Topologically invariant Chern numbers
of projective varieties

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Abstract

We prove that a rational linear combination of Chern numbers is an oriented diffeomorphism invariant of smooth complex projective varieties if and only if it is a linear combination of the Euler and Pontryagin numbers. In dimension at least three we prove that only multiples of the top Chern number, which is the Euler characteristic, are invariant under diffeomorphisms that are not necessarily orientation-preserving. These results solve a long-standing problem of Hirzebruch's. We also determine the linear combinations of Chern numbers that can be bounded in terms of Betti numbers.
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1. Introduction

1.1. Statement of results

In 1954, Hirzebruch [6, Problem 31] asked which linear combinations of Chern numbers of smooth complex projective varieties are topologically invariant. The purpose of this paper is to provide complete answers to this question. Of course, the answers depend on what exactly one means by topological invariance.

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Since the manifold underlying a complex-algebraic variety has a preferred orientation, it is most natural to examine which linear combinations of Chern numbers are invariant under orientation-preserving homeo- or diffeomorphisms. The answer is given by:

Theorem 1. *A rational linear combination of Chern numbers is an oriented diffeomorphism invariant of smooth complex projective varieties if and only if it is a linear combination of the Euler and Pontryagin numbers.*

In one direction, the Euler number, which is the top Chern number, is of course a homotopy invariant. Further, the Pontryagin numbers, which are special linear combinations of Chern numbers, are oriented diffeomorphism invariants. In fact, Novikov [21] proved that the Pontryagin numbers are also invariant under orientation-preserving homeomorphisms, and so Theorem 1 is unchanged if we replace oriented diffeomorphism-invariance by oriented homeomorphism-invariance. The other direction, proving that there are no other linear combinations that are oriented diffeomorphism-invariants, has proved to be quite difficult because of the scarcity of examples of diffeomorphic projective varieties with distinct Chern numbers.

Given Theorem 1, and the fact that Pontryagin numbers depend on the orientation, one might expect that only the Euler number is invariant under homeo- or diffeomorphisms that do not necessarily preserve the orientation. For diffeomorphisms this is almost but not quite true:

Theorem 2. *In complex dimension $n \geqslant 3$ a rational linear combination of Chern numbers is a diffeomorphism invariant of smooth complex projective varieties if and only if it is a multiple of the Euler number c_n. In complex dimension 2 both Chern numbers c_2 and c_1^2 are diffeomorphism-invariants of complex projective surfaces.*

The statement about complex dimension 2 is a consequence of Seiberg–Witten theory and was first proved in [12, Theorem 2]; see also [13, Theorem 1]. It is an exception due to the special nature of four-dimensional differential topology. The exception disappears if we consider homeomorphisms instead of diffeomorphisms:

Theorem 3. *A rational linear combination of Chern numbers is a homeomorphism invariant of smooth complex projective varieties if and only if it is a multiple of the Euler number.*

These theorems show that linear combinations of Chern numbers of complex projective varieties are not usually determined by the underlying manifold. This motivates the investigation of a modification of Hirzebruch's original problem, asking how far this indeterminacy goes. More precisely, we would like to know which linear combinations of Chern numbers are determined up to finite ambiguity by the topology. The obvious examples for which this is true are the numbers

$$\chi_p = \chi\left(\Omega^p\right) = \sum_{q=0}^{n} (-1)^q h^{p,q},$$

where n is the complex dimension. By the Hirzebruch–Riemann–Roch theorem [7] these are indeed linear combinations of Chern numbers. By their very definition, together with the Hodge decomposition of the cohomology, the numbers χ_p are bounded above and below by linear combinations of Betti numbers. It turns out that this property characterizes the linear combinations of the χ_p, as shown by the following theorem:

Theorem 4. *A rational linear combination of Chern numbers of smooth complex projective varieties can be bounded in terms of Betti numbers if and only if it is a linear combination of the χ_p.*

As a consequence of this result, most linear combinations of Chern numbers are independent of the Hodge structure:

Corollary 5. *A rational linear combination of Chern numbers of smooth complex projective varieties is determined by the Hodge numbers if and only if it is a linear combination of the χ_p.*

The span of the χ_p includes the Euler number $c_n = \sum_p (-1)^p \chi_p$ and the signature, which, according to the Hodge index theorem, equals $\sum_p \chi_p$. It also includes the Chern number $c_1 c_{n-1}$, by a result of Libgober and Wood [16, Theorem 3]. Nevertheless, the span of the χ_p is a very small subspace of the space of linear combinations of Chern numbers. The latter has dimension equal to $\pi(n)$, the number of partitions of n, which grows exponentially with n. The former has dimension $[(n + 2)/2]$, the integral part of $(n + 2)/2$. This follows from the symmetries of the Hodge decomposition, which imply $\chi_p = (-1)^n \chi_{n-p}$, together with the fact that $\chi_0, \ldots, \chi_{[n/2]}$ are linearly independent, as can be easily checked by evaluating on products of projective spaces.

Hirzebruch's problem is of central importance in the applications of the Hirzebruch–Riemann–Roch formula, for example in the classification theory of algebraic varieties. The right-hand side of HRR is given by the χ_p, and, by the results of this paper, this right-hand side is not usually a topological invariant. In fact, as explained in [14, Theorem 6], the only linear combinations of the χ_p that are also linear combinations of Euler and Pontryagin numbers are the multiples of the signature.

1.2. History and outline

The first example of a pair of diffeomorphic projective varieties with distinct Chern numbers was found by Borel and Hirzebruch [1] in 1959. Using Lie theory, they showed that the homogeneous space $SU(4)/S(U(2) \times U(1) \times U(1))$ has two invariant structures as a Hodge manifold with different values for the Chern number c_1^5. At the time this may well have seemed to be some sort of isolated curiosity. It was only comparatively recently that Calabi and Hirzebruch [9] clarified the geometric meaning of the example by identifying the two structures as the projectivised holomorphic tangent and cotangent bundles of $\mathbb{C}P^3$. Terzić and I [15] generalized this example to $SU(n + 2)/S(U(n) \times U(1) \times U(1))$, where the two structures correspond to the projectivised holomorphic tangent and cotangent bundles of $\mathbb{C}P^{n+1}$. For all $n \geqslant 2$, the Chern numbers c_1^{2n+1}, and many others, are different for the two structures, although the underlying smooth manifold is the same, and the Hodge numbers agree [15].

Over the years a number of attempts have been made to find further examples of diffeomorphic varieties with distinct Chern numbers. However, until recently they were all unsuccessful.

It turns out that a key ingredient for the proofs of Theorems 1, 2 and 3, and thus for the solution of Hirzebruch's problem, is the existence of homeomorphic simply connected algebraic surfaces with different signatures, equivalently with different c_1^2. The existence of such pairs of surfaces, which I proved in [11], depends on deep results in both surface geography and four-dimensional topology. Of course, the homeomorphism in question cannot preserve the orientation. By the

main result of [12], the homeomorphism is not smoothable. In Theorem 11 below we recall the main result of [11] in the form we shall use it here.

In [13], I used the examples from [11] as building blocks to manufacture certain pairs of diffeomorphic three-folds and four-folds, for which I could compute all the Chern numbers explicitly. Although this was sufficient to prove Theorem 1 in complex dimensions $\leqslant 4$, it is clear that such a brute force approach cannot work in general. In this paper we use cobordism theory to minimize the need for concrete calculations. This approach is inspired by the work of Kahn [10], who used a similar strategy to prove the analogue of Theorem 1 for almost complex manifolds in place of algebraic varieties. Of course the implementation of the strategy is a lot easier in that case, because almost complex structures are much more flexible and exist in abundance. Essentially we shall have to find only one pair of examples with specific properties in each dimension. In complex dimension n the examples will be algebraic $\mathbb{C}P^{n-2}$-bundles over the orientation-reversingly homeomorphic algebraic surfaces found in [11].

The proof of Theorem 4 is independent of the results of [11], but, like the proofs of the other main theorems, it also uses cobordism theory. However, the way in which cobordism theory is used in that proof is new; there is no parallel for that argument in earlier work. The results of the present paper were announced in [14], which also contains some related observations.

2. Complex cobordism

In the proofs of the main theorems we shall make use of the rational complex cobordism ring $\Omega_\star^U \otimes \mathbb{Q}$. (We use the classical terminology, calling this a *cobordism ring*, rather than the newer convention referring to $\Omega_\star^U \otimes \mathbb{Q}$ as a *bordism ring*.) This ring is a polynomial ring with one generator β_i in every even degree $2i$; see [17,20,27] and also [18,2]. Two stably almost complex manifolds of the same dimension have the same Chern numbers if and only if they represent the same class in $\Omega_\star^U \otimes \mathbb{Q}$. Equivalently, in complex dimension i the vector space of rational linear combinations of Chern numbers is the dual space of $\Omega_{2i}^U \otimes \mathbb{Q}$.

We define three ideals in $\Omega_\star^U \otimes \mathbb{Q}$, each one corresponding to the situation considered in one of the first three main theorems. The first definition is:

Definition 6. Let $\mathcal{D}O_{2i} \subset \Omega_{2i}^U \otimes \mathbb{Q}$ be the linear subspace spanned by the set

$$\left\{ [M] - [N] \in \Omega_{2i}^U \otimes \mathbb{Q} \mid M, N \text{ orientation-preservingly diffeomorphic projective varieties} \right\}$$

of differences of complex projective varieties of complex dimension i that are orientation-preservingly diffeomorphic.

Clearly the direct sum of the $\mathcal{D}O_{2i}$ is an ideal $\mathcal{D}O_\star$ in the ring $\Omega_\star^U \otimes \mathbb{Q}$. We shall prove the following:

Theorem 7. *There is a sequence of ring generators $\beta_1, \beta_2, \beta_3, \ldots$ for the rational complex cobordism ring $\Omega_\star^U \otimes \mathbb{Q}$ with the property that for each odd index $i \geqslant 3$ the elements $\beta_1 \cdot \beta_{i-1}$ and β_i are contained in $\mathcal{D}O_{2i}$.*

We now explain how this result implies Theorem 1. The vector space of rational linear combinations of Chern numbers that are oriented diffeomorphism-invariants of complex projective varieties of complex dimension i is the annihilator of the subspace $\mathcal{D}O_{2i} \subset \Omega_{2i}^U$. Therefore, its

dimension equals the codimension of $\mathcal{D}O_{2i}$ in Ω^U_{2i}. Now Theorem 7 implies that a monomial of degree i in the β_j is in $\mathcal{D}O_{2i}$ if it contains a β_j with j odd and $j > 1$, or if it contains β_1 and a β_j with j even. If i is odd, then the only monomial that is not obviously in $\mathcal{D}O_{2i}$ is β_1^i. This means that the codimension of $\mathcal{D}O_{2i}$ in $\Omega^U_{2i} \otimes \mathbb{Q}$ is at most one, and so the only linear combinations of Chern numbers invariant under orientation-preserving diffeomorphism are the multiples of the Euler number c_i. If i is even, then, in addition to β_1^i, all the monomials containing only β_{j_l} with all j_l even may be outside $\mathcal{D}O_{2i}$. The number of these other monomials is exactly $\pi(i/2)$, the number of partitions of $i/2$. This is also the number of Pontryagin numbers in dimension $2i$. This completes the deduction of Theorem 1 from Theorem 7.

To formulate cobordism statements that will imply Theorems 2 and 3 we require the following:

Definition 8. Let $\mathcal{D}_{2i} \subset \Omega^U_{2i} \otimes \mathbb{Q}$ be the linear subspace spanned by the set

$$\left\{ [M] - [N] \in \Omega^U_{2i} \otimes \mathbb{Q} \,\middle|\, M, N \text{ diffeomorphic projective varieties} \right\}$$

of differences of complex projective varieties of complex dimension i that are diffeomorphic, and $\mathcal{H}_{2i} \subset \Omega^U_{2i} \otimes \mathbb{Q}$ be the linear subspace spanned by the set

$$\left\{ [M] - [N] \in \Omega^U_{2i} \otimes \mathbb{Q} \,\middle|\, M, N \text{ homeomorphic projective varieties} \right\}$$

of differences of complex projective varieties of complex dimension i that are homeomorphic, in both cases without any assumption about orientations.

Again the direct sums of the $\mathcal{D}_{2i}$, respectively of the $\mathcal{H}_{2i}$, define an ideal $\mathcal{D}_\star$, respectively $\mathcal{H}_\star$, in the ring $\Omega^U_\star \otimes \mathbb{Q}$. We shall prove:

Theorem 9. *There is a sequence of ring generators $\beta_1, \beta_2, \beta_3, \ldots$ for the rational complex cobordism ring $\Omega^U_\star \otimes \mathbb{Q}$ with the properties $\beta_1 \cdot \beta_2 \in \mathcal{D}_6$, $\beta_2 \cdot \beta_2 \in \mathcal{D}_8$ and $\beta_i \in \mathcal{D}_{2i}$ for all $i \geqslant 3$.*

Theorem 10. *There is a sequence of ring generators $\beta_1, \beta_2, \beta_3, \ldots$ for the rational complex cobordism ring $\Omega^U_\star \otimes \mathbb{Q}$ with the property $\beta_i \in \mathcal{H}_{2i}$ for all $i \geqslant 2$.*

Note that, because the Chern numbers of algebraic surfaces are diffeomorphism-invariant [12,13], it is not possible to choose the generator β_2 to be in $\mathcal{D}_4$.

Dimension counts similar to the one above show that Theorem 3 and the case of complex dimension $\geqslant 3$ in Theorem 2 follow from Theorem 10 and Theorem 9 respectively.

The proof that certain differences of projective varieties can be taken as generators uses Milnor's characterization of ring generators for $\Omega^U_\star \otimes \mathbb{Q}$. Let M be a closed almost complex manifold of real dimension $2k$ with total Chern class

$$c(TM) = \prod_i (1 + x_i).$$

Then, following Thom [26], one defines the number $s_k(M)$ as

$$s_k(M) = \sum_i \langle x_i^k, [M] \rangle.$$

By the splitting principle s_k is a linear combination of Chern numbers. Milnor [17,27] proved that one can take the generator β_k in degree $2k$ to be $[M]$ if and only if $s_k(M) \neq 0$.

The proof of Theorem 4 will also use special sequences of ring generators for $\Omega_\star^U \otimes \mathbb{Q}$. For that proof we will choose generators belonging to families on which certain linear combinations of Chern numbers are unbounded, although the Betti numbers are fixed.

3. Some algebraic surfaces with useful properties

The following theorem is the starting point for the proofs of the first three main theorems of this paper.

Theorem 11. *(See [11].) There exist pairs (X, Y) of simply connected complex projective surfaces of non-zero signature, which are orientation-reversingly homeomorphic with respect to the orientations defined by their complex structures.*

Moreover, one may take X to be the four-fold blowup $X' \# \overline{\mathbb{C}P^2} \# \overline{\mathbb{C}P^2} \# \overline{\mathbb{C}P^2} \# \overline{\mathbb{C}P^2}$ of some other surface X'.

The main theorem of [11] in fact provides infinitely many such pairs, at least if one does not insist on the property that X be a four-fold blowup. The proof combines geography results due to Persson [22] in the case of negative signature and Chen [3] and Moishezon and Teicher [19] in the case of positive signature to find candidate pairs whose Chern numbers are related by

$$c_2(Y) = c_2(X), \tag{1}$$

$$c_1^2(Y) = 4c_2(X) - c_1^2(X). \tag{2}$$

These equations are equivalent to requiring that X and Y have the same Euler characteristic and have opposite signatures. As soon as both surfaces are simply connected and non-spin, they are orientation-reversingly homeomorphic by Freedman's classification result for simply connected four-manifolds [4]. The geography results used are flexible enough to allow one to make X non-minimal and, in fact a four-fold blowup. In [11, Theorem 3.10] such a result was stated for double blowups, and the case of four-fold blowups is exactly the same.

For the construction of diffeomorphic projective varieties of higher dimension, whose differences will be used as generators of the rational complex cobordism ring, we shall need the following.

Lemma 12. *Let (X, Y) be a pair of algebraic surfaces as in Theorem 11. Then for every $n \geqslant 1$ there exist holomorphic vector bundles $\mathcal{E} \to X$ and $\mathcal{F} \to Y$ of rank $n + 1$ with trivial first Chern classes and*

$$\langle c_2(\mathcal{E}), [X] \rangle = -\langle c_2(\mathcal{F}), [Y] \rangle \neq 0.$$

Proof. It is enough to prove the case $n = 1$, for one can then stabilize by direct summing with trivial line bundles.

For $n = 1$ we take $\mathcal{F} = \mathcal{O}(K_Y) \oplus \mathcal{O}(-K_Y)$, where K_Y is a canonical divisor of Y. Then $\mathcal{F}$ has trivial first Chern class and

$$\langle c_2(\mathcal{F}), [Y] \rangle = (K_Y) \cdot (-K_Y) = -c_1^2(Y) < 0,$$

where the final inequality follows, for example, from (2) and the Miyaoka–Yau inequality $c_1^2(X) \leqslant 3c_2(X)$ using $c_2(X) > 0$. To prove the lemma we now have to find a holomorphic rank two bundle $\mathcal{E} \to X$ with trivial first Chern class and with

$$\langle c_2(\mathcal{E}), [X] \rangle = -\langle c_2(\mathcal{F}), [Y] \rangle = c_1^2(Y) > 0.$$

Let E_i be the exceptional divisors in X. Since every positive integer is a sum of four squares, we can find integers a_i such that the divisor

$$D = a_1 E_1 + a_2 E_2 + a_3 E_3 + a_4 E_4$$

satisfies

$$D^2 = -a_1^2 - a_2^2 - a_3^2 - a_4^2 = -c_1^2(Y).$$

Now $\mathcal{E} = \mathcal{O}(D) \oplus \mathcal{O}(-D)$ has the desired property. $\quad\square$

We shall take projectivisations of such holomorphic bundles to construct high-dimensional examples.

Remark 13. The holomorphic bundles constructed in the lemma may seem rather arbitrary. For $n \geqslant 3$ one can take instead the stabilized direct sums of the holomorphic tangent and cotangent bundles $\mathcal{E} = TX \oplus T^*X \oplus \mathcal{O}_X^{\oplus(n-3)}$ and $\mathcal{F} = TY \oplus T^*Y \oplus \mathcal{O}_Y^{\oplus(n-3)}$. These bundles are in some sense canonical, and have the nice property that their second Chern number is a universal multiple of the signature. Moreover, one can use them without arranging one of the surfaces to be a four-fold blowup. But then one would still need other examples for $n = 1$ and 2.

4. Projective space bundles over algebraic surfaces

Let S be a smooth complex projective surface, and $\mathcal{E} \to S$ a holomorphic vector bundle of rank $n + 1 \geqslant 2$. Then the projectivisation $\mathbb{P}(\mathcal{E})$ is a $\mathbb{C}P^n$-bundle over S with an induced complex structure on the total space. We now calculate the Thom–Milnor number $s_{n+2}(\mathbb{P}(\mathcal{E}))$ for certain special choices of $\mathcal{E}$.

The cohomology ring of $\mathbb{P}(\mathcal{E})$ is described by the following consequence of the Leray–Hirsch theorem and of Grothendieck's definition of Chern classes [5]:

Proposition 14. *The integral cohomology ring of $\mathbb{P}(\mathcal{E})$ is generated as an $H^*(S)$-module by a class $y \in H^2(\mathbb{P}(\mathcal{E}))$ that restricts to every $\mathbb{C}P^n$-fibre as a generator, subject to the relation*

$$y^{n+1} + c_1(\mathcal{E}) y^n + c_2(\mathcal{E}) y^{n-1} = 0.$$

The next result provides the only Chern number calculation required for the proofs of our main theorems.

Proposition 15. *Let $\mathcal{E} \to S$ be a holomorphic vector bundle of rank $n + 1$ with $c_1(\mathcal{E}) = 0$ and $\langle c_2(\mathcal{E}), [S] \rangle = c$. Then $s_{n+2}(\mathbb{P}(\mathcal{E})) = -(n + 1)(n + 3) \cdot c$.*

Proof. Let

$$c(S) = (1 + x_1)(1 + x_2),$$

$$c(\mathcal{E}) = (1 + y_1) \cdot \ldots \cdot (1 + y_{n+1})$$

be formal factorizations of the total Chern classes in the sense of the splitting principle.

Since the projection $\pi : \mathbb{P}(\mathcal{E}) \to S$ is holomorphic, we can use the Whitney sum formula to conclude

$$c\big(\mathbb{P}(\mathcal{E})\big) = c(T\pi) \cdot \pi^* c(S),$$

where $T\pi$ is tangent bundle along the fibers of π. In what follows we shall suppress the cohomological pullback π^* in the notation.

To compute $c(T\pi)$ we use the tautological exact sequence

$$1 \to \mathcal{L}^{-1} \to \pi^* \mathcal{E} \to \mathcal{L}^{-1} \otimes T\pi \to 1,$$

where $\mathcal{L}$ is the fiberwise hyperplane bundle on the total space. Tensoring with $\mathcal{L}$, we conclude

$$c(T\pi) = c\big(\mathcal{L} \otimes \pi^* \mathcal{E}\big) = (1 + y + y_1) \cdot \ldots \cdot (1 + y + y_{n+1}),$$

where the last equality comes from $c_1(\mathcal{L}) = y$. Thus

$$c\big(\mathbb{P}(\mathcal{E})\big) = (1 + y + y_1) \cdot \ldots \cdot (1 + y + y_{n+1})(1 + x_1)(1 + x_2).$$

Now to calculate $s_{n+2}(\mathbb{P}(\mathcal{E}))$ we have to evaluate the following expression in the Chern roots on the fundamental class of $\mathbb{P}(\mathcal{E})$:

$$(y_1 + y)^{n+2} + \ldots + (y_{n+1} + y)^{n+2} + x_1^{n+2} + x_2^{n+2}.$$

Since $n + 2 \geqslant 3$ and symmetric functions of the x_i, respectively of the y_j, vanish in degrees $\geqslant 3$, this expression equals

$$(n + 1)y^{n+2} + (n + 2)y^{n+1} \sum_j y_j + \binom{n + 2}{2} y^n \sum_j y_j^2.$$

In the first summand we may, by Proposition 14, substitute $y^{n+2} = -c_2(\mathcal{E})y^n$. The second sum-

mand vanishes since $\sum_j y_j = c_1(\mathcal{E}) = 0$. Finally, in the last summand we use

$$\sum_j y_j^2 = \left(c_1^2 - 2c_2\right)(\mathcal{E}) = -2c_2(\mathcal{E}).$$

This gives us

$$(n+1)y^{n+2} + (n+2)y^{n+1} \sum_j y_j + \binom{n+2}{2} y^n \sum_j y_j^2 = -(n+1)(n+3)c_2(\mathcal{E})y^n.$$

Now the conclusion follows since y^n evaluates as $+1$ on the fibre of π. $\square$

Remark 16. The results of this section are purely topological, and hold in greater generality, assuming only that S is an almost-complex four-manifold and E is some complex vector bundle.

5. Proofs of the main theorems

In this section we first prove Theorems 7, 9 and 10, which in turn imply Theorems 1, 2 and 3 stated in the introduction. For the proofs we fix a pair of complex projective surfaces X and Y as in Theorem 11. They are simply connected with non-zero signature, and they are orientation-reversingly homeomorphic. For every $n \geqslant 1$ we consider holomorphic bundles $\mathcal{E}_n \to X$ and $\mathcal{F}_n \to Y$ of rank $n+1$ with trivial first Chern classes and with opposite second Chern numbers, compare Lemma 12.

For $k \geqslant 3$ let $X_k = \mathbb{P}(\mathcal{E}_{k-2})$ and $Y_k = \mathbb{P}(\mathcal{F}_{k-2})$. Since X and Y are projective-algebraic the holomorphic bundles $\mathcal{E}_{k-2}$ and $\mathcal{F}_{k-2}$ and their projectivisations X_k and Y_k are algebraic as well by Serre's GAGA principle [24].

We now define the generators for the rational complex cobordism ring that we shall use for the proofs of the first three main theorems.

Definition 17. Let $\beta_1 = [\mathbb{C}P^1]$, $\beta_2 = [X] - [Y]$ and $\beta_k = [X_k] - [Y_k]$ for $k \geqslant 3$.

First of all, these are indeed generators.

Proposition 18. *The elements $\beta_1 = [\mathbb{C}P^1]$, $\beta_2 = [X] - [Y]$ and $\beta_k = [X_k] - [Y_k]$ for $k \geqslant 3$ form ring generators for $\Omega_\star^U \otimes \mathbb{Q}$.*

Proof. By the result of Milnor [17,27] mentioned earlier we only have to check that $s_k(\beta_k) \neq 0$ for all k. Clearly there is nothing to prove for $k = 1$.

For $k = 2$ we have

$$s_2(\beta_2) = s_2(X) - s_2(Y) = \left(c_1^2 - 2c_2\right)(X) - \left(c_1^2 - 2c_2\right)(Y) = 3\left(\sigma(X) - \sigma(Y)\right) = 6\sigma(X) \neq 0,$$

where σ denotes the signature. We have used the fact that X and Y have opposite signatures since they are orientation-reversingly homeomorphic, and that these signatures are non-zero.

For $k \geqslant 3$ Proposition 15 and Lemma 12 give

$$s_k(\beta_k) = s_k(X_k) - s_k(Y_k) = -(k-1)(k+1)\big(\langle c_2(\mathcal{E}), [X] \rangle - \langle c_2(\mathcal{F}), [Y] \rangle\big)$$
$$= -2\big(k^2 - 1\big)\langle c_2(\mathcal{E}), [X] \rangle \neq 0.$$

This completes the proof.   □

Now we can prove the first three main theorems.

Proof of Theorem 9. Denote by $\bar{Y}$ the smooth manifold underlying Y equipped with the orientation that is opposite to the one defined by the complex structure. By a result of Wall [28], the four-manifolds X and $\bar{Y}$ are smoothly h-cobordant. Let W be any h-cobordism between them.

The product $\mathbb{C}P^1 \times W$ is a 7-dimensional h-cobordism between $\mathbb{C}P^1 \times X$ and $\mathbb{C}P^1 \times \bar{Y}$. By Smale's h-cobordism theorem [25] these two manifolds are diffeomorphic. This shows that $\beta_1 \cdot \beta_2 \in \mathcal{D}_6$.

Similarly $X \times W$ is an h-cobordism between $X \times X$ and $X \times \bar{Y}$, and $\bar{Y} \times W$ is an h-cobordism between $\bar{Y} \times X$ and $\bar{Y} \times \bar{Y}$. Therefore, by the h-cobordism theorem, $X \times X$, $X \times \bar{Y} = \bar{Y} \times X$ and $\bar{Y} \times \bar{Y}$ are all diffeomorphic. Thus $\beta_2 \cdot \beta_2 \in \mathcal{D}_8$.

As the inclusion of X into W is a homotopy equivalence, the complex vector bundle E_{k-2} underlying the holomorphic bundle $\mathcal{E}_{k-2}$ has a unique extension $\mathbb{E}_{k-2} \to W$. The restriction of $\mathbb{E}_{k-2}$ to $\bar{Y}$ is isomorphic to the complex vector bundle F_{k-2} underlying $\mathcal{F}_{k-2}$, if we think of F_{k-2} as a bundle over $\bar{Y}$ rather than over Y. This follows from the conditions imposed on $\mathcal{E}_{k-2}$ and $\mathcal{F}_{k-2}$ in Lemma 12, since they imply that the two complex vector bundles have the same Chern classes. A direct obstruction theory argument then shows that they are isomorphic, a conclusion that can also be reached by appealing to the work of Peterson [23]. Now the projectivisation $\mathbb{P}(\mathbb{E}_{k-2})$ is an h-cobordism between X_k and $\bar{Y}_k$. Applying Smale's h-cobordism theorem [25] again, we conclude $\beta_k \in \mathcal{D}_{2k}$ for $k \geqslant 3$.   □

Proof of Theorem 10. We have $\beta_2 \in \mathcal{H}_4$ by the assumption that X and Y are homeomorphic, and $\beta_k \in \mathcal{D}_{2k} \subset \mathcal{H}_{2k}$ for $k \geqslant 3$ by the proof of Theorem 9. This is all we had to do to prove Theorem 10.   □

Proof of Theorem 7. The only thing left to discuss is orientations.

In the proof of Theorem 9 we have shown that X_k and Y_k are orientation-reversingly diffeomorphic. If the complex dimension k is odd, then the complex structure on Y_k that is the complex conjugate of the given one induces the opposite orientation $\bar{Y}_k$. This conjugate structure has the same Chern numbers and is also projective algebraic. Thus, if we complex conjugate the complex structure of Y_k, but not the one on X_k, then the modified β_k has the property $\beta_k \in \mathcal{D}O_{2k}$ for k odd.

We showed in the proof of Theorem 9 that $\mathbb{C}P^1 \times X$ and $\mathbb{C}P^1 \times \bar{Y}$ are orientation-preservingly diffeomorphic. As $\mathbb{C}P^1$ admits orientation-reversing self-diffeomorphisms, it follows that $\mathbb{C}P^1 \times X$ and $\mathbb{C}P^1 \times Y$ are also orientation-preservingly diffeomorphic. This means that $\beta_1 \cdot \beta_2 \in \mathcal{D}O_6$. The same argument shows that $\beta_1 \cdot \beta_{k-1} \in \mathcal{D}O_{2k}$ for all $k \geqslant 3$.   □

It remains to prove Theorem 4 from the introduction. For this we shall use the following notation. The Hirzebruch χ_y-genus combines all the χ_p into the polynomial

$$\chi_y = \sum_{p=0}^{n} \chi_p y^p.$$

The compatibility of the Hodge and Künneth decompositions for the cohomology of a product shows that χ_y defines a ring homomorphism

$$\chi_y : \Omega_\star^U \otimes \mathbb{Q} \to \mathbb{Q}[y],$$

whose kernel is an ideal $\mathcal{I}_\star \subset \Omega_\star^U \otimes \mathbb{Q}$.

Proof of Theorem 4. In complex dimension 1 there is nothing to prove. In dimension 2 both Chern numbers are linear combinations of $\chi_0 = \frac{1}{12}(c_1^2 + c_2)$ and $\chi_1 = \frac{1}{6}(c_1^2 - 5c_2)$, so again there is nothing to prove. For the rest of the proof we work in dimension $i \geqslant 3$. In these dimensions we have to prove that a linear combination of Chern numbers that is not a linear combination of the χ_p is unbounded on some complex projective varieties with bounded Betti numbers. For this we consider the rational complex cobordism ring $\Omega_\star^U \otimes \mathbb{Q}$ and choose a convenient generating sequence γ_i by setting $\gamma_1 = [\mathbb{C}P^1]$, $\gamma_2 = [\mathbb{C}P^2]$ and $\gamma_i = [\mathbb{P}(\mathcal{E}_c)]$ for $i \geqslant 3$, where $\mathcal{E}_c \to A$ is a holomorphic bundle of rank $i - 1$ over an Abelian surface A with $c_1(\mathcal{E}_c) = 0$ and with second Chern number $c \neq 0$. Proposition 15 shows that this is indeed a sequence of ring generators.

For definiteness we may take $\mathcal{E}_c = \mathcal{H} \oplus \mathcal{H}^{-1} \oplus \mathcal{O}_A^{\oplus(i-3)}$, where $\mathcal{H}$ is an ample line bundle. Then

$$c = \big(c_1(\mathcal{H})\big) \cdot \big(-c_1(\mathcal{H})\big) = -c_1^2(\mathcal{H}) < 0.$$

The subspace $\mathcal{I}_{2i} \subset \Omega_{2i}^U \otimes \mathbb{Q}$ is the intersection of the kernels of all the χ_p and is therefore of codimension $[(i+2)/2]$. A $\mathbb{Q}$-vector space basis for $\Omega_{2i}^U \otimes \mathbb{Q}$ is given by the elements $\gamma_I = \gamma_{i_1} \cdot \cdots \cdot \gamma_{i_l}$, where $I = (i_1, \ldots, i_l)$ ranges over all partitions of i. Among these basis vectors there are $[(i+2)/2]$ many corresponding to partitions with all $i_j \leqslant 2$, and these are clearly not contained in $\mathcal{I}_{2i}$. However, all the other basis vectors are in the subspace $\mathcal{I}_{2i}$. To check this it is enough to check that each $[\mathbb{P}(\mathcal{E}_c)]$ is in $\mathcal{I}_{2i}$. This follows directly from looking at the Hodge decomposition of the cohomology of $\mathbb{P}(\mathcal{E}_c)$. This cohomology is given by Proposition 14, with the class y being of type $(1,1)$. Here we use the assumption that the base of $\mathcal{E}_c$ is an Abelian surface A, with $\chi_p(A) = 0$ for all p.

It follows by counting dimensions that the γ_I corresponding to partitions I containing an $i_j \geqslant 3$ form a vector space basis of $\mathcal{I}_{2i}$.

Let $f : \Omega_{2i}^U \otimes \mathbb{Q} \to \mathbb{Q}$ be any linear combination of Chern numbers. If f is not a linear combination of the χ_p, then $\ker(f) \cap \mathcal{I}_{2i}$ is a proper subspace of $\mathcal{I}_{2i}$. It follows that at least one of the γ_I with I containing an $i_j \geqslant 3$ is not in $\ker(f)$, i.e. $f(\gamma_I) \neq 0$ for this particular I. The projective variety representing γ_I fibers holomorphically over the Abelian surface A. Pulling back under a finite covering $A' \to A$ of degree d we obtain a complex projective variety on which f evaluates as $d \cdot f(\gamma_I)$, which is unbounded as we increase d. However, the Betti numbers of these coverings are bounded. This completes the proof. $\quad\square$

Proposition 15 shows that the number s_i is unbounded on complex projective varieties of dimension i with fixed Betti numbers. As another concrete instance for the unboundedness of Chern numbers we prove:

Proposition 19. *In every complex dimension $i \geqslant 3$ there are sequences of complex projective varieties with bounded Betti numbers on which the Chern number c_1^i is unbounded.*

Proof. We consider the same $\mathbb{P}(\mathcal{E}_c)$ with $c \neq 0$ as in the proof of Theorem 4. The formula for the total Chern class from the proof of Proposition 15 shows that $c_1(\mathbb{P}(\mathcal{E}_c)) = (i-1)y$. Thus the relation $y^i = -c_2(\mathcal{E}_c)y^{i-2}$ from Proposition 14 gives

$$\left\langle c_1^i\big(\mathbb{P}(\mathcal{E}_c)\big), \big[\mathbb{P}(\mathcal{E})\big]\right\rangle = (i-1)^i \left\langle y^i, \big[\mathbb{P}(\mathcal{E}_c)\big]\right\rangle = -(i-1)^i \left\langle c_2(\mathcal{E}_c), [A]\right\rangle = -(i-1)^i \cdot c \neq 0.$$

Pulling back to finite coverings of A shows that c_1^i is unbounded on these coverings, although their Betti numbers are bounded. $\square$

Remark 20. I proved in [13, Proposition 2] that in odd complex dimensions > 1 the Hirzebruch–Riemann–Roch formulae for χ_p do not involve c_1^i. This shows of course that c_1^i is not contained in the span of the χ_p. However, in even complex dimensions the Todd genus expressing χ_0 does contain c_1^i, so that, without Proposition 19 and Theorem 4, one would not know that c_1^i is not contained in the span of the χ_p.

Acknowledgments

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