A note on “Resolution of fuzzy relation equations (I) based on Boolean-type implications”\textsuperscript{*}

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A R T I C L E   I N F O

Article history:
Received 20 June 2008
Accepted 4 December 2008

Keywords:
Boolean-type implications
Fuzzy relation equations
Maximal solutions

A B S T R A C T

This paper shows, by examples, that Lemma 2.1, Lemma 3.1, Corollary 3.1, Corollary 3.2, Theorem 3.1, Corollary 3.3, Theorem 3.3 and Lemma 4.1 in [Y. Luo, Resolution of fuzzy relation equation (I) based on Boolean-type implications, Comput. Math. Appl. 52 (2006) 421–428] about resolution of fuzzy relation equations based on Boolean-type implications, are false. We also point out that the solution set of Example 5.1 is not right and we give the right solution set of Example 5.1.

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1. Introduction

Luo and Li studied the decomposition and resolution of finite min-implication fuzzy relation equations based on R-implications and S-implications in [1,2]. The complete solution sets are obtained, respectively, Luo also investigated the resolution of finite fuzzy relation equations (I) based on Boolean-type implications in [3]. An effective method to solve fuzzy relation equations (I) is given, and some conditions for existence of solutions to equations (I) are also discussed. The complete solution set of fuzzy relation equations (I) can be determined by the smallest solution and a finite number of maximal solutions.

Throughout Refs. [1–5], the definitions of R-implication, S-implication and QL-implication are as follows:

R-implication [1,4,5]. Let \( T \) be a \( t \)-norm. The function \( \theta : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is defined by \( \theta(a, b) = \sup \{c \in [0, 1] | T(a, c) \leq b \} \) for \( a, b \in [0, 1] \). Then, \( \theta \) is called an R-implication induced by \( T \).

S-implication [2,4,5]. Let \( S \) be a \( t \)-conorm and \( N \) be a negation. The function \( \theta : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is defined by \( \theta(a, b) = S(N(a), b) \) for \( a, b \in [0, 1] \). Then, \( \theta \) is called an S-implication.

QL-implication [4,5]. The function \( \theta : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is defined by \( \theta(a, b) = S(N(a), T(a, b)) \) for \( a, b \in [0, 1] \), where \( S \) is a \( t \)-conorm, \( T \) is a \( t \)-norm, \( N \) is a negation and \( S \) and \( T \) are dual through \( N \). Then, \( \theta \) is called QL-implication.

For example, Zadeh implication \( R_Z(a, b) = (1 - a) \vee (a \wedge b) \) is a QL-implication.

In [2,3], the negation can also be denoted as \( h \). So, in order to consist with [3], \( h \) denotes a negation in this note.

In [3], the definition of “Boolean-type implication” is not given in detail. The author said that “the Boolean-type implications mainly contain R-implications, S-implications and QL-implications” (See Introduction of [3]). In order to avoid misunderstanding and consist with [3], the Boolean-type implications used in this note are R-implications, S-implications or QL-implications.

\textsuperscript{*} This paper was supported by National Natural Science Foundation of China (60774049), the National 973 Fundamental Research Project of China (2002CB312200), and the National 863 High-Tech Program of China (No. 2006AA04Z163).

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Throughout [3], $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$ denote the universe of discourse. $F(X), F(Y)$ and $F(X \times Y)$ denote the set of all fuzzy subsets in $X, Y$ and $X \times Y$, respectively. $A \in F(X), B \in F(Y)$, and $R \in F(X \times Y)$. $\theta(a, b)$ is also denoted as $a \theta b$. $N_n = \{1, 2, \ldots, n\}$ and $N_m = \{1, 2, \ldots, m\}$. Any $A \in F(X)$ and $R \in F(X \times Y)$ are denoted by a vector and $n \times m$ matrix, respectively.

The min-implication fuzzy relation equation based on Boolean-type implication $\theta$ in [2,3] is as follows:

$$A \circ^\theta R = B$$

where matrix $A = (a_{ij})_{n \times m}$, matrix $R = (r_{ij})_{m \times k}$, matrix $B = (b_{ij})_{n \times k}$, and

$$b_{ij} = \inf_{k \in \mathbb{N}} a_{ik} \theta r_{kj} \quad \text{for} \ i \in N_n, j \in N_k.$$  

In [3], the decomposition of Eq. (1) is given, namely, Eq. (1) can be decomposed as the set of $k$ simpler fuzzy relation equations,

$$A \circ^\theta r = b$$

where $r_{m \times 1}$ and $b_{n \times 1}$ are the column vectors of $R$ and $B$, respectively. $\eta(A, b)$ [3] denotes the solution set of Eq. (2), i.e., $\eta(A, b) = \{r | A \circ^\theta r = b\}$. In [3], the decomposition of Eq. (2) is given as follows:

Eq. (2) can be further decomposed as a system of $n$ fuzzy relation equations,

$$a \circ^\theta r = b$$

where $a_{1 \times m}$ is a row vector of $A$ and $b \in [0, 1]$. $\eta(a, b)$ [3] denotes the solution set of Eq. (3), i.e., $\eta(a, b) = \{r | a \circ^\theta r = b\}$. Definition 2.1 and Definition 3.1 in [3] are as follows.

**Definition 2 [3]** Let $\theta$ be a Boolean-type implication. $\theta$ is called a nice Boolean-type implication if $\theta$ is continuous with respect to the second variable, i.e., $\forall a \in [0, 1]$, the function $f(x) = a \theta x$ is continuous.

In [3], $f(a)$ denotes the image set of the function $f(x) = a \theta x$. Also, $a \hat{\otimes}^\theta b$, $a \hat{\otimes}^\theta b$, $\hat{\mu}_\theta(a, b)$, $\hat{\nu}_\theta(a, b)$, and $w_{\max}(a, b)$ are defined in [3] as follows:

$$a \hat{\otimes}^\theta b = \sup\{x \in [0, 1] | a \theta x = b\},$$

$$\hat{\mu}_\theta(a, b) = \begin{cases} a \hat{\otimes}^\theta b, & b \in I(a); \\ 0, & \text{otherwise}, \end{cases}$$

$$\hat{\nu}_\theta(a, b) = \begin{cases} a \hat{\otimes}^\theta b, & b \in I(a); \\ 0, & \text{otherwise}, \end{cases}$$

$$w_{\max}(a, b) = \begin{cases} 1, & a > b; \\ b, & \text{otherwise}. \end{cases}$$

**Definition 3.1 [3]** Let $A \circ^\theta r = b$ be a fuzzy relation equation of form (2), and $\theta$ be a nice Boolean-type implication. The matrix $(\hat{R}_y)_{m \times n}$ is called a mean solution matrix (mean-SM) of the fuzzy relation equation, where

$$\hat{R}_y = \hat{\nu}_\theta(a_{ij}, b_j), \quad \forall i \in N_m, \forall j \in N_n.$$ 

The matrix $(\hat{R}_y)_{m \times n}$ is called the minimal solution matrix (min-SM) of the fuzzy relation equation, where

$$\hat{R}_y = \hat{\nu}_\theta(a_{ij}, b_j), \quad \forall i \in N_m, \forall j \in N_n.$$ 

The matrix $(\hat{R}_y)_{m \times n}$ is called the maximal solution matrix (max-SM), where

$$\hat{R}_y = w_{\max}(\sup_{k \in N} \hat{R}_y, \hat{R}_y), \quad \forall i \in N_m, \forall j \in N_n.$$ 

In [3], the vector $t = (t_1, t_2, \ldots, t_m)$ is defined by $t_i = \sup_j \hat{\nu}_\theta(a_{ij}, b_j), \forall i \in N_m$, namely, $t = \sup \hat{R}_y$.

2. Counterexamples

Lemma 2.1 in [3] is as follows:

**Lemma 2.1 [3]** Let $a, b \in [0, 1]$, and $\theta$ be any nice Boolean-type implication. Then, $a \theta \hat{w}_\theta(a, b) > b$.

**Example 2.1.** Let $\theta$ be Zadeh implication, i.e., $\theta(a, b) = (1 - a) \lor (a \land b)$, and let $a = 0.5, b = 0.6$. Obviously, Zadeh implication is a QL-implication and is a nice Boolean-type implication. Known by $\theta$ is nondecreasing with respect to the second variable, we have $I(0.5) = [\theta(0.5, 0), \theta(0.5, 1)] = [0.5, 0.5]$. It is obvious that $b = 0.6 \not\in I(0.5)$. It follows that $\hat{w}(0.5, 0.6) = 0$ and $0.5 \theta \hat{w}(0.5, 0.6) = 0.5 \theta 0 = 0.5 < 0.6$. This implies Lemma 2.1 in [3] is false.
Lemma 3.1, Corollary 3.1, Corollary 3.2 in [3] are as follows:

**Lemma 3.1** [3]. Let \( \mathbf{a} \circ^\theta \mathbf{r} = \mathbf{b} \) be a fuzzy relation equation of form (3). Then, \( \eta(\mathbf{a}, \mathbf{b}) \neq \emptyset \) iff there exists \( j \in N_m \) such that \( b_j \in I(a_j) \).

**Corollary 3.1** [3]. Let \( \mathbf{a} \circ^\theta \mathbf{r} = \mathbf{b} \) be a fuzzy relation equation of form (3). Then, \( \eta(\mathbf{a}, \mathbf{b}) \neq \emptyset \) iff there exists \( j \in N_m \) such that \( b_j \in [a \circ^\theta 0, a \circ^\theta 1] \).

**Corollary 3.2** [3]. Let \( \mathbf{a} \circ^\theta \mathbf{r} = \mathbf{b} \) be a fuzzy relation equation of form (3). If there exits \( j \in N_m \) such that \( a_j \neq 0 \), then \( \eta(\mathbf{a}, \mathbf{b}) \neq \emptyset \).

**Example 2.2.** Let \( \theta \) be Zadeh implication and the equation be \( \mathbf{a} \circ^\theta \mathbf{r} = \mathbf{b} \) where \( \mathbf{a} = (a_1, a_2, a_3) = (0.8, 0.3, 0.5) \) and \( b = 0.6 \). It is easy to verify that \( I(0.8) = [0.2, 0.8], I(0.3) = [0.7, 0.7], \) and \( I(0.5) = [0.5, 0.5] \). Obviously, \( b \in I(a_1) \). But for any \( \mathbf{r} = (r_1, r_2, r_3)(r_i \in [0, 1], i = 1, 2, 3) \), we have

\[
\inf_{i=1,2,3} a_i r_i = (0.8 \theta r_1) \land 0.7 \land 0.5 \leq 0.5 < 0.6.
\]

Hence, \( \eta(\mathbf{a}, \mathbf{b}) = \emptyset \). This implies that Lemma 3.1, Corollary 3.1 and Corollary 3.2 in [3] are false.

In [3], under the assumption that, for any QL-implication \( \theta = S(h(a), T(a, b)), \theta(a, 1) = S(h(a), a) = 1 \), Theorem 3.1 are obtained as follows:

**Theorem 3.1** [3]. Let \( \mathbf{a} \circ^\theta \mathbf{r} = \mathbf{b} \) be a fuzzy relation equation of form (2). If for any \( i \in N_n \), there exists \( j \in N_m \) such that \( b_j \in I(a_{ij}) \) and \( b_j \notin I(a_i), \forall l \in N_n - \{i\} \), then \( \eta(\mathbf{A}, \mathbf{b}) \neq \emptyset \).

**Example 2.3.** Let \( S \) and \( T \) be \( t \)-conorm and \( t \)-norm, defined as follows:

\[
S(a, b) = \begin{cases} 
  b, & a = 0; \\
  a, & b = 0; \\
  1, & \text{otherwise},
\end{cases} \quad T(a, b) = 1 - S(1 - a, 1 - b).
\]

Let the QL-implication be \( \theta(a, b) = S(1 - a, T(a, b)) \) and the equation be \( \mathbf{A} \circ^\theta \mathbf{r} = \mathbf{b} \) where

\[
\mathbf{A} = (a_{ij})_{2 \times 2} = \begin{pmatrix} 0.2 & 0.4 \\ 0.4 & 0.3 \end{pmatrix}, \quad \mathbf{b} = (b_1, b_2)^\top = \begin{pmatrix} 0.8 \\ 0.7 \end{pmatrix}.
\]

It is easy to get

\[
T(a, b) = \begin{cases} 
  b, & a = 1; \\
  a, & b = 1; \\
  0, & \text{otherwise},
\end{cases} \quad \theta(a, b) = S(1 - a, T(a, b)) = \begin{cases} 
  1, & b = 1; \\
  b, & b \neq 1 \text{ and } a = 1; \\
  1 - a, & a \neq 1 \text{ and } b \neq 1,
\end{cases}
\]

and

\[
I(a_{11}) = \{0.8, 1\}, \quad I(a_{12}) = \{0.6, 1\},
\]
\[
I(a_{21}) = \{0.6, 1\}, \quad I(a_{22}) = \{0.7, 1\}.
\]

It follows that

\[
b_1 = 0.8 \in I(a_{11}), \quad b_2 = 0.7 \notin I(a_{21}),
\]
\[
b_2 = 0.7 \in I(a_{22}), \quad b_1 = 0.8 \notin I(a_{12}).
\]

However, it is easy to verify that the solution set of equation above is \( \emptyset \). This implies that Theorem 3.1 in [3] is false.

**Corollary 3.3** [3]. Let \( \mathbf{A} \circ^\theta \mathbf{r} = \mathbf{b} \) be a fuzzy relation equation of form (2). If for any \( i \in N_n \), there exists \( j \in N_m \) such that \( a_{ij} = 0 \) and \( a_{ij} = 1, \forall l \in N_n - \{i\} \), then \( \eta(\mathbf{A}, \mathbf{b}) \neq \emptyset \).

**Example 2.4.** Let \( \theta = S(1 - a, T(a, b)) \) where \( S(a, b) = (a + b) \land 1 \) and \( T(a, b) = 1 - S(1 - a, 1 - b) \). The equation is \( \mathbf{A} \circ^\theta \mathbf{r} = \mathbf{b} \) where

\[
\mathbf{A} = (a_{ij})_{3 \times 3} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \mathbf{b} = (b_1, b_2, b_3)^\top = \begin{pmatrix} 0.2 \\ 0.5 \\ 0.8 \end{pmatrix}.
\]

It is easy to get \( T(a, b) = (a + b - 1) \lor 0 \) and \( \theta(a, b) = (1 - a) \lor b \). It is easy to verify that for any \( i \in N_3 \), there exists \( j \in N_3 \) such that \( a_{ij} = 0 \) and \( a_{ij} = 1 \) for \( l \in N_3 - \{i\} \). But the equation above has no solution. This implies that Corollary 3.3 in [3] is false.
Theorem 3.3 in [3] is as follows.

**Theorem 3.3** [3]. Let \( A \circ \theta r = b \) be a fuzzy relation equation of form (2). The following properties are equivalent.

(i) \( \eta(A, b) = \emptyset \);

(ii) There exits \( j \in N_n \) such that \( \hat{f} \cdot j = 1 \) and \( b_j \neq 1 \).

**Example 2.5.** Let equation be 0.8 \( \circ \theta r = 1 \) where \( \theta \) is Zadeh implication. It is easy to verify that \( \eta(0.8, 1) = \emptyset \). But \( b = 1 \). So (i) \( \implies \) (ii) is false. We also construct equation 0.2 \( \circ \theta x = 0.8 \) where \( \theta \) is same as Example 2.3, i.e.,

\[
\theta(a, b) = S(1 - a, T(a, b)) = \begin{cases} 
1, & b = 1; \\
\sup \{a \mid \theta(n, x) \leq b\}, & b \neq 1 \text{ and } a = 1; \\
1 - a, & a \neq 1 \text{ and } b \neq 1.
\end{cases}
\]

It is easy to verify that the solution set \( \eta(0.2, 0.8) = \{0, 1\} \). Furthermore, we can obtain \( \hat{f}(0.2, 0.8) = \hat{w}_0(0.2, 0.8) = \sup\{x \mid 0.2 \circ \theta x = 0.8\} = 1 \). It follows that \( \hat{f} \cdot x = 1 \). So, (ii) \( \implies \) (i) is false. The two equations imply that Theorem 3.3 in [3] is false.

Example 2.5 also shows that the solvability criteria (c) and (d) based on solution matrices on page 426 in [3] are false.

**Lemma 4.1** [3]. (i): The fuzzy relation equation of form (2) has a solution iff \( A \circ \theta t \leq b \).

(ii): If the fuzzy relation equation of form (2) has a solution, then \( t \) is the smallest solution.

We use an example to show that (i) in Lemma 4.1 is not right.

**Example 2.6.** Let equation be 0.5 \( \circ \theta r = 0.6 \) where \( \theta \) is Zadeh implication. It is easy to verify that \( t = 0 \). So 0.5 \( \circ \theta 0 = 0.5 < 0.6 \). But the equation 0.5 \( \circ \theta r = 0.6 \) has no solution, since \( f(0.5) = [0.5, 0.5] \). This implies (i) is not right.

The Example 5.1 in [3] is as follows.

Example 5.1 [3]. Let \( A \circ \theta b \) be a fuzzy relation equation of form (2), where

\[
A = (a_{ij})_{4 \times 4} = \begin{pmatrix}
0.3 & 0.8 & 0.6 & 0.2 \\
0.4 & 0.6 & 0.7 & 0.3 \\
0.5 & 0.4 & 0.8 & 0.5 \\
0.8 & 0.3 & 0.5 & 0.4
\end{pmatrix}, \quad b = (b_1, b_2, b_3, b_4)^T = \begin{pmatrix}
0.2 \\
0.3 \\
0.3 \\
0.4
\end{pmatrix},
\]

and \( \theta \) is Zadeh implication.

Its complete solution set given in [3] is \( \eta(A, b) = \{(0.4, 0, 0.3, a)^T : a \in [0, 1]\} \).

However, the complete solution set above is not right, since it is easy to verify that \( (0.4, 0.1, 0.3, 1)^T \) is a solution. Furthermore the maximal solution of fuzzy relation equation of Example 5.1 is \( (0.4, 0.2, 0.3, 1) \). So, the complete solution set of Example 5.1 is \( \{(0.4, a, 0.3, b)^T : a \in [0, 0.2], b \in [0, 1]\} \).

**References**


