A Double-Sequence Iteration Process for Fixed Points of Continuous Pseudocontractions

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Abstract—Let C be a bounded closed convex nonempty subset of a (real) Hilbert space H. The idea of a double-sequence iteration is introduced, and it is proved that a Mann-type double-sequence iteration process converges strongly to a fixed point of a continuous pseudocontractive map T which maps C into C. Related results deal with the strong convergence of the iteration process to fixed points of nonexpansive maps. © 2002 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Recall that a map $A : H \rightarrow H$ is said to be accretive if $\forall x, y \in D(A)$, we have that

$$\langle Ax - Ay, x - y \rangle \geq 0,$$  \hfill (1)

and is said to be strongly accretive if $A - kI$ is accretive where $k \in (0, 1)$ is a constant and $I$ denotes the identity operator on $H$. The map $A$ is said to be $\phi$-strongly accretive if $\forall x, y \in E$, $\exists$ a strictly increasing function $\phi : [0, \infty) \rightarrow [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \phi(||x - y||)||x - y||$$

and is called uniformly accretive if there exists a strictly increasing function $\psi : [0, \infty) \rightarrow [0, \infty)$ with $\psi(0) = 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \psi(||x - y||).$$
Let \( \mathcal{N}(A) = \{ x^* \in H : Ax^* = 0 \} \) denote the null space (set of zeros) of \( A \). If \( \mathcal{N}(A) \neq \emptyset \) and (1) holds for all \( x \in D(A) \) and \( y \in \mathcal{N}(A) \), then \( A \) is said to be quasi-accretive. The notions of strongly, \( \phi \)-strongly, uniformly quasi-accretive are similarly defined. \( A \) is said to be \( m \)-accretive if \( \forall r > 0 \) the operator \( (I + rA) \) is surjective. Closely related to the class of accretive maps is the class of pseudocontractive maps. A map \( T : H \to H \) is said to be pseudocontractive if \( \forall x, y \in D(T) \) we have that

\[
\langle (I - T)x - (I - T)y, x - y \rangle \geq 0.
\]

Observe that \( T \) is pseudocontractive if and only if \( A = (I - T) \) is accretive. (The map \( T \) is said to be hemicontractive if and only if \( A = (I - T) \) is quasi-accretive.) Similarly, the concepts of strong, \( \phi \)-strong, uniform pseudocontraction (hemicontraction) are easily deduced from the corresponding accretive definitions. Thus, the mapping theory of accretive maps is intimately tied to the fixed point theory of pseudocontractions. For ease of reference, we shall refer to the class of strongly, \( \phi \)-strongly, uniformly accretive (pseudocontractive, quasi-accretive, hemicontractive) maps as maps of strong\( \) type in the sequel.

2. PRELIMINARIES

Several iteration processes have been established for the constructive approximation of solutions to several classes of (nonlinear) operator equations and several convergence results established using these iterative processes (see, e.g., [1-23] and the references cited therein). Most of these convergence results have required that the operator be of the strong (accretive or pseudocontractive) type whereas several known existence theorems do not require the strong type property. Most of these convergence results for the iterative solution of nonlinear operator equations or approximation of fixed points of nonlinear maps have used iteration processes of the Mann and Ishikawa types.

Let \( C \) be a compact convex subset of a real Hilbert space and let \( T : C \to C \) be a Lipschitz pseudocontraction. It has remained an open question whether the Mann iteration process always converges to a fixed point of \( T \). In [14], Ishikawa introduced a new iteration method (now known as the Ishikawa iteration process) and proved that it converges strongly to a fixed point of \( T \). This result of Ishikawa has been extended to the continuous hemicontractions mapping a compact convex subset of a Hilbert space into itself (see [8,20]). Recently, Mutangadura and Chidume [19] constructed an example of a Lipschitz pseudocontraction mapping the closed unit ball \( B_1 \) in \( \mathbb{R}^2 \) into itself for which the Mann iteration process fails to converge.

It is, therefore, our purpose in this paper to introduce the concept of a Mann-type double-sequence iteration process and prove that it converges strongly to a fixed point of a continuous pseudocontraction which maps a bounded closed convex nonempty subset of a real Hilbert space into itself.

**DEFINITION 2.1.** Let \( \mathcal{N} \) denote the set of all the nonnegative integers (the natural numbers) and let \( E \) be a normed linear space. By a double sequence in \( E \) is meant a function \( f : \mathcal{N} \times \mathcal{N} \to E \) defined by \( f(n, m) := x_{n,m} \in E \). The double sequence \( \{ x_{n,m} \} \) is said to converge strongly to \( x^* \) if given any \( \varepsilon > 0 \) there exist integers \( N, M > 0 \) such that \( \forall n \geq N, m \geq M \), we have that \( \| x_{n,m} - x^* \| < \varepsilon \). If \( \forall n, r \geq N, m, t \geq M \), we have that \( \| x_{n,r} - x_{m,t} \| < \varepsilon \), then the double sequence is said to be Cauchy. Furthermore, if for each fixed \( n \), \( x_{n,m} \to x_n^* \) as \( m \to \infty \) and then \( x_n^* \to x^* \) as \( n \to \infty \), then \( x_{n,m} \to x^* \) as \( n, m \to \infty \).

3. MAIN THEOREMS

**THEOREM 3.1.** Let \( C \) be a bounded closed convex nonempty subset of a (real) Hilbert space \( H \), and let \( T : C \to C \) be a continuous pseudocontractive map. Let \( \{ \alpha_n \}_{n \geq 0}, \{ a_k \}_{k \geq 0} \subset (0,1) \) be real sequences satisfying the following conditions:

(i) \( \lim_{k \to \infty} a_k = 1 \) (monotonically);
(ii) lim_{k,r \to \infty}((a_k - a_r)/(1 - a_k)) = 0, \forall 0 < r \leq k;
(iii) \lim_{n \to \infty} a_n = 0;
(iv) \sum_{n \geq 0} a_n = \infty.

For an arbitrary but fixed \( \omega \in C \), and for each \( k \geq 0 \), define \( T_k : C \to C \) by \( T_kx := (1 - a_k)\omega + a_k Tx, \forall x \in C \). Then, the double sequence \( \{x_{k,n}\}_{k \geq 0, m \geq 0} \) generated from an arbitrary \( x_{0,0} \in C \) by

\[
x_{k,n+1} := (1 - a_n)x_{k,n} + a_nT_kx_{k,n}, \quad k, n \geq 0,
\]

converges strongly to a fixed point \( x^*_\infty \) of \( T \) in \( C \).

**Proof.** Clearly, \( F(T) \neq \emptyset \) (see, e.g., [24]). Now,

\[
\langle T_kx - T_ky, x - y \rangle = a_k\langle Tx - Ty, x - y \rangle \leq a_k \|x - y\|^2,
\]

so that \( \forall k \geq 0, T_k \) is continuous and strongly pseudocontractive. Also, \( C \) is invariant under \( T_k \), \( \forall k \) by convexity. Hence, \( T_k \) has a unique fixed point \( x_k^* \in C, \forall k \geq 0 \). It thus suffices to prove the following:

1. for each fixed \( k \geq 0 \), \( x_{k,n} \to x_k^* \in C \) as \( n \to \infty \);
2. \( x_k^* \to x^*_\infty \in C \) as \( k \to \infty \);
3. \( x^*_\infty \in F(T) \).

The first is known, but for completeness we give the details. Now, let \( d = \text{diam} C \) and \( b_k := 1 - a_k \in (0,1), \forall k \). Then,

\[
\|x_{k,n+1} - x_k^*\|^2 = \|x_{k,n} - x_k^* - a_n(x_{k,n} - T_kx_{k,n})\|^2
\]

\[
= \|x_{k,n} - x_k^*\|^2 - 2a_n \langle x_{k,n} - T_kx_{k,n}, x_{k,n} - x_k^* \rangle + a_n^2 \|x_{k,n} - T_kx_{k,n}\|^2
\]

\[
\leq (1 - 2b_k a_n) \|x_{k,n} - x_k^*\|^2 + d^2a_n^2.
\]

If we set

\[
\theta_{k,n} := \|x_{k,n} - x_k^*\|, \quad \delta_{k,n} := 2b_k a_n, \quad \sigma_{k,n} := d^2a_n^2,
\]

then we have that

\[
\theta_{k,n+1}^2 \leq (1 - \delta_{k,n})\theta_{k,n}^2 + \sigma_{k,n},
\]

so that observing that

\[
\sigma_{k,n} = o(\delta_{k,n}), \quad \lim_{n \to \infty} \delta_{k,n} = 0, \quad \text{and} \quad \sum_{n \geq 0} \sigma_{k,n} = \infty,
\]

we then have that \( \theta_{k,n} \to 0 \) as \( n \to \infty \). So the first part is proved. Now,

\[
\|x_k^* - Tx_k^*\| = \|x_k^* - a_k^{-1} x_k^* - a_k^{-1} (1 - a_k)\omega\|
\]

\[
\leq \left( \frac{1 - a_k}{a_k} \right) (\|x_k^*\| + \|\omega\|) \leq 2d \left( \frac{1 - a_k}{a_k} \right),
\]

so that

\[
\lim_{k \to \infty} \|x_k^* - Tx_k^*\| = 0,
\]

and hence, \( \{x_k^*\} \) is an approximate fixed point sequence for \( T \). Also, supposing that \( x^*_\infty \) is a fixed point of \( T \), then

\[
\lim_{k \to \infty} \|x^*_\infty - Tx^*_\infty\| \leq 2d \lim_{k \to \infty} (1 - a_k) = 0.
\]
Now, for all $0 < r \leq k$,
\[
\|x_k^* - x_r^*\|^2 = (x_k^* - x_r^*, x_k^* - x_r^*) = (T_kx_k^* - T_rx_r^*, x_k^* - x_r^*)
\]
\[
= (a_k - a_r) (\omega, x_k^* - x_r^*) + (a_kT_kx_k^* - a_rT_rx_r^*, x_k^* - x_r^*)
\]
\[
- (a_r - a_k) (\omega, x_k^* - x_r^*) + (a_kx_k^* - a_rx_r^*, T_rx_r^* - T_kx_k^*)
\]
\[
+ a_k (T_kx_k^* - T_kx_k^*, x_k^* - x_r^*)
\]
\[
\leq |a_k - a_r| \|x_k^* - x_r^*\| \cdot (\|\omega\| + \|T_rx_r^*\|) + a_k \|x_k^* - x_r^*\|^2,
\]
so that
\[
(1 - a_k) \|x_k^* - x_r^*\|^2 \leq |a_k - a_r| (\|\omega\| + \|T_rx_r^*\|) \|x_k^* - x_r^*\|,
\]
and hence,
\[
\lim_{k,r \to \infty} \|x_k^* - x_r^*\| \leq 2d \lim_{k,r \to \infty} \left( \frac{a_k - a_r}{1 - a_k} \right) = 0.
\]
Thus, $\{x_k^*\}$ is a Cauchy sequence, and hence, there exists $x_k^* \in C$ such that $x_k^* \to x^\infty$ as $k \to \infty$.
So, the second part is proved. By continuity, $T_kx_k^* \to Tx^\infty$ as $k \to \infty$. But $x_k^* - Tx_k^* \to 0$ as $k \to \infty$. Hence, $x_k^* \in F(T)$. This completes the proof.

**COROLLARY 3.2.** Let $C$ be a bounded closed convex nonempty subset of a Hilbert space $H$ with $0 \in C$. Let $T, \{a_k\}, \{a_n\}, \{x_{k,n}\}$ be as in Theorem 3.1 and $\forall k \geq 0$ define $T_k := a_kT$. Then, $T_k$ maps $C$ into itself and $\{x_{k,n}\}$ converges strongly to a fixed point of $T$.

**PROOF.** This follows from Theorem 3.1 on setting $\omega = 0 \in C$.

**COROLLARY 3.3.** In Theorem 3.1, let $T$ be a nonexpansive map. Then, the same conclusion is obtained.

**PROOF.** Observe that every nonexpansive map is a continuous pseudocontraction.

**REMARK 3.4.** Mutangadura and Chidume [19] recently constructed the following example to demonstrate that the Mann iteration process is not guaranteed to converge to a fixed point of a Lipschitz pseudocontraction mapping a compact convex subset of a real Hilbert space into itself: Let $H = \mathbb{R}^2$ with the usual Euclidean inner product, and for $x = (a, b) \in H$ define $x^1 = (b, -a)$. Now, let $C = B_1(0)$, the closed unit ball in $H$ and let $C_1 = \{x \in H : \|x\| \leq 1/2\}$, $C_2 = \{x \in H : 1/2 < \|x\| < 1\}$. Define the map $T : C \to C$ by
\[
Tx := \begin{cases} 
  x + x^1, & \text{if } x \in C_1, \\
  \frac{x}{\|x\|} - x + x^1, & \text{if } x \in C_2.
\end{cases}
\]
Observe that $T$ is pseudocontractive (but not strongly so), Lipschitz continuous (with Lipschitz constant $5$), and has the origin $(0,0)$ as its unique fixed point, $C$ is compact and convex. However, $\forall x \in C_1$ we have that
\[
\|(1 - \lambda)x + \lambda Tx\|^2 = (1 + \lambda^2) \|x\|^2 > \|x\|^2, \quad \forall \lambda \in (0, 1),
\]
while $\forall x \in C_2$, we have that
\[
\|(1 - \lambda)x + \lambda Tx\|^2 \geq \frac{1}{2} \|x\|^2, \quad \forall \lambda \in (0, 1),
\]
and so, no Mann sequence can converge to $(0,0)$ the unique fixed point of $T$ (unless the initial guess is the fixed point itself).

**REMARK 3.5.** We make the following remarks.

1. Whereas the strong convergence of the Ishikawa iteration process to a fixed point of a Lipschitz pseudocontraction (and its generalizations) is established in compact convex
subsets of a Hilbert space, no compactness condition is imposed in establishing the strong convergence of the iteration process (3) to a fixed point of a continuous pseudocontraction.

(2) Our theorems extend to the slightly more general classes of continuous hemicontractive and continuous quasi-nonexpansive maps.

(3) Prototypes of the sequence \( \{a_k\} \) are: \( \forall k \geq 0, \)
\[
 a_k := \exp \left\{ -\frac{1}{k+1} \right\}, \quad a_k := \log_e \left( e - \frac{1}{k+1} \right), \quad a_k := 1 - \frac{1}{\log_m(k+m+1)}, \quad m > 1.
\]

REFERENCES