets $\pm F_i, \pm F_j$.

We denote each belt by the set of indices of its generating facets. So, we have the following two types of belts: $\{i, j, k\}$ and $\{i, j\}$.

Therefore some facet vectors of the pair of the sets $\{q_j : j \in \mathcal{I}(v)\}$ and $\{q_k : k \in \mathcal{I}(v^s)\}$ are joined into pairs of two types such that

(1) either the facets F_i, F_j, F_k belong to the 3-belt $\{i, j, k\}$;
(2) or the facets F_j and F_k are opposite, i.e. $F_k = -F_j$, and belong to the 2-belt $\{i, j\}$.

Let $\mathcal{B}$ be the set of all 3-belts. Consider a belt $\{i, j, k\} \in \mathcal{B}$. The facets vectors q_i, q_j, q_k lie in a two-dimensional plane, which is orthogonal to $(n-2)$-faces of the belt. Hence they are linearly dependent. Obviously, the associated lattice vectors t_i, t_j, t_k are also linearly dependent. Moreover, this dependence has the following form:

$$t_i - \varepsilon_j t_j - \varepsilon_k t_k = 0, \qquad \{i, j, k\} \in \mathcal{B}, \tag{4}$$

where $\varepsilon_j, \varepsilon_k \in \{\pm 1\}$. Since each facet vector q_i is defined up to a scalar multiplier β_i, we can choose lengths of the associated facet vectors such that the new facet vector $\beta_i q_i$ satisfies the equality similar to (4)

$$\beta_i q_i - \varepsilon_j \beta_j q_j - \varepsilon_k \beta_k q_k = 0, \qquad \{i, j, k\} \in \mathcal{B}. \tag{5}$$

Following [12] and [10], we say that, for the belt $\{i, j, k\}$, the facet vectors q_i, q_j, q_k are defined *canonically with respect to the 3-belt* $\{i, j, k\}$ if they satisfy the same equality as the associated lattice vectors t_i, t_j, t_k.

Definition. A parallelotope P *is defined canonically* by (1) if the facet vectors q_i, $i \in \mathcal{I}_P$, are defined canonically simultaneously with respect to all belts of P.

In other words, a parallelotope P *is defined canonically* if the system of Eq. (5) determining multipliers β_i, $i \in \mathcal{I}_P$, has a non-zero solution.

Voronoi proves in [12] that a primitive parallelotope can be defined canonically.

5. Relations between the lattice and facet vectors

We suppose that the facet vectors q_i, q_j and q_k determine the facets F_i, F_j and F_k, respectively. Hence the vector $\varepsilon_j q_j$ defines the facet $\varepsilon_j F_j$.

In Proposition 1 below, for the sake of simplicity, we suppose that $\varepsilon_j = \varepsilon_k = 1$. To apply the results below to the general dependencies (4) and (5), it is sufficient to change q_j and q_k by $\varepsilon_j q_j$ and $\varepsilon_k q_k$, respectively.

So the lattice vecors, corresponding to the belt $\{i, j, k\}$, satisfy the equality

$$t_i = t_j + t_k. \tag{6}$$

Hence the defined canonically facet vectors satisfy the equality

$$q_i = q_j + q_k \tag{7}$$

and the intersections $F_i \cap F_j$ and $F_i \cap F_k$ are non-empty and define two opposite facets of F_i.

For $i \in \mathcal{I}_P$, let $\mathcal{I}_i = \{j \in \mathcal{I}_P : F_j \cap F_i$ is an $(n-2)$-face of $P\}$ (which is a facet of F_i). Let $\mathcal{B}_i$ be the set of 3-belts containing the facet F_i. Now, using the property (ii) of parallelotopes, we prove an important fact.

Proposition 1. *For all $j \in \mathcal{I}_i$, the vectors q_j can be defined canonically with respect to all 3-belts of $\mathcal{B}_i$. For these canonical facet vectors we have*

(1) *for a 3-belt* $\{i, j, k\} \in \mathcal{B}_i$ *the following equalities hold:*

$$q_i^{\mathrm{T}} t_j = q_j^{\mathrm{T}} t_i, \qquad q_i^{\mathrm{T}} t_k = q_k^{\mathrm{T}} t_i, \qquad q_j^{\mathrm{T}} t_k = q_k^{\mathrm{T}} t_j;$$

(2) *for a 2-belt* $\{i, j\}$, $j \in \mathcal{I}_i$ *the following equalities hold:*

$$q_j^{\mathrm{T}} t_i = q_i^{\mathrm{T}} t_j = 0.$$

Proof. Obviously, the vectors q_j, $j \in \mathcal{I}_i$, can be defined canonically with respect to all 3-belts of $\mathcal{B}_i$, since the belts of $\mathcal{B}_i$ have only one common vector q_i.

Recall that opposite vertices v and v^s of the facet F_i are determined by facet vectors some of which form belts with q_i. Consider the equations

$$q_j^{\mathrm{T}} v = \tfrac{1}{2} q_j^{\mathrm{T}} t_j, \qquad q_k^{\mathrm{T}} v^s = \tfrac{1}{2} q_k^{\mathrm{T}} t_k,$$

related either to a 3-belt $\{i, j, k\}$ or to a 2-belt $\{i, j\}$, and then $q_k = -q_j$.

For the case (1), substituting the expressions $q_k = q_i - q_j$, $t_k = t_i - t_j$, $v^s = t_i - v$, in the second equation, and using the equalities $q_i^{\mathrm{T}} v = \frac{1}{2} q_i^{\mathrm{T}} t_i$, $q_j^{\mathrm{T}} v = \frac{1}{2} q_j^{\mathrm{T}} t_j$, we obtain

$$\begin{aligned} q_k^{\mathrm{T}} v^s = \ & \tfrac{1}{2} q_k^{\mathrm{T}} t_k \Rightarrow (q_i - q_j)^{\mathrm{T}} (t_i - v) = \tfrac{1}{2} (q_i - q_j)^{\mathrm{T}} (t_i - t_j) \\ & \Rightarrow q_i^{\mathrm{T}} (t_i - v - \tfrac{1}{2} t_i + \tfrac{1}{2} t_j) = q_j^{\mathrm{T}} (t_i - v - \tfrac{1}{2} t_i + \tfrac{1}{2} t_j) \Rightarrow q_i^{\mathrm{T}} t_j = q_j^{\mathrm{T}} t_i. \end{aligned}$$

Similarly, beginning with $q_j^{\mathrm{T}} v = \frac{1}{2} q_j^{\mathrm{T}} t_j$ and using the equality $v = t_i - v^s$, we obtain the equality $q_i^{\mathrm{T}} t_k = q_k^{\mathrm{T}} t_i$. Now, this equality implies

$$(q_j + q_k)^{\mathrm{T}} t_k = q_k^{\mathrm{T}} (t_j + t_k) \Rightarrow q_j^{\mathrm{T}} t_k = q_k^{\mathrm{T}} t_j.$$

In the case (2) we have $q_k = -q_j$, $t_k = -t_j$. Hence we obtain

$$q_k^{\mathrm{T}} v^s = \tfrac{1}{2} q_k^{\mathrm{T}} t_k \Rightarrow q_j^{\mathrm{T}} (v - t_i) = \tfrac{1}{2} q_j^{\mathrm{T}} t_j \Rightarrow q_j^{\mathrm{T}} (\tfrac{1}{2} t_j - t_i) = \tfrac{1}{2} q_j^{\mathrm{T}} t_j \Rightarrow q_j^{\mathrm{T}} t_i = 0.$$

Using the facet F_j instead of F_i, we obtain the equality $q_i^{\mathrm{T}} t_j = 0$. $\square$

Note that the equalities $q_j^{\mathrm{T}} t_i = 0 = q_i^{\mathrm{T}} t_j$ do not depend on whether q_i and q_j are canonical or not.

Let $|\mathcal{I}_P| = m$ and let Q and T be $n \times m$ matrices whose columns are the vectors q_i and t_i for $i \in \mathcal{I}_P$, respectively. Then the product $q_i^{\mathrm{T}} t_j$ is the (ij)th element of the matrix product $Q^{\mathrm{T}} T$. If the equalities

$$q_i^{\mathrm{T}} t_j = t_i^{\mathrm{T}} q_j \text{ hold for all pairs } i, j \in \mathcal{I}_P, \tag{8}$$

then the $m \times m$ matrix $Q^{\mathrm{T}} T$ is symmetric, i.e. $Q^{\mathrm{T}} T = (Q^{\mathrm{T}} T)^{\mathrm{T}} = T^{\mathrm{T}} Q$.

Lemma 2. *The following assertions are equivalent:*

(i) *the equalities* $q_i^{\mathrm{T}} t_j = t_i^{\mathrm{T}} q_j$ *hold for all pairs* $i, j \in \mathcal{I}_P$;

(ii) *there is a unique symmetric non-degenerate* $n \times n$ *matrix* D *such that* $q_i = D t_i$ *for all* $i \in \mathcal{I}_P$.

Proof. (i) $\Rightarrow$ (ii) Let $\mathcal{I}_b \subseteq \mathcal{I}_P$ be an n-subset of $\mathcal{I}_P$ such that the set $\{t_i : i \in \mathcal{I}_b\}$ is linearly independent. Let T_b and Q_b be the submatrices of T and Q composed by column vectors t_i and q_i for $i \in \mathcal{I}_b$, respectively. Then T_b is an $n \times n$ nondegenerate matrix. If (8) is true, then it implies the equality $T_b^{\mathrm{T}} Q = Q_b^{\mathrm{T}} T$. This equality is equivalent to the equality

$$Q = DT,$$

where $D = (T_b^{\mathrm{T}})^{-1} Q_b^{\mathrm{T}} = (Q_b T_b^{-1})^{\mathrm{T}}$. The matrix D is symmetric. In fact, take a restriction of the equality $Q = DT$ onto the columns t_i, q_i for $i \in \mathcal{I}_b$. The restriction is $Q_b = DT_b$, i.e. $Q_b T_b^{-1} = D = (Q_b T_b^{-1})^{\mathrm{T}}$. The matrix D does not depend on a choice of T_b. In fact, if there is another symmetric matrix D' such that $Q = D'T$, then $D = Q_b T_b^{-1} = D' T_b T_b^{-1} = D'$.

(ii) $\Rightarrow$ (i) Conversely, let $q_i = Dt_i$. Then $q_i^{\mathrm{T}} t_j = (Dt_i)^{\mathrm{T}} t_j = t_i^{\mathrm{T}} D t_j = t_i^{\mathrm{T}} q_j$. $\square$

6. Graphs related to tilings

Recall that the centers of parallelotopes $P(t)$ form a lattice L. Consider the points of L (i.e. the endpoints of lattice vectors) as vertices of a graph G_L. Two vertices $t, t' \in L$ are adjacent in G_L if and only if $t - t' \in \{\pm t_i : i \in \mathcal{I}_P\}$. We can consider G_L as a directed graph, where the direction of the edge t_i is the direction of the vector t_i. Hence edges of G_L are the vectors $\pm t_i$, $i \in \mathcal{I}_P$. Therefore the set of all edges of G_L is partitioned into $m = |\mathcal{I}_P|$ classes E_i. We suppose that all edges of E_i are vectors t_i (with the same sign, say $+$). In other words, all edges of E_i are obtained from one by translations. For each $i \in \mathcal{I}_P$, a vertex of G_L is incident to two edges t_i, one of which goes out and another comes in the vertex.

For any collection of integers $\{z_i : i \in \mathcal{I}_P\}$, we set $t(z) = \sum_{i \in \mathcal{I}_P} z_i t_i$ and $q(z) = \sum_{i \in \mathcal{I}_P} z_i q_i$. Let $\mathcal{I}(z) = \{i \in \mathcal{I}_P : z_i \neq 0\}$ be the *support* of $z = \{z_i : i \in \mathcal{I}_P\}$.

Any two vertices t^0 and t are connected in G_L by an oriented path $\mathcal{P}$ directed from t^0 to t. We denote such a path as a sequence $\mathcal{P} = (t_1, t_2, \ldots, t_s)$ of vectors $t_k \in \{\pm t_i : i \in \mathcal{I}_P\}$ corresponding to edges of the path in the natural order. Here $t_k = t_i$ or $t_k = -t_i$ according to the directions of the path and the corresponding edge coincide or not, respectively. Then, obviously, $t = t^0 + \sum_{k=1}^{s} t_k$. In particular, if the path is closed, i.e. it is a circuit and $t = t^0$, then $\sum_{k=1}^{s} t_k = 0$. We rewrite the sum $\sum_{k=1}^{s} t_k$ as $\sum_{i \in \mathcal{I}_P} z_i(\mathcal{P}) t_i = t(z(\mathcal{P})) = t(\mathcal{P})$. So, $t = t^0 + t(\mathcal{P})$. We set $\mathcal{I}(\mathcal{P}) = \mathcal{I}(z(\mathcal{P}))$. Note that there are many paths with the same collection $z(\mathcal{P})$. All of them are obtained from $\mathcal{P}$ by permutations of the edges t_k.

Since the set $\{\pm t_i : i \in \mathcal{I}_P\}$ generates the lattice L, any lattice vector t has a (non-unique) representation $t = t(\mathcal{P})$, where $\mathcal{P}$ is a path in G_L connecting 0 with t. We associate the vector $q(\mathcal{P}) = \sum_{t_k \in \mathcal{P}} q_k$ to the vector $t(\mathcal{P})$. We call the vector $q(\mathcal{P})$ the *vector associated* to the vector $t(\mathcal{P})$. If the equalities $q_i^{\mathrm{T}} t_j = q_j^{\mathrm{T}} t_i$ hold for all $i, j \in \mathcal{I}_P$, then using Lemma 2 we see that $q(\mathcal{P}) = Dt(\mathcal{P})$ does not depend on $\mathcal{P}$. The uniqueness of q in this case follows from the fact that the equality $\sum_k t_k = 0$ implies the equality $\sum_k q_k = 0$.

Consider some important subgraphs of the graph G_L. Let $t \in L$ and let $G(t)$ be the graph induced by all vertices $t' \in L$ adjacent to t. The union of t and $G(t)$ is the *suspension* $\nabla G(t)$. In the graph $\nabla G(t)$, the vertex t is adjacent to all vertices of $G(t)$.

For $t = 0$, the vertices of $G(0)$ are endpoints of the vectors $\pm t_i$, $i \in \mathcal{I}_P$. In other words, the graph $G(0)$ is determined on two copies of the set $\mathcal{I}_P$. In fact, any edge t_i of $G(0)$

belongs to a triangle $\triangle = (t_i, t_j, t_k)$ of $\nabla G(0)$ such that $t_i - \varepsilon_j t_j - \varepsilon_k t_k = 0$. This triangle corresponds to the belt $\{i, j, k\}$ of $P(0)$. Moreover, there are six such triangles in $\nabla G(0)$. These six triangles form a hexagon spanning a two-dimensional plane.

Let $G(F^k)$ be the graph induced by the centers of all parallelotopes having with $P(0)$ a common k-face F^k. So, the graph $G(F^{n-1})$ is an edge t_i with end-vertices corresponding to adjacent parallelotopes $P(0)$ and $P(t_i)$. There are only two types of graphs $G(F^{n-2})$, namely, triangles and quadrangles, according to two types of belts with six and four facets, respectively.

If P is m-primitive, then, for $m \le k \le n$, $G(F^k) = K_{n-k+1}$, where K_s is the complete graph on s vertices.

The following reformulation of canonical definity of a parallelotope P is obvious.

Lemma 3. *For a parallelotope P, the following assertions are equivalent:*

(i) *P is defined canonically;*
(ii) *$q(\triangle) = 0$ for all triangles $\triangle \subset G_L$ such that $\triangle = G(F^{n-2})$.* $\square$

Call a 4-circuit $(t_i, t_j, -t_i, -t_j)$ by a *quadrangle* $\mathcal{Q}_{ij}$. It is a parallelogram and spans a two-dimensional plane. Among quadrangles of G_L there are quadrangles $G(F^{n-2})$, where F^{n-2} is an $(n-2)$-face which is common to four parallelotopes. Obviously, for each quadrangle, we have trivially $q(\mathcal{Q}_{ij}) = q_i + q_j - q_i - q_j = 0$.

The technique used in [11] can be applied for a proof of Proposition 2 below (see also Theorem 1 of [10]).

Proposition 2. *Any circuit of G_L can be represented as a sum modulo 2 of circuits of type $G(F^{n-2})$.*

Hence Lemma 3 and Proposition 2 imply the following lemma.

Lemma 4. *For a parallelotope P, the following assertions are equivalent:*

(i) *P is defined canonically;*
(ii) *$q(\mathcal{C}) = 0$ for all circuits $\mathcal{C} \subset G_L$.*

But we give here an explicit proof of a weaker result which we will use later.

Lemma 5. *Any quadrangle $\mathcal{Q}_{ij}$ can be represented as a sum modulo 2 of triangles and quadrangles, both of type $G(F^{n-2})$.*

Proof. We span a two-dimensional surface S on the quadrangle $\mathcal{Q}_{ij}$ as follows. The four edges $t_i, t_j, -t_i$ and $-t_j$ form the boundary of S. Recall that the vertices of the quadrangle $\mathcal{Q}_{ij}$ are centers of four parallelotopes, say $P(0)$, $P(t_i)$, $P(t_j)$ and $P(t_i + t_j)$. Hence the surface S intersects a number of parallelotopes. We choose S such that it intersects boundaries of these parallelotopes only by facets and $(n-2)$-faces, and these intersections are transversal. Hence if the intersection $S \cap F$ is not empty, then it is a segment or a point depending on whether F is a facet or an $(n-2)$-face, respectively.

These segments and points form a planar graph Γ drawn on S. This graph Γ has four half-edges corresponding to the four facets intersected by the four edges of $\mathcal{Q}_{ij}$. Vertices of Γ have degrees 3 and 4 only. The dual of Γ is a planar subgraph $G(\Gamma)$ of the graph G_L. Minimal circuits of $G(\Gamma)$ are just triangles and quadrangles of type $G(F^{n-2})$. Since $G(\Gamma)$

is planar, each edge of it (excluding the four boundary edges) belongs to two minimal circuits. So we obtain the wanted representation of the quadrangle $\mathcal{Q}_{ij}$ as a sum of graphs $G(F^{n-2})$ modulo 2. □

7. Pegged tilings and their duals

McMullen in [9] defines a pegged tiling as follows:

A tiling $\{Q(t) : t \in \mathcal{T}\}$ is called *pegged* if with each tile $Q(t)$, $t \in \mathcal{T}$, is associated a point $v^*(t) \in \mathbf{R}^n$, the *peg* of $Q(t)$, such that if the tile $Q(t')$ is adjacent to $Q(t)$, and so meets it on a facet F, then $v^*(t') - v^*(t)$ is an outer normal vector to $Q(t)$ at the facet F. The equation $x^{\mathrm{T}}(v^*(t') - v^*(t)) = \alpha(t, t')$ defines the hyperplane supporting F.

Note that the pegs are defined up to a shift on an arbitrary vector. Hence we can suppose that $v^*(t^0) = 0$ for some $t^0 \in \mathcal{T}$.

Recall that parallelotopes form a tiling $\{P(t) : t \in L\}$. Suppose that this tiling is pegged. Then the peg $v^*(t)$ relates to the vertex t of the graph G_L. In particular, the pegs $v^*(0) = 0$, $v^*(\pm t_i)$ relate to the vertices $0, \pm t_i$, $i \in \mathcal{I}_P$, of the graph $\nabla G(0)$. Since, by definition of pegs, $v^*(t_i) - v^*(0) = v^*(t_i)$ are proportional to q_i, and since the vectors q_i are defined up to a scalar multiple, we set $q_i = v^*(t_i)$ if the tiling $\{P(t) : t \in L\}$ is pegged.

By definition of the lattice vector t_i, the tile $P(t)$ is adjacent to the tile $P(t + t_i)$ by the facet F_i which is orthogonal to the facet vector q_i. Hence $v^*(t + t_i) - v^*(t) = \beta_i(t)q_i$ for some scalar $\beta_i(t) > 0$, where $\beta_i(0) = 1$.

In Lemma 6 below, we show that such defined facet vectors give a canonical representation of the parallelotope $P(0)$.

Lemma 6. *The following assertions are equivalent*

(i) *the tiling $\{P(t); t \in L\}$ is pegged;*
(ii) *the parallelotope $P = P(0)$ is defined canonically.*

Proof. (i) $\Rightarrow$ (ii) Let $\{t_i, t_j, t_k\}$ be a 3-belt such that $t_i = t_j + t_k$. We show that $q_i = q_j + q_k$. Consider the hexagon of $\nabla G(0)$ corresponding to this belt. The vertices of the hexagon are $0, \pm t_i, \pm t_j$ and $\pm t_k$. The corresponding pegs are 0, $v^*(t_r) = q_r$ and $v^*(-t_r)$, $r \in \{i, j, k\}$. All these pegs lie in the 2-plane spanned by the facet vectors q_i, q_j, q_k.

Consider the quadrangle with vertices $0 = v^*(0)$, $v^*(t_j)$, $v^*(t_k)$ and $v^*(t_j + t_k) = v^*(t_i)$. The pairs of opposite edges of this quadrangle are $(v^*(t_j) - v^*(0), v^*(t_i) - v^*(t_k))$ and $(v^*(t_k) - v^*(0), v^*(t_i) - v^*(t_j))$. They are parallel to the vectors q_j and q_k, respectively. Hence this quadrangle is a parallelogram. We have

$$q_i - q_k = v^*(t_i) - v^*(t_k) = v^*(t_j + t_k) - v^*(t_k) = v^*(t_j) - v^*(0) = v^*(t_j) = q_j.$$

So, we obtain the wanted equality $q_i = q_j + q_k$. Since a similar reasoning is true for every 3-belt, we see that the parallelotope $P(0)$ is defined canonically.

(ii) $\Rightarrow$ (i) Let $\mathcal{P} = \{t_0, t_1, \ldots, t_s\}$ be a path connecting the point $t = t_s$ with origin $t_0 = 0$. Lemma 4 implies that $q(\mathcal{P})$ does not depend on the path $\mathcal{P}$, i.e. $q(\mathcal{P}) = q(t)$. It is easy to verify that the points $q(t)$ are pegs of the tiling $\{P(t) : t \in L\}$, i.e. $v^*(t) = q(t)$. □

For a pegged tiling $\mathcal{Q} = \{Q(t) : t \in \mathcal{T}\}$, it is natural to determine a tiling $\mathcal{Q}^* = \{Q^*(t) : t \in \mathcal{T}^*\}$ which is combinatorially and topologically dual to the tiling

$\mathcal{Q}^*$ (see [9, 10]). The combinatorial duality means that, for $0 \le k \le n$, there is a one-to-one correspondence between k-faces of $\mathcal{Q}$ and $(n-k)$-faces of $\mathcal{Q}^*$. The topological duality means that the affine spaces spanning the corresponding k-face of $\mathcal{Q}$ and $(n-k)$-face of $\mathcal{Q}^*$ are orthogonal.

So, the peg $v^*(t)$ is a vertex of the tiling $\mathcal{Q}^*$. The convex hull of all pegs $v^*(t)$ corresponding to tiles $Q(t) \in \mathcal{Q}$ having a fixed common vertex v is a tile $Q^*(v)$ of $\mathcal{Q}^*$, and each tile of $\mathcal{Q}^*$ is obtained in this way. It is proved in [9] (see Theorem 3.1) that $\mathcal{Q}^*$ is a tiling. Obviously, the dual tiling $\mathcal{Q}^*$ is pegged with pegs which are vertices of the tiling $\mathcal{Q}$. Moreover, we have $(\mathcal{Q}^*)^* = \mathcal{Q}$.

If $\mathcal{Q}$ is a pegged tiling by parallelotopes then the tiles of the dual tiling $\mathcal{Q}^*$ are called *Delaunay polytopes*. In [10], a k-face of the dual tiling is called the dual convex polytope $D^k(st)$.

Since the mutual dual tilings $\mathcal{Q}$ and $\mathcal{Q}^*$ are equivalent, we have the following obvious assertion.

Lemma 7. *The following assertions are equivalent*

(i) *a tiling $\mathcal{Q}$ is pegged;*
(ii) *a tiling $\mathcal{Q}$ has the dual tiling $\mathcal{Q}^*$.*

8. Generatrissa of a tiling

For a tiling $\{P(t) : t \in L\}$, whose tiles are parallelotopes, Voronoi [12] defines on the space $\mathbf{R}^n \otimes L$ a function $l(x;t)$. He calls this function *generatrissa* and defines it as follows:

(i) $l(x;0) = 0$;
(ii) if $P(t')$ is adjacent to $P(t)$ by the facet F_i defined by the equation $q_i^{\mathrm{T}} x = \alpha_i$, then

$$l(x;t') = l(x;t) + q_i^{\mathrm{T}} x - \alpha_i. \tag{9}$$

(Recall that, by (2), $q_i^{\mathrm{T}} x \le \alpha_i = q_i^{\mathrm{T}}(t + \frac{1}{2}t_i)$ for $x \in P(t)$.) In fact, Voronoi uses vectors $-q_i$, $i \in \mathcal{I}_P$, and therefore defines $-l(x;t)$. Voronoi proves that for primitive canonically defined parallelotopes the conditions (i) and (ii) above determine uniquely $l(x;t)$ for each $t \in L$. The obtained generatrissa has the following property:

$$l(x;t^0) \ge l(x;t) \text{ for all } x \in P(t^0) \text{ and all } t^0, t \in L, \tag{10}$$

with strict inequality if $x \in \mathrm{int} P(t^0)$.

This property implies that the function

$$l(x) = \max_{t \in L} l(x;t)$$

is a convex piecewise affine function on $\mathbf{R}^n$.

Consider in $(n+1)$-dimensional space $\mathbf{R}^n \oplus \mathbf{R}$ a convex surface defined as $\{(x,z) : x \in \mathbf{R}^n, z = l(x)\}$. The main property of this surface is that its projection in the space containing the tiling $\{P(t) : t \in L\}$ is this tiling.

But the above definition of generatrissa works for any pegged locally finite tiling (see, for example, [9]).

For $t \in \mathcal{T}$, define $\varphi^*(t)$ as follows:

(1) $\varphi^*(t^0) = 0$;
(2) $\varphi^*(t) = \sum_{i=0}^{s-1} \alpha(t^i, t^{i+1})$, where $t^s = t$, and $P(t^0), P(t^1), \dots, P(t^s) = P(t)$ is a chain such that $P(t^i)$ and $P(t^{i+1})$, $0 \le i \le s-1$, are adjacent by a facet.

Here $P(t^i)$ lies but $P(t^{i+1})$ does not lie in the halfspace $\{x \in \mathbf{R}^n : x^{\mathrm{T}}(v^*(t^{i+1}) - v^*(t^i)) \le \alpha(t^i, t^{i+1})\}$.

It is proved in [9] that the function $\varphi^*(t)$ is well defined. In fact, it is sufficient to prove that if the chain $P(t^0), P(t^1), \dots, P(t^s) = P(t^0)$ is closed, then the sum in (2) is equal to 0. It is so if $P(t^i)$, $0 \le i \le s-1$, have a common face F, since, for $x_0 \in F$, we have

$$\alpha(t^i, t^{i+1}) = x_0^{\mathrm{T}}(v^*(t^{i+1}) - v^*(t^i)), \qquad 0 \le i \le s-1.$$

Now, any closed chain can be contracted to a point avoiding $(n-3)$-faces, and in contracting over an $(n-2)$-face we can appeal to the above reasoning.

So, the function

$$f(x; t) = x^{\mathrm{T}} v^*(t) - \varphi^*(t) \tag{11}$$

is a generatrissa such that

$$f(x; t) > f(x; t') \text{ for all } x \in \operatorname{int} P(t) \text{ and all } t, t' \in \mathcal{T}.$$

Obviously if a tiling has a generatrissa of the form (11), then this tiling is pegged. The essential part of the papers [4] and [2] is devoted to a proof of the following proposition.

Proposition 3. *The following assertions are equivalent for a tiling* $\{Q(t) : t \in \mathcal{T}\}$

(i) *the tiling is pegged with pegs* $v^*(t)$, $t \in \mathcal{T}$;
(ii) *the tiling has the generatrissa* $f(x; t) = x^{\mathrm{T}} v^*(t) - \varphi^*(t)$.

The following general result is proved in [4] and [2].

Theorem 1. *If a tiling is primitive then it is pegged and has a generatrissa.*

This theorem implies the main result of Voronoi [12] asserting that Voronoi's conjecture is true for primitive parallelotopes. The proof of Theorem 1 in [4] is similar to the proof of Voronoi: both authors construct explicitly a generatrissa. The author of [2] constructs explicitly pegs.

9. The case of parallelotopes

If $P(0)$ is defined canonically, then by Lemma 6 the tiling $\{P(t) : t \in L\}$ is pegged with pegs $v^*(t) = q(t)$. Now, by Proposition 3, the tiling $\{P(t) : t \in L\}$ by parallelotopes has the following generatrissa:

$$f(x; t) = x^{\mathrm{T}} q(t) - \varphi^*(t).$$

But to obtain another equivalence and an explicit form of $f(x; t)$, we use the recursion (9) not supposing that P is defined canonically. We want to know, when the recursion (9) determines uniquely the generatrissa $l(x; t)$.

For parallelotopes, we know an explicit form of α_i in the recursion (9). In fact, since the point $t+\frac{1}{2}t_i$ belongs to the facet F_i, we have $\alpha_i = q_i^{\mathrm{T}}(t+\frac{1}{2}t_i)$. Hence the recursion (9) takes the following form:

$$l(x;t+t_i) = l(x;t) + q_i^{\mathrm{T}}(x-(t+\tfrac{1}{2}t_i)). \tag{12}$$

Let us have an arbitrary parallelotope $P = P(t_0)$, and let a linear on x function $l_{t^0}(x)$ be given. Then, using the recursive expression (12) and going out from $l(x;t) = l_{t^0}(x)$ by a path connecting t^0 with $t^1 = t^0 + t \in L$ we can, for all $t^1 \in L$, find a function $l(x;t^1)$ related to L.

Let $t = t(\mathcal{P}) = \sum_{k=1}^{s} t_k$, where $(t_1, t_2, \dots, t_s)$ is an arbitrary path $\mathcal{P}$ in G_L connecting t^0 with t^1. Let $q(\mathcal{P}) = \sum_{k=1}^{s} q_k$, where each q_k is associated to t_k. Obviously, $t(\mathcal{P}) = t^1 - t^0$ does not depend on the path $\mathcal{P}$.

Using (12) with $l(x;t) = l_{t^0}(x)$, and going along the path $\mathcal{P}$ from t^0 we obtain

$$l(x;t^1) = l_{t^0}(x) + x^{\mathrm{T}}q(\mathcal{P}) - \phi(t^0,\mathcal{P}),$$

where

$$\phi(t^0,\mathcal{P}) = \sum_{k=1}^{s} q_k^{\mathrm{T}}\left(t^0 + \sum_{r=1}^{k-1} t_r + \tfrac{1}{2}t_k\right), \tag{13}$$

and the sum $\sum_{r=1}^{k-1} t_r$ is empty for $k=1$ (cf., $l(x;t^1)$ with $f(x;t)$ in (11)).

Obviously, $q(\mathcal{P})$ is additive on $\mathcal{P}$. It is easy to verify that the function $\phi(t^0,\mathcal{P})$ is additive on the variable $\mathcal{P}$, too. In fact, let $\mathcal{P} = (t_1,\dots,t_s)$, $\mathcal{P}' = (t_{s+1},\dots,t_w)$. We set $\mathcal{P}+\mathcal{P}' = (t_1,\dots,t_s,t_{s+1},\dots,t_w)$. Hence the sum of two paths is its join. Let $t = t(\mathcal{P})$, $t' = t(\mathcal{P}')$, $t^1 = t^0 + t = t^0 + \sum_{r=1}^{s} t_r$. Let $-\mathcal{P} = (-t_s, -t_{s-1}, \dots, -t_1)$ be the path from t^1 to t^0 opposite to $\mathcal{P}$. Then

$$\phi(t^0,\mathcal{P}+\mathcal{P}') = \phi(t^0,\mathcal{P}) + \phi(t^1,\mathcal{P}'), \tag{14}$$

since

$$\begin{aligned}
\phi(t^0,\mathcal{P}+\mathcal{P}') &= \left(\sum_{k=1}^{s} + \sum_{k=s+1}^{w}\right) q_k^{\mathrm{T}}\left(t^0 + \sum_{r=1}^{k-1} t_r + \tfrac{1}{2}t_k\right) \\
&= \phi(t^0,\mathcal{P}) + \sum_{k=s+1}^{w} q_k^{\mathrm{T}}\left(t^0 + \sum_{r=1}^{s} t_r + \sum_{r=s+1}^{k-1} t_r + \tfrac{1}{2}t_k\right) \\
&= \phi(t^0,\mathcal{P}) + \phi(t^1,\mathcal{P}').
\end{aligned}$$

Now,

$$\phi(t^1,-\mathcal{P}) = \sum_{k=1}^{s}(-q_{s+1-k}^{\mathrm{T}})\left(t^1 - \sum_{r=1}^{k-1} t_{s+1-r} - \tfrac{1}{2}t_{s+1-k}\right).$$

Since $t^1 = t^0 + \sum_{r=1}^{s} t_{s+1-r}$, for $k \le s$, we have

$$t^1 - \sum_{r=1}^{k-1} t_{s+1-r} = t^0 + \sum_{r=1}^{s} t_{s+1-r} - \sum_{r=1}^{k-1} t_{s+1-r} = t^0 + \sum_{r=k}^{s} t_{s+1-r} = t^0 + \sum_{r=1}^{s+1-k} t_r.$$

If we perform the substitution $s + 1 - k \to k$ here and in $\phi(t^1, -\mathcal{P}; x)$, we obtain

$$\phi(t^1, -\mathcal{P}) = -\phi(t^0, \mathcal{P}). \tag{15}$$

Let $\mathcal{C}_1 = \mathcal{P}_1 + \mathcal{P}'$, $\mathcal{C}_2 = -\mathcal{P}' + \mathcal{P}_2$, where $\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset$. Let $\mathcal{P}_1$ go from t^0 to t^1, and the paths $\mathcal{P}'$, $\mathcal{P}_2$ go from t^1 to t^0. Then $\mathcal{C}_1 \oplus \mathcal{C}_2 = \mathcal{P}_1 + \mathcal{P}_2$ is the sum modulo 2.

Lemma 8.

$$\phi(t^0, \mathcal{C}_1 \oplus \mathcal{C}_2) = \phi(t^0, \mathcal{C}_1) + \phi(t^0, \mathcal{C}_2).$$

Proof. Using (14) and (15), we have $\phi(t^0, \mathcal{C}_1) = \phi(t^0, \mathcal{P}_1 + \mathcal{P}') = \phi(t^0, \mathcal{P}_1) + \phi(t^1, \mathcal{P}')$ and $\phi(t^0, \mathcal{C}_2) = \phi(t^0, -\mathcal{P}' + \mathcal{P}_2) = \phi(t^0, -\mathcal{P}') + \phi(t^1, \mathcal{P}_2) = -\phi(t^1, \mathcal{P}') + \phi(t^1, \mathcal{P}_2)$. Hence

$$\begin{aligned}\phi(t^0, \mathcal{C}_1) + \phi(t^0, \mathcal{C}_2) &= \phi(t^0, \mathcal{P}_1) + \phi(t^1, \mathcal{P}_2) = \phi(t^0, \mathcal{P}_1 + \mathcal{P}_2)\\ &= \phi(t^0, \mathcal{C}_1 \oplus \mathcal{C}_2). \quad \square\end{aligned}$$

Using Lemmas 5 and 8, for any quadrangle $\mathcal{Q}_{ij}$, we can represent $\phi(t^0, \mathcal{Q}_{ij})$ as a sum of functions $\phi(t, G(F^{n-2}))$.

Lemma 9. *Let $\phi(t^0, \mathcal{P})$ be given by (13). Then, for $i, j \in \mathcal{I}_P$ and $\mathcal{P} = \mathcal{Q}_{ij}$,*

$$\phi(t^0, \mathcal{Q}_{ij}) = q_j^{\mathrm{T}} t_i - q_i^{\mathrm{T}} t_j.$$

Proof. Using the equality (13) for $\mathcal{P} = \mathcal{Q}_{ij} = (t_i, t_j, -t_i, -t_j)$, we obtain

$$\begin{aligned}\phi(t^0, \mathcal{Q}_{ij}) = {}& q_i^{\mathrm{T}}(t^0 + \tfrac{1}{2}t_i) + q_j^{\mathrm{T}}(t^0 + t_i + \tfrac{1}{2}t_j) - q_i^{\mathrm{T}}(t^0 + t_i + t_j - \tfrac{1}{2}t_i)\\ &- q_j^{\mathrm{T}}(t^0 + t_i + t_j - t_i - \tfrac{1}{2}t_j) = q_j^{\mathrm{T}} t_i - q_i^{\mathrm{T}} t_j. \quad \square\end{aligned}$$

Lemma 9 implies the following result.

Lemma 10. *Let $i, j \in \mathcal{I}_P$, and let $\phi(t^0, \mathcal{P})$ be given by (13). The following assertions are equivalent:*

(i) $\phi(t^0, \mathcal{Q}_{ij}) = 0$;

(ii) $q_i^{\mathrm{T}} t_j = q_j^{\mathrm{T}} t_i$. $\square$

Let $\mathcal{P} = (t_1, t_2, \ldots, t_s)$ be a path. Recall that $t(\mathcal{P}) = \sum_{k=1}^{s} t_k$, $q(\mathcal{P}) = \sum_{k=1}^{s} q_k$, and $\mathcal{I}(\mathcal{P})$ is the set of all indices $i \in \mathcal{I}_P$ of t_i in the path $\mathcal{P}$.

Lemma 11. *Let the equalities $q_i^{\mathrm{T}} t_j = q_j^{\mathrm{T}} t_i$ hold for all pairs $i, j \in \mathcal{I}(\mathcal{P})$. Then the function $\phi(t^0, \mathcal{P})$ is given by the expression*

$$\phi(t^0, \mathcal{P}) = q^{\mathrm{T}}(\mathcal{P})(t^0 + \tfrac{1}{2}t(\mathcal{P})). \tag{16}$$

Proof. Using the equalities $q_i^T t_j = t_i^T q_j$ for $i, j \in \mathcal{I}(\mathcal{P})$, we obtain the equality

$$q_k^T \sum_{r=1}^{k-1} t_r = t_k^T \sum_{r=1}^{k-1} q_r.$$

Using this equality and setting $\sum_{k=1}^{s} q_k = q(\mathcal{P})$, we rewrite $\phi(t^0, \mathcal{P})$ from (13) as follows:

$$\phi(t^0, \mathcal{P}) = \sum_{k=1}^{s} q_k^T t^0 + \frac{1}{2}\sum_{k=1}^{s} q_k^T \sum_{r=1}^{k-1} t_r + \frac{1}{2}\sum_{k=1}^{s} q_k^T \left(\sum_{r=1}^{k-1} t_r + t_k\right)$$
$$= q^T(\mathcal{P})t^0 + \frac{1}{2}\sum_{k=1}^{s} t_k^T \sum_{r=1}^{k-1} q_r + \frac{1}{2}\sum_{k=1}^{s} q_k^T \sum_{r=1}^{k} t_r.$$

A permutation of the order of summation in the first double sum, gives

$$\frac{1}{2}\sum_{k=1}^{s} t_k^T \sum_{r=1}^{k-1} q_r = \frac{1}{2}\sum_{k=1}^{s-1} q_k^T \sum_{r=k+1}^{s} t_r.$$

Recall that $\sum_{k=1}^{s} t_k = t(\mathcal{P})$. Hence $\phi(t^0, \mathcal{P})$ takes the form

$$\phi(t^0, \mathcal{P}) = q^T(\mathcal{P})t^0 + \frac{1}{2}\sum_{k=1}^{s} q_k^T \sum_{r=1}^{s} t_r = q^T(\mathcal{P})(t^0 + \frac{1}{2}t(\mathcal{P})).$$

So, we obtain the wanted expression. □

Using Lemma 11, we can prove the following important result.

Lemma 12. *The following assertions are equivalent:*

(i) *a parallelotope P is defined canonically;*

(ii) $q_i^T t_j = q_j^T t_i$ *for all* $i, j \in \mathcal{I}_P$.

Proof. (i) ⇒ (ii). If P is defined canonically, then Proposition 1 assert that $q_i^T t_j = q_j^T t_i$ for $i, j \in \mathcal{I}(G(F^{n-2}))$. Obviously, $q(\mathcal{Q}_{ij}) = 0$, and by Lemma 3, $q(\triangle) = 0$ if $\triangle = G(F^{n-2})$. Hence $q(G(F^{n-2})) = 0$ and Lemma 11 implies that $\phi(t^0, G(F^{n-2})) = 0$. Lemma 5 asserts that each quadrangle $\mathcal{Q}_{ij}$ is a sum modulo 2 of circuits $G(F^{n-2})$. By Lemma 8 we obtain that $\phi(t^0, \mathcal{Q}_{ij}) = 0$. Now, Lemma 10 gives the wanted equality for all pairs i, j.

(ii) ⇒ (i) By Lemma 2, $q_i = Dt_i$ for all $i \in \mathcal{I}_P$. Hence any linear equality between lattice vectors implies the corresponding equality between the associated facet vectors. This implies that the facet vectors are defined canonically with respect to each belt of P, i.e., P is defined canonically. □

The obtained function $\phi(t^0, \mathcal{P})$ depends on the chosen path $\mathcal{P}$. Since $\phi(t^0, \mathcal{P})$ satisfies the condition (15), it does not depend on $\mathcal{P}$ if and only if $\phi(t^0, \mathcal{C}) = 0$ for every circuit $\mathcal{C}$. In Lemma 13 below we give conditions when this property is true.

Lemma 13. *Let $\mathcal{C}$ be a circuit and $t^0 \in \mathcal{C}$. The following assertions are equivalent:*

(i) *for all t^0 and all circuits $\mathcal{C} \ni t^0$, the function $\phi(t^0, \mathcal{C}) = 0$;*
(ii) *the equalities $q_i^{\mathrm{T}} t_j = q_j^{\mathrm{T}} t_i$ hold for all pairs $i, j \in \mathcal{I}_P$.*

Proof. (i) $\Rightarrow$ (ii) The item (i) implies that $\phi(t^0, \mathcal{Q}_{ij}) = 0$ for any quadrangle $\mathcal{Q}_{ij}$. Now Lemma 10 gives the wanted implication.

(ii) $\Rightarrow$ (i) Recall that if $q_i^{\mathrm{T}} t_j = q_j^{\mathrm{T}} t_i$ for all $i, j \in \mathcal{I}_P$, then $q_i = Dt_i$. Since the sum $\sum_{k=1}^{s} t_k = t(\mathcal{C}) = 0$ for any circuit $\mathcal{C}$, the associated vector $q(\mathcal{C}) = \sum_{k=1}^{s} q_k = \sum_{k=1}^{s} Dt_k = Dt(\mathcal{C})$ is also equal to zero. Hence $\phi(t^0, \mathcal{C}) = q^{\mathrm{T}}(\mathcal{C})(t^0 + \frac{1}{2}t(\mathcal{C})) = 0$. $\square$

10. Generatrissa for parallelotopes

By Lemma 13 the generatrissa $l(x; t) = q^{\mathrm{T}}(t)(x - \frac{1}{2}t)$ is uniquely determined for all $x \in \mathbf{R}^n$ and $t \in L$. This is the function $q^{\mathrm{T}}(\mathcal{P})x - \phi(0, \mathcal{P})$, where $q(\mathcal{P}) = q(t)$ and $\phi(0, \mathcal{P}) = \frac{1}{2} q^{\mathrm{T}}(t)t$, both do not depend on $\mathcal{P}$. In other words, we have the following assertion:

Lemma 14. *The following assertions are equivalent:*

(i) *the function $q^{\mathrm{T}}(t)(x - \frac{1}{2}t)$, $t \in L$, is the generatrissa of the tiling obtained by translations of a parallelotope P;*
(ii) *the equalities $q_i^{\mathrm{T}} t_j = q_j^{\mathrm{T}} t_i$ hold for all $i, j \in \mathcal{I}_P$.* $\square$

Recall that the facet vector $q(t)$ is associated to the lattice vector t, when (8) is true. Hence, by Lemma 2, $q(t) = Dt$ and we see that $l(x; t)$ is given by

$$l(x; t) = t^{\mathrm{T}} D(x - \tfrac{1}{2}t).$$

We see that $l(x; 0) = 0$. Using standard arguments of [12] and [13], one can prove that $l(x; t) \leq 0$ for all $x \in P(0)$ and all $t \in L$, with strict inequality for $x \in \mathrm{int}P(0)$. Hence (10) (with $t^0 = 0$) is true. But Lemma 2 does not assert that the matrix D is positive definite. The next lemma proves that D is positive definite if the vectors q_i and t_i are related to a parallelotope.

Lemma 15. *Let $P(0)$ be a parallelotope with center in origin, and let (8) be true. Let D be a non-singular matrix. The following assertions are equivalent:*

(i) *the inequality $l(x; t) = t^{\mathrm{T}} D(x - \frac{1}{2}t) < 0$ holds for all $x \in \mathrm{int}P(0)$ and all $t \in L$, $t \neq 0$;*
(ii) *the matrix D is positive definite.*

Proof. (i) $\Rightarrow$ (ii) For $x = 0$ and $t \neq 0$, we have $-l(0; t) = \frac{1}{2} t^{\mathrm{T}} Dt > 0$. This inequality holds for all vectors $t \in L - \{0\}$. Any rational combination of basic vectors of L is equal to $\frac{1}{p} t$ for some $t \in L$ and an integer p. We obtain that the above inequality holds also for rational vectors. By continuity, this inequality holds for all $t \in \mathbf{R}^n$. This implies, that the matrix D is positive definite.

(ii) $\Rightarrow$ (i) Let D be positive definite. Consider the quadratic function $x^{\mathrm{T}}Dx - 2l(x;t) = (x-t)^{\mathrm{T}}D(x-t) = f(x-t)$. Using the quadratic form $f(x)$ we can define the parallelotope

$$P_f(t^0) = \{x \in \mathbf{R}^n : l(x;t^0) - l(x;t) \geq 0,\ t \in L\}.$$

We show that $P_f(t^0) = P(t^0)$. Then for $t^0 = 0$ we will have $l(x;t) \leq 0$, $t \in L - \{0\}$, and $l(x;t) < 0$ for interior points of $P(t^0)$. The infinite system inequalities

$$l(x;t^0) - l(x;t) \geq 0, \qquad t \in L,$$

contain the subsystem for $t = t^0 \pm t_i$, $i \in \mathcal{I}_P$, describing the parallelotope $P(t^0)$ of type (2). Hence $P_f(t^0) \subseteq P(t^0)$ for every $t^0 \in L$. But we have here an equality. In fact, if there is $t \in L$ such that $P_f(t) \subset P(t)$ strictly, then there is an adjacent to $P_f(t)$ parallelotope $P_f(t')$ such that $P_f(t')$ and $P(t)$ have a common interior point. Since $P(t')$ contains $P_f(t')$, the parallelotopes $P(t)$ and $P(t')$ have a common interior point. This is a contradiction. □

The above proof of Lemma 15 is also a proof of the following.

Lemma 16. *The following assertions are equivalent*

(i) *the parallelotope P is a Voronoi polytope with respect to the positive quadratic form $f(x) = x^{\mathrm{T}}Dx$;*
(ii) *the function $l(x;t) = t^{\mathrm{T}}D(x - \frac{1}{2}t)$ is a generatrissa of the tiling $\{P(t) : t \in L\}$.*

11. Main theorem

If we collect Lemmas 1, 2, 6 and 7, Proposition 3 and Lemmas 12, 14 and 16 together, we obtain the following.

Theorem 2. *Let P be a parallelotope defined by facet vectors q_i, and defining lattice vectors t_i, $i \in \mathcal{I}_P$. Let A be a non-degenerate $n \times n$ matrix, and $D = A^{\mathrm{T}}A$. Then the following assertions are equivalent:*

(i) *Voronoi's conjecture holds for P, i.e. the affine transformation $x \to Ax$ transforms the parallelotope P into a Voronoi polytope;*
(ii) *the parallelotope P is a Voronoi polytope with respect to the positive quadratic form $f(x) = x^{\mathrm{T}}Dx$;*
(iii) *the parallelotope P is defined canonically;*
(iv) *the equality $q_i = Dt_i$ holds for all $i \in \mathcal{I}_P$;*
(v) *for all pairs $i, j \in \mathcal{I}_P$, the equalities $t_i^{\mathrm{T}}q_j = q_i^{\mathrm{T}}t_j$ hold;*
(vi) *the tiling $\{P(t) : t \in L\}$ is pegged with pegs $v^*(t) = Dt$;*
(vii) *the function $l(x;t) = t^{\mathrm{T}}D(x - \frac{1}{2}t)$ is a generatrissa of the tiling $\{P(t) : t \in L\}$;*
(viii) *the tiling $\{P(t) : t \in L\}$ has a dual tiling.*

Theorem 2 implies a result of [6] that there is a unique (up to isomorphism of P) map which transforms a primitive parallelotope P into a Voronoi polytope. Our Theorem 2 gives an explicit matrix A of the corresponding affine map. Therefore we have the following.

Proposition 4. *If a parallelotope P is affinely equivalent to a Voronoi polytope, then this affinity is uniquely (up to the aftomorphism of P) determined by the parallelotope P.*

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