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ON THE DIMENSION OF PARTIALLY ORDERED SETS*

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We study the topic of the title in some detail. The main results are summarized in the first four paragraphs of this paper.

The *dimension* [4] of a partially ordered set (poset) is the minimum number of linear orders whose intersection is the ordering of the poset. For an integer $d \geqslant 2$, a poset is *d-irreducible* [13] if it has dimension d and removal of any element lowers its dimension; calling a poset *irreducible* means it is d-irreducible for some $d \geqslant 2$. (Irreducible posets are finite and the dimension of any finite poset is finite.)

In Section 2, we show that planar posets have arbitrary finite dimension. In Section 3, we present two new families of irreducible posets and show that finite dismantlable lattices have arbitrary finite dimension.

We introduce the *dimension product* construction in Section 4. In Section 6, we show that $P \otimes \mathbf{2}$, the dimension product of a 3-irreducible poset P and a 2-element chain, is 4-irreducible. (The complete list of 3-irreducible posets is given in Kelly [6] or Trotter and Moore [17].) Using the dimension product, we construct, for any $d \geqslant 3$ and $l \geqslant 1$, a d-irreducible poset of length l, answering Problem 3 of Trotter [14].

A d-irreducible poset P has the *embedding property* iff for any integer $n > d$, there is an n-irreducible poset containing P as a subposet. The unique 2-irreducible poset obviously has the embedding property. Theorem 4.9 shows that every 3-irreducible poset has the embedding property, as do the irreducible posets we introduce in Section 3.

1. Preliminaries

For a poset P, the pair $\langle a, b\rangle \in P^2$ is called a *critical pair* iff $a \parallel b$, $x < b$ implies $x < a$, and $x > a$ implies $x > b$. (Such a pair is also called "nonforcing".) All the results of this section are elementary or trivial extensions of known results.

Lemma 1.1. *If a and b are incomparable elements of a finite poset P, then there is a critical pair $\langle a_1, b_1\rangle$ for P with $a \leqslant a_1$ and $b_1 \leqslant b$.*

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Proof. First, choose a_1 maximal such that $a \leq a_1$ and $a_1 \parallel b$; then, choose b_1 minimal such that $b_1 \leq b$ and $a_1 \parallel b_1$.

Henceforth, we shall usually write a critical pair $\langle a, b\rangle$ as "$a < b$" and call it a *critical inequality*. (Note, however, that a critical inequality for a poset P is *not* an inequality that holds in P.) The set of all critical inequalities for a poset P is denoted by Crit(P).

A linear extension of a subposet of a poset P will be called a *partial linear extension* of P. The following lemma is a slight generalization of the well-known theorem of E. Szpilrajn [11].

Lemma 1.2. *For any partial linear extension C of a poset P, there is a linear extension C' of P that extends C.*

Proof. It is easily shown that the transitive closure of $C \cup P$ is an order relation which we denote by R. By Szpilrajn [11], there is a linear extension C' of R. Clearly, C' satisfies the conditions of the lemma.

We shall say that the partial linear extensions C_i $(i \in I)$ *realize* P when the ordering on P is $\bigcap (C_i' \mid i \in I)$ for any choice of linear extensions C_i' extending C_i $(i \in I)$. The dimension of a poset P is denoted by dim P. The following result will reduce the "bookkeeping" involved in calculating dimension.

Proposition 1.3. *Let $C_1, C_2, \ldots, C_n$ be partial linear extensions of a finite poset P. If each critical inequality for P holds in some C_i $(1 \leq i \leq n)$, then $C_1, C_2, \ldots, C_n$ realize P. In particular,* dim $P \leq n$.

Proof. Let C_i' be a linear extension of P that extends C_i for $1 \leq i \leq n$. Clearly, $P \subseteq C_1' \cap C_2' \cap \cdots \cap C_n'$ as order relations. Let $a \parallel b$ in P. It remains to show that $a < b$ in some C_i'. By Lemma 1.1, there is a critical pair $\langle a_1, b_1\rangle$ for P such that $a \leq a_1$ and $b_1 \leq b$. If $a_1 < b_1$ holds in C_i, then $a < b$ holds in C_i'.

Corollary 1.4. *The dimension of a finite poset P is the minimum (nonzero) number of partial linear extensions of P such that critical inequality for P holds in one of the partial linear extensions.*

Let P be a finite poset. An element a of P is *join-reducible* if $a = \bigvee S$ for some $S \subseteq P$ with $a \notin S$; otherwise a is *join-irreducible*. In particular, taking $S = \emptyset$, a smallest element (zero) is always join-reducible. $\mathbf{J}(P)$ denotes the set of all join-irreducible elements of P; dually, $\mathbf{M}(P)$ is the set of *meet-irreducible* elements of P. $\mathbf{P}(P) = \mathbf{J}(P) \cup \mathbf{M}(P)$, the set of *irreducible* elements; $\mathrm{Irr}(P) = \mathbf{J}(P) \cap \mathbf{M}(P)$, the set of *doubly irreducible* elements. ($\mathrm{Irr}(P)$ is not necessarily the set of elements with a unique lower and upper cover.)

Proposition 1.5. *For a finite poset P,* $\mathrm{Crit}(P) \subseteq \mathbf{M}(P) \times \mathbf{J}(P)$.

Proof. Let $\langle a, b\rangle \in \mathrm{Crit}(P)$ and suppose that $a = \bigwedge S$ with $a \notin S$. For all $x \in S$, $x > a$, and therefore, $x > b$. Consequently, $a = \bigwedge S \geqslant b$, a contradiction.

Corollary 1.6. *For a finite nontrivial poset P,* $\dim P = \dim \mathbf{P}(P)$.

Proof. By Lemma 1.1 and Proposition 1.5, $\mathrm{Crit}(\mathbf{P}(P)) = \mathrm{Crit}(P)$. Now apply Corollary 1.4.

Consequently, for any irreducible poset P, $P = \mathbf{P}(P)$; in other words, P contains no doubly reducible element.

The *completion* of a poset P, denoted by $\mathbf{L}(P)$, is also called the "completion by cuts" [3] or "MacNeille completion". P is a subposet of $\mathbf{L}(P)$. Recall that $\mathbf{J}(P) = \mathbf{J}(\mathbf{L}(P))$ and $\mathbf{M}(P) = \mathbf{M}(\mathbf{L}(P))$; thus, $\mathbf{P}(P) = \mathbf{P}(\mathbf{L}(P))$. Combining the last equality and Corollary 1.6, we obtain the following result for finite P.

Proposition 1.7 (Baker [1]). *For any poset P,* $\dim \mathbf{L}(P) = \dim P$.

Proof. Let $\mathscr{C} = (C_i \mid i \in I)$ be a family of linear extensions realizing P. We show that $\mathscr{C}$ realizes $L = \mathbf{L}(P)$. Let $a \parallel b$ in L. Since there are subsets A and B of P such that $a = \bigwedge A$ and $b = \bigvee B$, we can choose $x \in A$ and $y \in B$ such that $x \ngeqslant y$. Therefore, $x < y$ in C_i for some $i \in I$. In any linear extension C_i' of L that extends C_i, $a < b$ holds.

Henceforth, all posets will be finite.

The next results follows from the characterization of the completion given by B. Banaschewski [2, p. 123], and independently, by J. Schmidt [10, p. 246].

Lemma 1.8. *For any finite lattice L,* $\mathbf{L}(\mathbf{P}(L)) = L$.

2. Planar posets

A poset is *planar* if it is finite and its diagram can be drawn in the plane without any crossing of lines. For each positive integer n, we shall construct a planar poset P_n of dimension n. If a planar poset P contains both a zero and one, then P is a lattice and $\dim P \leqslant 2$. (The first part appears in [3, p. 32, ex. 7(a)] and is proved in [8, Corollary 2.4]. The second part was proved by K.A. Baker [1] and is a combination of results of J. Zilber [3, p. 32, ex. 7(c)] and B. Dushnik and E.W. Miller [4, Theorem 3.61].) If a planar poset P contains a zero, W.T. Trotter, Jr., and J.T. Moore, Jr. [16] showed that $\dim P \leqslant 3$.

We shall define the planar poset P_n as a subposet of the power set $\mathbf{2}^n$. Let

$Q_n(R_n)$ be the set of atoms (coatoms) of $\mathbf{2}^n$. Then $Q_n = \{\{i\} \mid 1 \leq i \leq n\}$. We set

$$P_n = Q_n \cup R_n \cup \{\{1, 2, \ldots, i\} \mid 2 \leq i \leq n-2\} \\ \cup \{\{i, i+1, \ldots, n\} \mid 3 \leq i \leq n-1\}.$$

Since $\mathbb{P}(P_n) = Q_n \cup R_n = \mathbb{P}(\mathbf{2}^n)$, dim $P_n = n$ by Corollary 1.6. Fig. 1 shows a planar diagram for P_6, where i denotes $\{i\}$ and i' denotes $\{j \mid 1 \leq j \leq n, j \neq i\}$ for $1 \leq i \leq 6$.

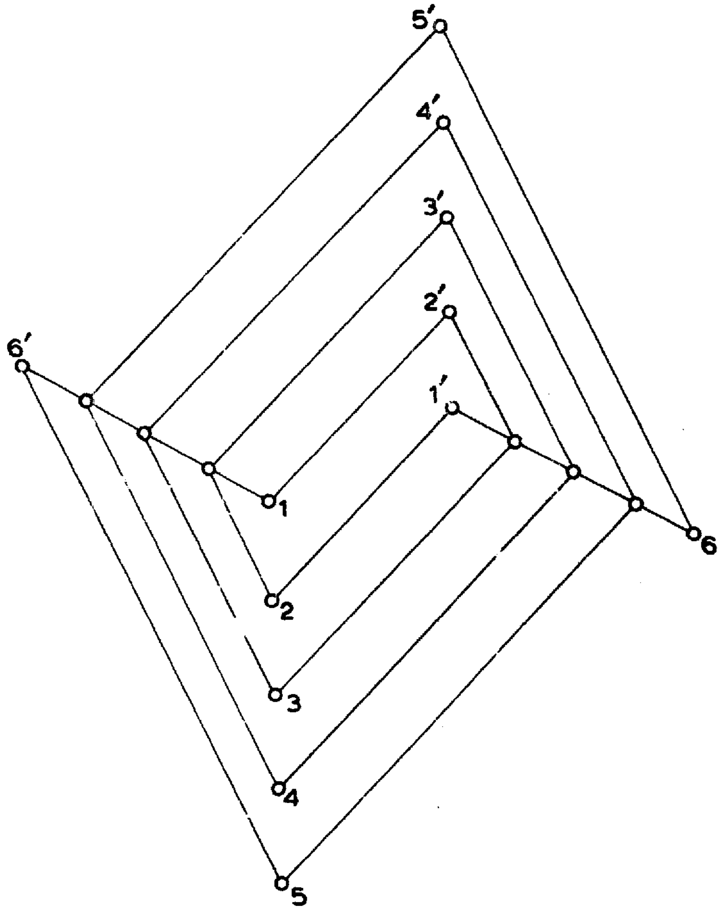

Fig. 1. A planar poset of dimension 6.

3. Two new families of irreducible posets

In this section, we shall define posets $P_{n,k}$ and $Q_{n,k}$, and show that $P_{n,k}$ ($Q_{n,k}$) is irreducible of dimension $2k$ $(2k-1)$ when n is suitably chosen. These posets will both be subposets of the lattices $L_{n,k}$ which we now define.

Let n and k be positive integers. The lattice

$$L_{n,k} = \{e^{\lambda}_{i,j} \mid 0 \leq i \leq j \leq k, 1 \leq \lambda \leq n\}$$

where $e^{\lambda}_{i,i} = d_i$ $(0 \leq i \leq k, 1 \leq \lambda \leq n)$ and all other elements with distinct indices are distinct. Thus, $|L_{n,k}| = \frac{1}{2}nk(k+1)+k+1$. The ordering is defined by:

$$e^{\lambda}_{i,j} \leq e^{\lambda}_{r,s} \quad \text{iff} \quad i \leq r \quad \text{and} \quad j \leq s;$$

$$\text{if} \quad \lambda \neq \mu, \quad \text{then} \quad e^{\lambda}_{i,j} \leq e^{\mu}_{r,s} \quad \text{iff} \quad j \leq r.$$

One "flap" of $L_{n,k}$ is shown in Fig. 2. (The flaps are "pasted together" at the d_i's to form $L_{n,k}$.) As indicated in Fig. 2, we set $a^{\lambda}_i = e^{\lambda}_{0,i}$, $b^{\lambda} = e^{\lambda}_{0,k}$ and $c^{\lambda}_j = e^{\lambda}_{j,k}$ for $1 \leq i \leq k$, $0 \leq j \leq k-1$ and $1 \leq \lambda \leq n$. K.A. Baker has observed that $L_{n,k} = M^{\mathbf{k}}_n$, an ordinal power, where M_n has the atoms $x_1, x_2, \ldots, x_n$ and $\mathbf{k}$ is a k-element chain. For example, b^{λ} is the function that maps all of $\mathbf{k}$ to x_{λ}. Consequently, each $L_{n,k}$ is in the modular lattice variety $\mathbf{M}_{\omega}$ generated by M_{ω}.

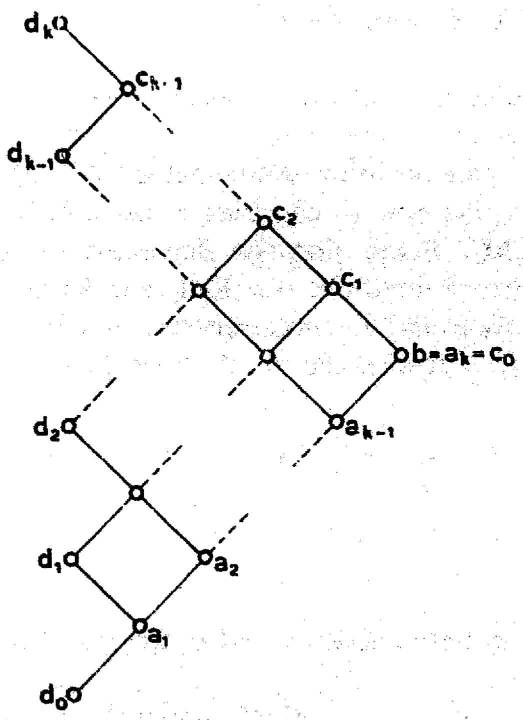

Fig. 2. One flap of $L_{n,k}$.

We now define the subposets of $L_{n,k}$.

$$P_{n,k} = \{a^{\lambda}_i \mid 1 \leq i \leq k-1, 1 \leq \lambda \leq n\} \cup \{b^{\lambda} \mid 1 \leq \lambda \leq n\}$$
$$\cup \{c^{\lambda}_i \mid 1 \leq i \leq k-1, 1 \leq \lambda \leq n\}.$$

Note that $P_{n,1} = M_n$. For $k \geq 2$,

$$Q_{n,k} = P_{n,k} \cup \{d_{k-1}\} - \{c^{\lambda}_{k-1} \mid 1 \leq \lambda \leq n\}.$$

Also, let

$$L'_{n,k} = L_{n,k} - \{c^{\lambda}_{k-1} \mid 1 \leq \lambda \leq n\}.$$

Clearly, $|P_{n,k}| = n(2k-1)$ and $|Q_{n,k}| = n(2k-2)+1$.

Let us list the four main results of this section.

Theorem 3.1. *If $k \geq 2$ and $n = 1+(2k-1)2^{2k-3}$, then $P_{n,k}$ is a $2k$-irreducible poset.*

Theorem 3.2. *If $k \geq 2$ and $m = (2k-1)2^{2k-3}$, then*
(i) $\dim L_{m,k} = 2k-1$, *and*
(ii) $\dim L_{n,k} = 2k$ *whenever* $n > m$.

Theorem 3.3. *If $k \geq 2$ and $n = 1+(k-1)2^{2k-3}$, then $Q_{n,k}$ is a $(2k-1)$-irreducible poset.*

Theorem 3.4. *If $k \geq 2$ and $m = (k-1)2^{2k-3}$, then*
(i) $\dim L'_{m,k} = 2k-2$,
(ii) $\dim L'_{n,k} = 2k-1$ *whenever* $n > m$.

A lattice is *dimantlable* [7] iff every sublattice with at least three elements[1] contains an element that is doubly irreducible in the sublattice. Since a single flap is planar, $L_{n,k}$ and $L'_{n,k}$ are obviously dismantlable (Kelly and Rival [8, Corollary 2.3]). Theorem 3.2 shows how to construct a dismantlable lattice of arbitrary finite dimension in $\mathbf{M}_\omega$. Recall that the dimension of a finite dismantlable *distributive* lattice cannot exceed two (see Kelly and Rival [7, Corollary 3.6]).

We postpone the proofs until the necessary preliminary results are established. For integers i and j, $[i, j]$ denotes the set of all integers l such that $i \leq l \leq j$. It is obvious that

$$\mathrm{Crit}(P_{n,k}) = \{c_{i-1}^{\lambda} < a_i^{\mu} \mid \lambda \neq \mu;\ 1 \leq \lambda, \mu \leq n;\ 1 \leq i \leq k\}$$

and

$$\mathrm{Crit}(Q_{n,k}) = \{d_{k-1} < b^{\lambda} \mid 1 \leq \lambda \leq n\} \cup \{c_{i-1}^{\lambda} < a_i^{\mu} \mid \lambda \neq \mu;\ 1 \leq \lambda, \mu \leq n;\ 1 \leq i \leq k-1\}.$$

Lemma 3.5. *For any positive integers k and n,* $\dim P_{n,k} \leq 2k$.

Proof. If $C_i = (a_i^1, c_{i-1}^1, a_i^2, c_{i-1}^2, \ldots, a_i^n, c_{i-1}^n)$ and $D_i = (a_i^n, c_{i-1}^n, \ldots, a_i^2, c_{i-1}^2, a_i^1, c_{i-1}^1)$, then each C_i and D_i is a partial linear extension of $P_{n,k}$ and each critical inequality for $P_{n,k}$ holds in some C_i or D_i $(1 \leq i \leq k)$. By Proposition 1.3, this completes the proof.

Lemma 3.6. *For $k \geq 2$, let m be a positive integer and let the functions $f_\lambda : [1, 2k-1] \to [1, k]$ be given for $1 \leq \lambda \leq m$. For $\lambda, \mu \in [1, m]$, it is further assumed that $\lambda = \mu$ whenever the following three conditions are satisfied for some $i \in [1, k]$ and some $j \in [1, 2k-1]$.*
(a) $f_\lambda^{-1}(i) \subseteq \{j\}$.
(b) $f_\mu^{-1}(i) \subseteq \{j\}$.
(c) *If $l \in [1, 2k-1] - \{j\}$, then $f_\lambda(l) < i$ iff $f_\mu(l) < i$.*

[1] This size restriction was not needed in [7] since the empty join and meet were excluded there.

Then, $m \leq (2k-1)2^{2k-3}$. *Moreover, functions* $f_\lambda (1 \leq \lambda \leq m)$ *can be defined with* $m = (2k-1)2^{2k-3}$ *so that the above conditions are satisfied.*

Proof. For $i \in [1, k]$, $j \in [1, 2k-1]$ and $A \subseteq [1, 2k-1] - \{j\}$ with $|A| = 2i-2$, let $F(i, j, A)$ denote the set of all functions $f:[1, 2k-1] \to [1, k]$ that satisfy one of the following conditions.

(i) $f^{-1}(i) = \{j\}$ and $f^{-1}([1, i-1]) = A$.

(ii) $f^{-1}(i) = \emptyset$ and $f^{-1}([1, i-1]) = A$.

(iii) $f^{-1}(i) = \emptyset$ and $f^{-1}([1, i-1]) = A \cup \{j\}$.

It is easily shown that every function from $[1, 2k-1]$ to $[1, k]$ lies in some $F(i, j, A)$ for i, j and A as above. The conditions of the lemma mean that $f_\lambda, f_\mu \in F(i, j, A)$ is possible only if $\lambda = \mu$. Thus, if m' is the number of such triples (i, j, A), then $m \leq m'$. Since

$$m' = (2k-1) \sum_{i=1}^{k} \binom{2k-2}{2i-2} = (2k-1)2^{2k-3},$$

the first statement of the lemma follows. Let $f:[1, 2k-1] \to [1, k]$ be a function such that $f^{-1}(i) = \{j\}$, $f^{-1}([1, i-1]) = A$, and $|f^{-1}(l)| = 2$ whenever $l \neq i$; then $f \in F(i', j', A')$ exactly when $i' = i$, $j' = j$ and $A' = A$. The second statement now follows.

We define $P^{\#}_{n,k} = P_{n,k} \cup \{d_0, d_1, \ldots, d_k\}$. Note that $\dim P^{\#}_{n,k} = \dim P_{n,k}$.

Proposition 3.7. *If* $k \geq 2$ *and* $n > (2k-1)2^{2k-3}$, *then* $\dim P_{n,k} = 2k$.

Proof. It is enough by Lemma 3.5 to show that $\dim P^{\#}_{n,k} \geq 2k$. Suppose C_j $(1 \leq j \leq 2k-1)$ are chains whose intersection is the ordering on $P^{\#}_{n,k}$. For $1 \leq \lambda \leq n$, we define the functions $f_\lambda:[1, 2k-1] \to [1, k]$ by

$$f_\lambda(j) = i \quad \text{iff} \quad d_{i-1} < b^\lambda < d_i \text{ in } C_j.$$

Thus, by Lemma 3.6, there are distinct elements λ and μ in $[1, n]$, together with $i \in [1, k]$ and $j \in [1, 2k-1]$ such that conditions (a), (b) and (c) of Lemma 3.6 are satisfied. The critical inequalities $c^\lambda_{i-1} < a^\mu_i$ and $c^\mu_{i-1} < a^\lambda_i$ cannot both hold in the same chain because $c^\lambda_{i-1} < a^\lambda_i$ would then follow using $a^\mu_i < c^\mu_{i-1}$. Thus, we can assume that $c^\lambda_{i-1} < a^\mu_i$ holds in C_l with $l \neq j$. Note that $f_\lambda(l) \neq i$ and $f_\mu(l) \neq i$ because $l \neq j$. Since $b^\lambda < d_i$ and $d_{i-1} < b^\mu$ hold in C_l, it follows that $f_\lambda(l) < i < f_\mu(l)$, contradicting condition (c).

Proposition 3.8. *If* $k \geq 2$ *and* $m = (2k-1)2^{2k-3}$, *then* $\dim P_{m,k} = 2k-1$.

Proof. We shall define partial linear extensions C_j $(1 \leq j \leq 2k-1)$ that realize $P^{\#}_{m,k}$. Expressing C_j as

$$(d_0, C_j^1, d_1, C_j^2, d_2, \ldots, d_{k-1}, C_j^k, d_k)$$

it is enough to define C_j^i for any $i \in [1, k]$ and $j \in [1, 2k-1]$. For $1 \leq \lambda \leq m$, let $f_\lambda : [1, 2k-1] \to [1, k]$ be functions for which $|f_\lambda^{-1}(i)| = 2$ for every $i \in [1, k]$ except one, and that satisfy the condition of Lemma 3.6. We now fix $i \in [1, k]$ and $j \in [1, 2k-1]$. For $1 \leq l \leq 2k-1$, we define functions $\varphi_l : [1, m] \to \{0, 1, 2\}$ as follows:

$$\varphi_l(\lambda) = \begin{cases} 0, & \text{if } f_\lambda(l) < i; \\ 1, & \text{if } f_\lambda(l) = i; \\ 2, & \text{if } f_\lambda(l) > i. \end{cases}$$

Let $D = \varphi_j^{-1}(1) = \{\lambda \mid f_\lambda(j) = i\}$. We define a binary relation δ between distinct elements $\lambda, \mu \in D$ by the following two rules:

(α) Suppose $\varphi_l(\lambda) = \varphi_l(\mu)$ for all $l \in [1, 2k-1]$ and $f_\lambda^{-1}(i) = f_\mu^{-1}(i) = \{j, h\}$ for some $h \neq j$. Let $\lambda' = \min(\lambda, \mu)$ and $\mu' = \max(\lambda, \mu)$ in the usual order on $[1, m]$. If $j < h$, set $\lambda' \delta \mu'$; otherwise, set $\mu' \delta \lambda'$.

(β) Suppose $\varphi_l(\lambda) \geq \varphi_l(\mu)$ for all $l \in [1, 2k-1]$ but $\varphi_h(\lambda) \neq \varphi_h(\mu)$ for some $h \in [1, 2k-1]$. In this case, set $\lambda \delta \mu$.

We *claim* there is a linear ordering Δ on D that extends δ. It is enough to show that the transitive closure of δ is a strict partial ordering. Suppose $\lambda_0 \delta \lambda_1 \delta \cdots \delta \lambda_n = \lambda_0$ holds in D for some $n \geq 1$. If $\lambda_\gamma \delta \lambda_{\gamma+1}$ by rule (α) for all $\gamma < n$, then for some $h \neq j$, $f_\lambda^{-1}(i) = \{j, h\}$ whenever $\lambda = \lambda_\gamma$ and $\gamma \leq n$. If $j < h$, then $\lambda_0 < \lambda_1 < \cdots < \lambda_n = \lambda_0$ in the usual order, which is impossible; the other case is similar. Thus, we can assume that $\lambda_0 \delta \lambda_1$ holds by rule (β). Let $h \in [1, 2k-1]$ be such that $\varphi_h(\lambda_0) > \varphi_h(\lambda_1)$. It follows that $\varphi_h(\lambda_0) > \varphi_h(\lambda_n)$, which is impossible since $\lambda_0 = \lambda_n$. With this contradiction, the proof of the claim is complete.

D^* denotes D endowed with linear ordering Δ. Let A and B be linear orderings of the sets $\varphi_j^{-1}(0)$ and $\varphi_j^{-1}(2)$ respectively. We set

$$C_j^i = ((c_{i-1}^\lambda \mid \lambda \in A), ((a_i^\lambda, c_{i-1}^\lambda) \mid \lambda \in D^*), (a_i^\lambda \mid \lambda \in B)).$$

C_j^i is obviously a partial linear extension of $P_{m,k}^{\#}$ and it follows easily that C_j is also one. (Observe that, in any linear extension of $P_{m,k}^{\#}$ that extends C_j, $d_{i-1} < b^\lambda < d_i$ iff $f_\lambda(j) = i$.)

We consider an arbitrary critical inequality $c_{i-1}^\lambda < a_i^\mu$ for $P_{m,k}$ where i, λ and μ are fixed ($1 \leq i \leq k$, $\lambda \neq \mu$); j is no longer fixed. By Proposition 1.3, it suffices to show that this inequality holds in C_j^i for some $j \in [1, 2k-1]$. There are three cases.

Case 1. $\varphi_l(\lambda) < \varphi_l(\mu)$ *for some* $l \in [1, 2k-1]$. For $j = l$, $\lambda \in A$ or $\mu \in B$, and it is immediate that $c_{i-1}^\lambda < a_i^\mu$ in C_j^i.

Case 2. $\varphi_l(\lambda) = \varphi_l(\mu)$ *for all* $l \in [1, 2k-1]$. Since $|f_\lambda^{-1}(i)| = 1$ is impossible by the conditions of Lemma 3.6, $f_\lambda^{-1}(i) = f_\mu^{-1}(i) = \{h_1, h_2\}$ for distinct $h_1, h_2 \in [1, 2k-1]$. Then, for $j = h_1$ or h_2, $\lambda \delta \mu$ by rule (α); consequently, $c_{i-1}^\lambda < a_i^\mu$ in C_j^i.

Case 3. $\varphi_l(\lambda) \geq \varphi_l(\mu)$ *for all* $l \in [1, 2k-1]$ *and* $\varphi_h(\lambda) \neq \varphi_h(\mu)$ *for some* $h \in [1, 2k-1]$. By rule (β), it is enough to find $j \in [1, 2k-1]$ so that $\lambda, \mu \in D$; in other words, $\varphi_j(\lambda) = \varphi_j(\mu) = 1$. Let us suppose, to the contrary, that there is no such j.

This means that $f_\mu(j)<i$ whenever $f_\lambda(j)=i$. Observe that $f_\lambda^{-1}([1,i-1])\subseteq f_\mu^{-1}([1,i-1])$. If $|f_\lambda^{-1}(i)|=1$, then $|f_\lambda^{-1}([1,i-1])|=2i-2$ and the above inclusion would imply that $|f_\mu^{-1}([1,i-1])|=2i-1$, an impossibility. Therefore, $|f_\lambda^{-1}(i)|=2$, and since $|f_\lambda^{-1}([1,i-1])|\geqslant 2i-3$, the above inclusion implies that $|f_\mu^{-1}([1,i-1])|\geqslant 2i-1$. This contradiction completes the proof that $\dim P_{m,k}\leqslant 2k-1$. Because adding one flap increases the dimension by at most one, it follows from Proposition 3.7 that $\dim P_{m,k}=2k-1$.

Proof of Theorem 3.1. Let $m=n-1=(2k-1)2^{2k-3}$. By Proposition 3.7, $\dim P_{n,k}=2k$. By duality, it is enough to show that $\dim Q\leqslant 2k-1$ where $Q=P^{\#}_{n,k}-\{a^n_g\}$ and $1\leqslant g\leqslant k$. We shall define partial linear extensions C'_j $(1\leqslant j\leqslant 2k-1)$ by adding the elements a^n_i $(1\leqslant i\leqslant k, i\neq g)$ and c^n_i $(1\leqslant i\leqslant k-1)$ to the chains C_j constructed in the proof of Proposition 3.8; we shall use the notation of that proof. We can assume that $f_1^{-1}(g)=\{1\}$. We first assume that $g<k$. Whenever either a^1_i or c^1_i appears alone in one of the original chains, add a^n_i (for $i\neq g$) or c^n_i, respectively, immediately after. If $i\neq g$, then (a^1_i, c^1_{i-1}) appears in two chains C_h and C_j with $h<j$; add (a^n_i, c^n_{i-1}) to both chains, immediately before a^1_i in C_h, and immediately after c^1_{i-1} in C_j. Finally, we add c^n_{g-1} immediately before a^1_g in C_1.

The set of critical inequalities for Q is

$$\{c^\lambda_{i-1}<a^\mu_i \mid \lambda\neq\mu,\ \mu\neq n \text{ when } i=g,\ 1\leqslant i\leqslant k\}.$$

Only the cases where λ or μ is n need to be checked. If $\mu\neq 1$, then $c^1_{i-1}<a^\mu_i$ holds in some chain C_j; in this case, $c^n_{i-1}<a^\mu_i$ holds in C'_j. The case where $\lambda\neq 1$, $\mu=n$ and $i\neq g$ is similar. For $i\neq g$, $c^n_{i-1}<a^1_i$ and $c^1_{i-1}<a^n_i$ also hold in one of the new chains. Finally, $c^n_{g-1}<a^1_g$ holds in C'_1.

We can now assume that $g=k$; in other words, $Q=P^{\#}_{n,k}-\{b^n\}$. We shall only consider the case that $k>2$. (These additional arguments are unnecessary if $k=2$.) We specify the function f_1 completely by stipulating that $f_1^{-1}(i)=\{2i,2i+1\}$ whenever $1\leqslant i\leqslant k-1$. We require the remaining functions to be chosen so that, for $2\leqslant\lambda\leqslant m$ and $2\leqslant i\leqslant k-1$,

$$f_\lambda(2i)\neq i \quad\text{or}\quad f_\lambda(2i+1)\neq i.$$

This means that a suitable function f must be chosen from each set $F(h,j,A)$ of functions defined in the proof of Lemma 3.6. Let $h\in[1,k]$, $j\in[1,2k-1]$ and $A\subseteq[1,2k-1]-\{j\}$ with $|A|=2h-2$ be fixed. We shall specify certain values of f that still allow f to be a function lying only in $F(h,j,A)$. We consider each $i\in[2,k-1]$. If $i\leqslant h$ and $j\neq 2i$, then set $f(2i)=i-1$ whenever $2i\in A$. (If $2i\notin A$, then $f(2i)>h\geqslant i$.) If $i\leqslant h$ and $j=2i$, then set $f(2i+1)=i-1$ whenever $(2i+1)\in A$. If $i>h$ and $j\neq 2i$, then set $f(2i)=i$ whenever $2i\notin A$. If $i>h$ and $j=2i$, then set $f(2i+1)=i$ whenever $(2i+1)\notin A$. Therefore, the above requirement can be met.

Let $i\in[2,k-1]$, $j\in\{2i,2i+1\}$, $\{j,h\}=\{2i,2i+1\}$, and $D=\{\lambda\mid f_\lambda(j)=i\}$. Observe that $f_1(j)=i=f_1(h)$. We show that the linear ordering Δ of Proposition 3.8

can be chosen so that:

(*) If $\lambda \in D$ satisfies $f_\lambda(h)<i$, then $1\Delta\lambda$.

Let $\lambda \in D$ satisfy $f_\lambda(h)<i$. If $\lambda\delta^*1$, where δ^* is the transitive closure of δ, then $\varphi_l(\lambda)\geqslant\varphi_l(1)$ whenever $1\leqslant l\leqslant 2k-1$. Since $\varphi_h(\lambda)=0$ and $\varphi_h(1)=1$, $\lambda\delta^*1$ cannot hold. The statement (*) now follows.

C_1' is formed from C_1 by adding a_i^n immediately after a_i^1 $(1\leqslant i\leqslant k-1)$ and by adding (c_1^n, c_{k-1}^n) immediately before b^1. C_2' is obtained from C_2 by adding (a_1^n, a_{k-1}^n) immediately before a_1^1 and (c_1^n, c_{k-1}^n) immediately after c_{k-1}^1. C_3' is obtained from C_3 by adding (a_1^n, a_{k-1}^n) immediately after b^1 and adding c_i^n immediately after c_i^1 $(1\leqslant i\leqslant k-1)$. The remaining chains C_j' $(4\leqslant j\leqslant 2k-1)$ are constructed in the same way as when $g<k$.

The set of critical inequalities for Q is

$$\{c_{i-1}^\lambda < a_i^\mu \mid \lambda\neq\mu, \lambda\neq n \text{ when } i=1, \mu\neq n \text{ when } i=k, 1\leqslant i\leqslant k\}$$
$$\cup\{a_{k-1}^n < a_1^\mu \mid n\neq\mu\}\cup\{c_{k-1}^\lambda < c_1^n \mid \lambda\neq n\}.$$

The inequality $c_{k-1}^\lambda < c_1^n$ $(\lambda\neq 1, n)$ holds in C_1'; $a_{k-1}^n < a_1^1$ and $c_{k-1}^1 < c_1^n$ hold in C_2'; $a_{k-1}^n < a_1^\mu$ $(\mu\neq 1, n)$ holds in C_j' where $j\in\{2,3\}$ is chosen so that $b^1 < a_1^\mu$ holds in C_j.

After applying the arguments used when $g<k$, it only remains to consider critical inequalities of the form $c_{i-1}^n < a_i^\mu$ with $\mu\neq 1$ or n, and $2\leqslant i\leqslant k-1$.[2] Let μ and i be fixed. Since $|f_\mu^{-1}([1,i-1])|\leqslant 2i-2$, there is $j\in[3,2i+1]$ such that $f_\mu(j)\geqslant i$. Note that $f_1(j)\leqslant i$. If j can be chosen so that $f_\mu(j)>i$, then $c_{i-1}^1 < a_i^\mu$ holds in C_j; consequently, $c_{i-1}^n < a_i^\mu$ holds in C_j'. We can now assume that $f_\mu(j)=i$ and $f_\mu(l)\leqslant i$ whenever $l\in[3,2i+1]$. If $j<2i$, then $c_{i-1}^n < a_i^\mu$ holds in C_j' because $c_{i-1}^1 < a_i^\mu$ holds in C_j (since $f_1(j)<i$). Thus, without loss of generality, $j\in\{2i, 2i+1\}$ and $f_\mu(h)<i$ where $\{j,h\}=\{2i,2i+1\}$. (Recall that f_μ was chosen so that $f_\mu(h)\neq i$.) By the statement (*) above, $c_{i-1}^1 < a_i^\mu$ holds in C_j. Therefore, $c_{i-1}^n < a_i^\mu$ holds in C_j', completing the proof of the theorem.

For $n\geqslant 2$, $P_{n,k}=\mathbf{P}(L_{n,k})$. Thus, by Lemma 1.8, $L_{n,k}=\mathbf{L}(P_{n,k})$ for $n\geqslant 2$. Theorem 3.2 now follows from Proposition 1.7, Proposition 3.7 and Proposition 3.8.

We now give the preliminary results for Theorems 3.3 and 3.4. Let $Q_{n,k}^\# = Q_{n,k}\cup\{d_0, d_1, \ldots, d_{k-2}, d_k\}$, a poset having the same dimension as $Q_{n,k}$.

Lemma 3.9. *For integers $k\geqslant 2$ and $n\geqslant 1$,* $\dim Q_{n,k}\leqslant 2k-1$.

Proof. Take the chain $(d_{k-1}, b^1, b^2, \ldots, b^n)$ in addition to the chains C_i and D_i $(1\leqslant i\leqslant k-1)$ of Lemma 3.5.

[2] Note that $d_{k-1} < c_1^n$ in C_2' whereas $c_1^1 < d_2$ held in C_2.

Lemma 3.10. *For $k \geq 2$, let m be a positive integer and let the functions $f_\lambda : [1, 2k-2] \to [1, k]$ be given so that $f_\lambda^{-1}(k) \neq \emptyset$ for $1 \leq \lambda \leq m$. For $\lambda, \mu \in [1, m]$ it is further assumed that $\lambda = \mu$ whenever the following three conditions are satisfied for some $i \in [1, k-1]$ and some $j \in [1, 2k-2]$.*

(a) $f_\lambda^{-1}(i) \subseteq \{j\}$.

(b) $f_\mu^{-1}(i) \subseteq \{j\}$.

(c) *If $l \in [1, 2k-2] - \{j\}$, then $f_\lambda(l) < i$ iff $f_\mu(l) < i$.*

Then, $m \leq (k-1)2^{2k-3}$. Moreover, functions f_λ $(1 \leq \lambda \leq m)$ *can be defined with $m = (k-1)2^{2k-3}$ so that the above conditions are satisfied.*

Proposition 3.11. *If $k \geq 2$ and $n > (k-1)2^{2k-3}$, then* $\dim Q_{n,k} = 2k-1$.

Proof. Suppose C_j $(1 \leq j \leq 2k-2)$ are linear extensions of $Q^{\#}_{n,k}$ that realize it. For $1 \leq \lambda \leq n$, we define functions $f_\lambda : [1, 2k-2] \to [1, k]$ similarly as in the proof of Proposition 3.7. Since $d_{k-1} < b^\lambda$ must hold in some C_j, $f_\lambda^{-1}(k) \neq \emptyset$. Let $\lambda \neq \mu$ in $[1, n]$, $i \in [1, k-1]$ and $j \in [1, 2k-2]$ satisfy conditions (a), (b) and (c) of Lemma 3.10. Now proceed as in the proof of Proposition 3.7.

Proposition 3.12. *If $k \geq 2$ and $m = (k-1)2^{2k-3}$, then* $\dim Q_{m,k} = 2k-2$.

Proof. For $\lambda \in [1, m]$, let $f_\lambda : [1, 2k-2] \to [1, k]$ be functions for which $|f_\lambda^{-1}(i)| = 2$ for every $i \in [1, k-1]$ except one and $|f_\lambda^{-1}(k)| = 1$, and that satisfy the conditions of Lemma 3.10. For $j \in [1, 2k-2]$, let $C_j = (d_0, C_j^1, d_1, C_j^2, d_2, \ldots, d_{k-1}, C_j^k, d_k)$, where C_j^i $(1 \leq i \leq k-1)$ are defined as in the proof of Proposition 3.8 (when $[1, 2k-1]$ is replaced by $[1, 2k-2]$) and C_j^k is $\{b^\lambda \mid f_\lambda(j) = k\}$ endowed with a linear ordering. Similarly as in the proof of Proposition 3.8, we can show that $C_1, C_2, \ldots, C_{2k-2}$ realize $Q^{\#}_{m,k}$, and conclude that $\dim Q_{m,k} = 2k-2$.

Proof of Theorem 3.3. Let $Q = Q^{\#}_{n,k} - \{x\}$, where x is a_g^1 $(1 \leq g \leq k-1)$, b^n, c_g^n $(1 \leq g \leq k-2)$, or d_{k-1}. Let $m = n-1 = (k-1)2^{2k-3}$. We must show that $\dim Q \leq 2k-2$. We consider only the cases that $x = b^n$ or $x = d_{k-1}$ since the proof of Theorem 3.1 can be modified slightly to handle the other cases (when Proposition 3.8 is replaced by Proposition 3.12).

Let $x = b^n$ and assume $k > 2$.

$$\begin{aligned}\mathrm{Crit}(Q) = &\{d_{k-1} < c_1^n\} \cup \{d_{k-1} < b^\lambda \mid \lambda \neq n\} \\ &\cup \{c_{i-1}^\lambda < a_i^\mu \mid \lambda \neq \mu, \lambda \neq n \text{ when } i = 1, 1 \leq i \leq k-1\} \\ &\cup \{a_{k-1}^n < a_1^\mu \mid \mu \neq n\}.\end{aligned}$$

We adopt the notation of the proof of Proposition 3.12. We specify f_1 by requiring that $f_1(1) = k$, $f_1(2) = 1$, and $f_1^{-1}(i) = \{2i-1, 2i\}$ whenever $2 \leq i \leq k-1$. Similarly as in the proof of Theorem 3.1, the remaining functions can be chosen so that, for $2 \leq \lambda \leq m$ and $2 \leq i \leq k-1$,

$$f_\lambda(2i-1) \neq i \quad \text{or} \quad f_\lambda(2i) \neq i.$$

Let $i \in [2, k-1]$, $j \in \{2i-1, 2i\}$, $\{j, h\} = \{2i-1, 2i\}$, and $D = \{\lambda \mid f_\lambda(j) = i\}$. The linear ordering of Proposition 3.12 can be chosen so that:

(*) If $\lambda \in D$ satisfies $i < f_\lambda(h)$, then $\lambda \Delta 1$.

C_1' is formed from C_1 by adding (a_1^n, a_{k-1}^n) just before a_1^1 and c_1^n just after d_{k-1}. C_2' is obtained from C_2 by adding (a_1^n, a_{k-1}^n) just after b^1 and c_i^n just after c_i^1 $(1 \leq i \leq k-2)$. C_j' $(3 \leq j \leq 2k-2)$ is formed from C_j by adding (a_i^n, c_{i-1}^n) for $i \in [2, k-1]$ immediately before (after) (a_i^1, c_{i-1}^1) when $j = 2i-1$ $(2i)$. Also, whenever a_i^1 (c_g^1) appears alone in C_j $(1 \leq i \leq k-1, 1 \leq g \leq k-2, 3 \leq j \leq 2k-2)$, it is immediately followed by a_i^n (c_g^n) in C_j'. It can be verified that $C_1', C_2', \ldots, C_{2k-2}'$ realize Q. (The only nontrivial part is showing, for $i \in [2k-1]$ and $\lambda \in [2, m]$, the existence of $j \in [3, 2k-2]$ such that $c_{i-1}^\lambda < a_i^1$ holds in C_j.)

Let $x = d_{k-1}$. If a_{k-1}^λ and b^λ are identified in Q for $1 \leq \lambda \leq n$, we obtain $P_{n,k-1}$. Thus $\dim Q = \dim P_{n,k-1}$, and therefore, $\dim Q \leq 2k-2$ by Lemma 3.5. This completes the proof of the theorem.

Since $Q_{3,2}$ is 3-irreducible by Theorem 3.3, it must occur in the list of all 3-irreducible posets in [6]. It does, under the name B^d.

Since $Q_{n,k} = \mathbf{P}(L'_{n,k})$ for $n \geq 2$, $L'_{n,k} = \mathbf{L}(Q_{n,k})$ by Lemma 1.8. Theorem 3.4 now follows by Proposition 1.7, Proposition 3.11 and Proposition 3.12.

Whenever $m < n$ and $k < l$, $P_{m,k}$ is isomorphic to a subposet of both $P_{n,l}$ and $Q_{n,l}$, and $Q_{m,k}$ is isomorphic to a subposet of both $P_{n,l}$ and $Q_{n,l}$. Consequently, any irreducible poset of the form $P_{n,k}$ or $Q_{n,k}$ has the embedding property. (This statement is also a consequence of Theorem 4.9.) For example, the 7-irreducible poset $Q_{97,4}$ is a subposet of the 8-irreducible poset $P_{225,4}$. Similar inclusions between the lattices $L_{n,k}$ and $L'_{n,k}$ allow us to conclude from Theorems 3.2 and 3.4 that:

$$\dim L_{n,k} = 2k-1$$

whenever $k \geq 2$ and $1 + (k-1)2^{2k-3} \leq n \leq (2k-1)2^{2k-3}$, and

$$\dim L'_{n,k} = 2k-2$$

whenever $k \geq 3$ and $1 + (2k-3)2^{2k-5} \leq n \leq (k-1)2^{2k-3}$.

4. Dimension product of irreducible posets

Recall that we treat **2**, the two-element chain, as a special case so that an irreducible poset is understood to have dimension at least two. Every known irreducible poset satisfies the conditions we shall give to be called *normal*. We shall define the *dimension product* $P \otimes Q$ of normal irreducible posets P and Q, so that $P \otimes Q$ is an irreducible poset of dimension $\dim P + \dim Q$. Our construction was motivated by—but differs from—the one given by W.T. Trotter, Jr. [12]. If P_1

and P_2 are posets of length one, his construction yields a poset P of length one satisfying $|P|=|P_1|+|P_2|$. Our construction does not satisfy this condition.

Let P be a nontrivial (finite) poset of dimension d and let $L=\mathbf{L}(P)$; 0 and 1 are the zero and one of L. We define $\mathbf{D}(P)$ to be the set of elements $x\in \mathrm{Irr}(P)=\mathrm{Irr}(L)$ such that $\dim(L-\{x\})=d$. Since $\mathrm{Irr}(\mathbf{2})=\emptyset$, $\mathbf{D}(\mathbf{2})=\emptyset$. $\mathbf{A}(P)$ denotes the set of all minimal elements of $P-\{0\}$ that lie in $\mathbf{D}(P)$; $\mathbf{B}(P)$ is defined dually. Equivalently, $\mathbf{A}(P)$ consists of those elements of $\mathbf{D}(P)$ whose lower cover in L is 0. We further define $\mathbf{L}^*(P)=L-\mathbf{D}(P)$. Observe that $L^*=\mathbf{L}^*(P)$ is a sublattice of L.

P is *normal* if $P=\mathbf{2}$ or if P satisfies the following four conditions.

(N0) If $a<b$ in L, then $a\notin \mathbf{M}(P)$ or $b\notin \mathbf{J}(P)$; $\mathbf{A}(P)\cap\mathbf{B}(P)=\emptyset$.

(N1) $\mathbf{D}(P)=\mathbf{A}(P)\cup\mathbf{B}(P)$.

(N2) $0\notin\mathbf{M}(L^*)$ and $1\notin\mathbf{J}(L^*)$.

(N3) $\dim L^*=d$.

P is *completely normal* if $P=\mathbf{2}$ or if P is normal and satisfies:

(N4) Let $x, y\in P$ with $x\leqslant y$. If $y\in\mathbf{J}(P)$, then $x\in\mathbf{J}(P)$; if $x\in\mathbf{M}(P)$, then $y\in\mathbf{M}(P)$.

Observe that P is normal if and only if $\mathbf{P}(P)$ is. Since adding a zero or one to a poset does not increase its dimension, every irreducible poset P satisfies $0\notin\mathbf{M}(P)$ and $1\notin\mathbf{J}(P)$. Note that these latter two conditions are consequences of (N2).

Lemma 4.1. *Every irreducible poset satisfies* (N0).

Proof. Let P be a poset such that $a<b$ in $L=\mathbf{L}(P)$ with $a\in\mathbf{M}(P)$ and $b\in\mathbf{J}(P)$. If $x>a$ in P, then $x\geqslant b$ in L, and therefore, also in P. Similarly, $x\leqslant a$ in P whenever $x<b$ in P. If $C_1, C_2, \ldots, C_n$ are linear extensions realizing $P-\{b\}$, then $C_1', C_2', \ldots, C_n'$ are linear extensions realizing P, where C_i' is formed from C_i by adding b immediately after a $(1\leqslant i\leqslant n)$. Thus, P is not irreducible. For the second clause, let P be a d-irreducible poset and suppose that $x\in\mathbf{A}(P)\cap\mathbf{B}(P)$. Because $\mathbf{D}(P)\neq\emptyset$, $d\geqslant 3$. Since x is incomparable with every element of $P-\{x\}$, the last clause now follows.

If P is an irreducible poset for which $\mathbf{D}(P)=\emptyset$, then P is obviously normal. Observe that $\mathbf{D}(P_{3,2})=\emptyset$ although $P_{3,2}$ is not irreducible. All known irreducible posets are completely normal. Note that $\mathbf{2}$ fails (N0), (N2) and (N4).

For each of the following 3-irreducible posets P (in the notation of Kelly [6]), $\mathbf{D}(P)=\emptyset$: A_n, B, C, D, E_n, F_n, G_n, H_n, EX_2 and I_n $(n\geqslant 0)$. After determining that $\mathbf{D}(CX_1)=\{b_1\}$, $\mathbf{D}(CX_2)=\{b_1, b_3\}$, $\mathbf{D}(CX_3)=\{a_1, b_1\}$, $\mathbf{D}(EX_1)=\{b_2\}$, $\mathbf{D}(FX_1)=\{a_1\}$, $\mathbf{D}(FX_2)=\{a_1, b_3\}$, $\mathbf{D}(J_n)=\{c, d\}$, it is easy to verify that the remaining 3-irreducible posets are normal. Thus, all 3-irreducible posets are completely normal. Since $\mathbf{P}(L_{n,k}-\{b^1\})=P_{n,k}-\{b^1\}$ for $n\geqslant 3$, it follows from Corollary 1.6 that $\mathbf{D}(P_{n,k})=\emptyset$ whenever $P_{n,k}$ is irreducible. Hence, any irreducible poset of the form $P_{n,k}$ or $Q_{n,k}$ is completely normal.

Let $\mathbf{J}^*(P) = \mathbf{J}(P) - \mathbf{B}(P)$ and $\mathbf{M}^*(P) = \mathbf{M}(P) - \mathbf{A}(P)$. Note that $\mathbf{M}^*(P)$ does *not* equal $\mathbf{M}(\mathbf{L}^*(P))$ in general, and dually. For example, if $P = CX_1$ (notation of [6]), then $b_1 \in \mathbf{M}^*(P)$ but $b_1 \notin \mathbf{L}^*(P)$, and $a_1 \vee a_2$ is in $\mathbf{M}(\mathbf{L}^*(P))$ but not in $\mathbf{P}(P)$.

Let $P_1, P_2, \ldots, P_n$ $(n \geq 2)$ be normal posets. (In particular, each poset is finite and nontrivial.) Let $L_i = \mathbf{L}(P_i)$, $J_i^* = \mathbf{J}^*(P_i)$ and $M_i^* = \mathbf{M}^*(P_i)$ for $1 \leq i \leq n$. The *dimension product* of $P_1, P_2, \ldots, P_n$, denoted by $P_1 \otimes P_2 \otimes \cdots \otimes P_n$, is the subposet $Q = Q_0 \cup Q_1$ of $L = L_1 \times L_2 \times \cdots \times L_n$ (direct product), where

$$Q_0 = J_1^* \times \{0\} \times \cdots \times \{0\} \cup \{0\} \times J_2^* \times \{0\} \times \cdots \times \{0\} \cup \cdots \cup \{0\} \times \cdots \times \{0\} \times J_n^*$$

and

$$Q_1 = M_1^* \times \{1\} \times \cdots \times \{1\} \cup \{1\} \times M_2^* \times \{1\} \times \cdots \times \{1\} \cup \cdots \cup \{1\} \times \cdots \times \{1\} \times M_n^*.$$

We also define $A_i = \mathbf{A}(P_i)$, $B_i = \mathbf{B}(P_i)$, $L_i^* = \mathbf{L}^*(P_i)$, $J_i = \mathbf{J}(P_i)$ and $M_i = \mathbf{M}(P_i)$ for $1 \leq i \leq n$. Let $R = R_0 \cup R_1$, a subposet of L where

$$R_0 = A_1 \times \{0\} \times \cdots \times \{0\} \cup \{0\} \times A_2 \times \{0\} \times \cdots \times \{0\} \cup \cdots \cup \{0\} \times \cdots \times \{0\} \times A_n$$

and

$$R_1 = B_1 \times \{1\} \times \cdots \times \{1\} \cup \{1\} \times B_2 \times \{1\} \times \cdots \times \{1\} \cup \cdots \cup \{1\} \times \cdots \times \{1\} \times B_n.$$

Let $K^\# = L_1^* \times L_2^* \times \cdots \times L_n^*$ and set

$$K = K^\# \cup R,$$

a subposet of L. Note that $K^\#$ and R are disjoint. Each element of R has a unique lower cover and a unique upper cover in K, both of which lie in $K^\#$. For example, if $x \in A_1$, let y be the unique upper cover of x in L_1. By (N0), $y \notin J_1$. Consequently, $y \in L_1^*$. The unique lower (upper) cover of $\langle x, 0, \ldots, 0\rangle$ in K is $\langle 0, 0, \ldots, 0\rangle$ ($\langle y, 0, \ldots, 0\rangle$). Thus, the next lemma shows $K^\#$ to be a sublattice of K.

Lemma 4.2. *Let $K = K^\# \cup R$ be a finite poset, where $K^\#$ is a lattice. If each element of R has a unique lower cover and a unique upper cover in K, then K is a lattice and $K^\#$ is a sublattice of K.*

Proof. (Cf. [8, Proposition 2.1].) By induction on $|R|$, it suffices to assume $R = \{a\}$. We can assume that $a \notin K^\#$. Let $b \in K^\#$ be the unique upper cover of a. If $x \in K^\#$ and $x \nleq a$, it is easily verified that $a \vee x = b \vee x$, where the left-hand join is calculated in K and the right-hand one in $K^\#$. Therefore, K is a lattice and $K^\#$ is a sublattice of K.

In general, K is not a sublattice of L. For $x \in K$, we define $x^\# \in K^\#$ by: $x^\#$ is the unique upper cover of x if $x \in R$; otherwise, $x^\# = x$. For incomparable $x, y \in K$,

$x \vee y = x^{\#} \vee y^{\#}$, where the left-hand join is calculated in K and the right-hand one in $K^{\#}$. In particular, every element of R is doubly irreducible in K.

We now show that $\mathbf{J}(K) = Q_0 \cup R = Q_0 \cup R_1$. We already know that $R \subseteq \mathbf{J}(K)$. By (N1), $Q_0 - R \subseteq K^{\#}$. Let $q \in Q_0 - R$; we can assume that $q = \langle x, 0, \ldots, 0\rangle$ with $x \in J_1$. If $y \prec x$ in L_1, then $\langle y, 0, \ldots, 0\rangle \in K^{\#}$ is the unique lower cover of q in K. Therefore, $Q_0 \cup R \subseteq \mathbf{J}(K)$. Suppose there exists $q \in \mathbf{J}(K) - (Q_0 \cup R)$. We can assume that $q = \langle x, 0, \ldots, 0\rangle$ with $x \in L_1^*$. If $x = 1$, then $P_1 \neq \mathbf{2}$, and by (N2), there are distinct lower covers y and z of 1 in L_1^*. Let $S = \{y, z\}$ in this case. Otherwise, there is $S \subseteq J_1$ such that $x = \bigvee S$ and $x \notin S$. Since $x < 1$, $S \cap B_1 = \emptyset$. In both cases, $T \subseteq K$ where $T = \{\langle s, 0, \ldots, 0\rangle \mid s \in S\}$. Then $q \notin T$ but $q = \bigvee T$ in K. This contradiction completes the proof that $\mathbf{J}(K) = Q_0 \cup R$.

By duality, $\mathbf{M}(K) = Q_1 \cup R$. Hence, $\mathbf{P}(K) = Q \cup R = Q = P_1 \otimes P_2 \otimes \cdots \otimes P_n$. Therefore, by Lemma 1.8, $K = \mathbf{L}(P_1 \otimes P_2 \otimes \cdots \otimes P_n)$. Since

$$L_1^* \times L_2^* \times \cdots \times L_n^* \subseteq K \subseteq L_1 \times L_2 \times \cdots \times L_n,$$

it now follows using (N3) and Proposition 1.7 that

Proposition 4.3. *If $P_1, P_2, \ldots, P_n$ $(n \geq 2)$ are normal posets, then*

$$\dim(P_1 \otimes P_2 \otimes \cdots \otimes P_n) = \dim P_1 + \dim P_2 + \cdots + \dim P_n.$$

Unless $n = 2$ and $P_1 = P_2 = \mathbf{2}$, $\mathrm{Irr}(K) = R$. Hence, in all cases, $\mathbf{D}(P_1 \otimes P_2 \otimes \cdots \otimes P_n) = R$. Since $\mathbf{L}^*(P_1 \otimes P_2 \otimes \cdots \otimes P_n) = K^{\#}$, it is easy to verify that

Lemma 4.4. *If $P_1, P_2, \ldots, P_n$ $(n \geq 2)$ are (completely) normal posets, then so is $P_1 \otimes P_2 \otimes \cdots \otimes P_n$.*

Since $\mathbf{J}^*(P_1 \otimes P_2 \otimes \cdots \otimes P_n) = Q_0$ and $\mathbf{M}^*(P_1 \otimes P_2 \otimes \cdots \otimes P_n) = Q_1$, it follows that

Lemma 4.5. *If $P_1, P_2, \ldots, P_n$ $(n \geq 3)$ are normal posets, then*

$$P_1 \otimes P_2 \otimes \cdots \otimes P_n = (\cdots (P_1 \otimes P_2) \otimes \cdots) \otimes P_n.$$

By virtue of Lemma 4.5, most statements about the dimension product need only be proved for two factors.

Proposition 4.6. *Let $Q = P_1 \otimes P_2$, where P_1 and P_2 are normal posets and $P_1 \neq \mathbf{2}$. If $\langle a, 0\rangle \in Q$, then* $\dim(Q - \{\langle a, 0\rangle\}) = \dim Q - 1$.

Proof. By Proposition 4.3, $\dim Q = d = d_1 + d_2$, where $d_i = \dim P_i$ $(i = 1, 2)$. Since removing any element from a poset lowers the dimension by at most one (Hiraguchi [5]), we only need to show that $\dim(Q - \{\langle a, 0\rangle\}) \leq d - 1$ whenever $a \in J_1^*$. If $a \notin M_1^*$ then $\langle a, 1\rangle \notin Q$. Thus, $Q - \{\langle a, 0\rangle\}$ is a subposet of

$((P_1-\{a\})\cup\{0,1\})\times(P_2\cup\{0,1\})$, whose dimension is $(d_1-1)+d_2=d-1$. We can assume that $a\in M_1^*$. Consequently, by (N1), $a\notin \mathbf{D}(P_1)$ although $a\in \mathrm{Irr}(P_1)$. This means that $\dim(L_1-\{a\})=d_1-1$. Let b be the (unique) lower cover of a in L_1. Let $C_i=(x_1^i, x_2^i, \ldots, x_l^i)$, $1\leq i\leq d_1-1$, be chains realizing $L_1-\{a,0,1\}$, where $l=|L_1|-3$. Also, let $E_j=(y_1^j, y_2^j, \ldots, y_m^j)$, $1\leq j\leq d_2$, be chains realizing $P_2-\{0,1\}$, where $m=|P_2-\{0,1\}|$. Finally, let $(z_1, z_2, \ldots, z_r)$ be a linear extension of the subposet $\{x\in P_1 \mid x\not\geq a\}$ and $(z_1, z_2, \ldots, z_n)$ be a linear extension of $P_1-\{a\}$. We now define some partial linear extensions of $L_1\times(P\cup\{0,1\})$ where $2\leq i\leq d_1-1$ and $2\leq j\leq d_2$.

$$C_1'=(\langle 0,y_1^1\rangle,\ldots,\langle 0,y_m^1\rangle,\langle 0,1\rangle,\langle x_1^1,0\rangle,\langle x_1^1,1\rangle,\ldots,\langle b,0\rangle,\langle b,1\rangle,$$
$$\langle a,1\rangle,\ldots,\langle x_l^1,0\rangle,\langle x_l^1,1\rangle,\langle 1,0\rangle,\langle 1,y_1^1\rangle,\ldots,\langle 1,y_m^1\rangle);$$
$$C_i'=(\langle x_1^i,0\rangle,\langle x_1^i,1\rangle,\ldots,\langle b,0\rangle,\langle b,1\rangle,\langle a,1\rangle,\ldots,\langle x_l^i,0\rangle,\langle x_l^i,1\rangle);$$
$$E_1'=(\langle z_1,0\rangle,\ldots,\langle z_n,0\rangle,\langle 1,0\rangle,\langle 0,y_1^1\rangle,\langle 1,y_1^1\rangle,\ldots,\langle 0,y_m^1\rangle,\langle 1,y_m^1\rangle,$$
$$\langle 0,1\rangle,\langle z_1,1\rangle,\ldots,\langle z_r,1\rangle,\langle a,1\rangle,\langle z_{r+1},1\rangle,\ldots,\langle z_n,1\rangle);$$
$$E_j'=(\langle 0,y_1^j\rangle,\langle 1,y_1^j\rangle,\ldots,\langle 0,y_m^j\rangle,\langle 1,y_m^j\rangle).$$

All of the above chains are obviously partial linear extensions of $L_1\times L_2$. (Note that $\langle a,1\rangle$ immediately follows $\langle 0,1\rangle$ in C_1' if $b=0$.) We shall show that these $(d-1)$ chains realize $\bar{Q}=Q-\{\langle a,0\rangle\}$ when restricted to $\bar{Q}$. Let $r\parallel s$ in $\bar{Q}$. It is enough to show that $r<s$ holds in one of the above chains. The letters x and y indicate arbitrary elements of $P_1-\{a\}$ and $P_2-\{0,1\}$ respectively. If $r=\langle 0,1\rangle$, then $r<\langle x,0\rangle$, $r<\langle 1,0\rangle$ and $r<\langle 1,y\rangle$ in C_1'. If $r=\langle 1,0\rangle$, then $r<\langle 0,y\rangle$, $r<\langle 0,1\rangle$, $r<\langle x,1\rangle$ and $r<\langle a,1\rangle$ in E_1'. The cases where s is $\langle 0,1\rangle$ or $\langle 1,0\rangle$ are similar. If $a<x$ in P_1, then $\langle a,1\rangle<\langle x,0\rangle$ in C_1' and $\langle x,0\rangle<\langle a,1\rangle$ in E_1'. If $a\parallel x$ in P_1, then $\langle x,1\rangle<\langle a,1\rangle$ in E_1' and $\langle a,1\rangle<\langle x,1\rangle$ in C_i' where $b<x$ holds in C_i (i is arbitrary if $b=0$). The remaining cases are easily checked.

Corollary 4.7. *If $Q=P\otimes 2$ where P is a normal d-irreducible poset, then one of the following four posets is $(d+1)$-irreducible:*

$$Q,\quad Q-\{\langle 0,1\rangle\},\quad Q-\{\langle 1,0\rangle\},\quad Q-\{\langle 0,1\rangle,\langle 1,0\rangle\}.$$

We call a normal irreducible poset P *regular* if $P\otimes 2$ is irreducible, and *irregular* otherwise. For $k\geq 2$, the k-irreducible poset

$$\mathbf{P}(2^k)=\mathbf{2}\otimes\mathbf{2}\otimes\cdots\otimes\mathbf{2}\quad (k \text{ times})$$

is clearly regular. In Section 6, we show that all 3-irreducible posets are regular. There is no known example of an irregular irreducible poset.

Theorem 4.8. *Let $P_1, P_2, \ldots, P_n$ $(n\geq 2)$ be normal posets, where each P_i equals $\mathbf{2}$ or is irreducible $(1\leq i\leq n)$, and let $d=\dim P_1+\dim P_2+\cdots+\dim P_n$. The dimension product $P_1\otimes P_2\otimes\cdots\otimes P_n$ is d-irreducible except possibly when both the* *following conditions are satisfied:*

(a) $P_i = \mathbf{2}$ *holds for exactly one* i, *say* $i = i_0$;

(b) *each* P_i *for* $i \neq i_0$ *is irregular.*

Proof. If $P_i = \mathbf{2}$ holds for k values of i with $k \geq 2$, then by Lemma 4.5, we can drop these posets and substitute $\mathbf{2} \otimes \mathbf{2} \cdots \otimes \mathbf{2}$ (k times), an irreducible poset. Therefore, if (a) fails, we can assume that each P_i is irreducible. In this case, the result follows by Proposition 4.6 and Lemma 4.5. We can now assume that (a) holds. If $P_i \otimes \mathbf{2}$ is irreducible for some $i \neq i_0$, then we again apply Proposition 4.6 and Lemma 4.5 in order to complete the proof.

Theorem 4.9 (The Embedding Theorem). *If* P *is a completely normal* d*-irreducible poset and* $k \geq 1$, *there is a* $(d+k)$*-irreducible poset* Q *that contains* P *as a subposet. In fact, if* $k \geq 2$, $Q = P \otimes \mathbf{P}(\mathbf{2}^k)$ *will serve.*

Proof. If $k = 1$, let Q be a $(d+1)$-irreducible poset of $P \otimes \mathbf{2}$ given by Corollary 4.7. Otherwise, let $Q = P \otimes \mathbf{P}(\mathbf{2}^k)$ which is $(d+k)$-irreducible by Theorem 4.8. For $x \in J^* = \mathbf{J}^*(P)$, let $\varphi(x) = \langle x, 0\rangle$, and for $x \in M^* = \mathbf{M}^*(P)$ but $x \notin J^*$, let $\varphi(x) = \langle x, 1\rangle$. Since $J^* \cup M^* = \mathbf{P}(P) = P$, φ is a one-to-one map from P to Q. Let $x < y$ in P. If $\varphi(y) = \langle y, 0\rangle$, then $x \in \mathbf{J}(P)$ by (N4). Since x is not a lower cover of 1 in L, $x \notin \mathbf{B}(P)$. Therefore, $\varphi(x) = \langle x, 0\rangle$. Hence, $x < y$ in P implies $\varphi(x) < \varphi(y)$ in Q, and since the converse is obvious, P is isomorphic to a subposet of Q.

Remarks. (1) Note that the above proof requires only one-half of condition (N4).

(2) For any regular normal irreducible poset P, $P \otimes \mathbf{2}$ is regular by Theorem 4.8.

(3) If $P_1, P_2, \ldots, P_n$ ($n \geq 2$) are irreducible and normal and P_1 is regular, then $P_1 \otimes P_2 \otimes \cdots \otimes P_n$ is regular by Theorem 4.8.

(4) Any irreducible poset of length one satisfies (N4).

5. Regularity of $P_{n,k}$ and $Q_{n,k}$

If $C = (c_1, c_2, \ldots, c_n)$ is a partial linear extension of a poset Q, then $C(0)$ denotes $(\langle c_1, 0\rangle, \langle c_2, 0\rangle, \ldots, \langle c_n, 0\rangle)$, a partial linear extension of $Q \times \mathbf{2}$. $C(1)$ is defined analogously. $C(0, 1)$ denotes the following partial linear extension of $Q \times \mathbf{2}$:

$$(\langle c_1, 0\rangle, \langle c_1, 1\rangle, \langle c_2, 0\rangle, \langle c_2, 1\rangle, \ldots, \langle c_n, 0\rangle, \langle c_n, 1\rangle).$$

Theorem 5.1. *For* k *and* n *as in Theorem* 3.1, $P_{n,k}$ *is completely normal and regular.*

Proof. We have already shown that $P_{n,k}$ is completely normal. Let $\bar{Q} = P_{n,k} \otimes \mathbf{2} - \{\langle 1, 0\rangle\}$. Clearly,

$$\mathrm{Crit}(\bar{Q}) = \{\langle b^{\lambda}, 0\rangle < \langle 0, 1\rangle \mid 1 \leq \lambda \leq n\} \cup \{\langle a^{\lambda}_{i-1}, 1\rangle < \langle a^{\mu}_{i}, 0\rangle \mid \lambda \neq \mu;\ 1 \leq \lambda, \mu \leq n;\ 1 \leq i \leq k\}.$$

For $g=1$ and $j\in[1, 2k-1]$, let C_j and C'_j be as in the proof of Theorem 3.1, and let E_j and F_j be obtained by deleting $d_0, d_1, \dots, d_k$ from C_j and C'_j respectively. Note that $F_1, F_2, \dots, F_{2k-1}$ realize $P_{n,k}-\{a_1^n\}$. Since a_1^1 is the first element of E_1, b^n is the first element of F_1. Let $F_1^{\#}$ be F_1 with b^n deleted and let $B=(b^1, b^2, \dots, b^{n-1})$. We define

$$F'_1=(\langle b^n, 0\rangle, \langle 0, 1\rangle, \langle b^n, 1\rangle, F_1^{\#}\,(0, 1)),$$

$$F'_j=F_j\,(0, 1)$$

for $2\le j\le 2k-1$, and

$$F'_{2k}=(B(0), \langle 0, 1\rangle, B(1), \langle a_1^n, 0\rangle).$$

We leave to the reader the verification that these $2k$ chains (when restricted to the underlying set of $\bar{Q}$) realize $\bar{Q}$. By duality, $P_{n,k}$ is regular.

Theorem 5.2. *For k and n as in Theorem 3.3, $Q_{n,k}$ is completely normal and regular.*

Proof. We know that $Q_{n,k}$ is completely normal. Let $Q=Q_{n,k}\otimes\mathbf{2}$ and $\bar{Q}=Q-\{\langle 1, 0\rangle\}$. Clearly,

$$\mathrm{Crit}(\bar{Q})=\{\langle b^\lambda, 0\rangle<\langle 0, 1\rangle \mid 1\le\lambda\le n\}\cup\{\langle d_{k-1}, 1\rangle<\langle b^\lambda, 0\rangle \mid 1\le\lambda\le n\}$$
$$\cup\{\langle c_{i-1}^\lambda, 1\rangle<\langle a_i^\mu, 0\rangle \mid \lambda\ne\mu;\ 1\le\lambda, \mu\le n;\ 1\le i\le k-1\}.$$

By the proof of Theorem 3.3, there are chains $F_1, F_2, \dots, F_{2k-2}$ that realize $Q_{n,k}-\{a_1^n\}$ such that b^n is the first element of F_1. Let $F_1^{\#}, B, F'_1$ and F'_j $(2\le j\le 2k-2)$ be defined as in the proof of Theorem 5.1. Also, let $F'_{2k-1}=(B(0), \langle 0, 1\rangle, B(1), \langle a_1^n, 0\rangle)$. $F'_1, F'_2, \dots, F'_{2k-1}$ realize $\bar{Q}$.

Let $R=Q-\{\langle 0, 1\rangle\}$. Clearly,

$$\mathrm{Crit}(R)=\{\langle 1, 0\rangle<\langle b^\lambda, 1\rangle \mid 1\le\lambda\le n\}\cup\{\langle d_{k-1}, 1\rangle<\langle b^\lambda, 0\rangle \mid 1\le\lambda\le n\}$$
$$\cup\{\langle c_{i-1}^\lambda, 1\rangle<\langle a_i^\mu, 0\rangle \mid \lambda\ne\mu;\ 1\le\lambda, \mu\le n;\ 1\le i\le k-1\}.$$

If $C_1, C_2, \dots, C_{2k-2}$ are linear extensions realizing $Q_{n,k}-\{d_{k-1}\}$, then R is realized by the restrictions to R of the following chains:

$$C_i(0, 1),\quad 1\le i\le 2k-2,$$

and

$$(\langle d_{k-1}, 1\rangle, \langle b^1, 0\rangle, \dots, \langle b^n, 0\rangle, \langle 1, 0\rangle, \langle b^1, 1\rangle, \dots, \langle b^n, 1\rangle).$$

Therefore, $Q_{n,k}$ is regular.

6. All 3-irreducible posets are regular

Lemma 6.1. *Let P be a 3-irreducible poset. Dropping $\langle 1, 0\rangle$ from $P\otimes\mathbf{2}$ lowers the dimension if there are elements $a, b\in P$ with $a \parallel b$ such that:*

(i) *If $x<a$ in P, then $x\notin \mathbf{M}^*(P)$.*

(ii) *If $x>a$ in P, then $x\notin \mathbf{J}^*(P)$.*

(iii) *In P, $x<a$ implies $x<b$, and $x>b$ implies $x>a$.*

(iv) *As binary relations, let $P^{\#}=P\cup\{\langle a,b\rangle\}$. (By (iii), $P^{\#}$ is a poset with the same underlying set as P.) $P^{\#}$ is a subposet of a planar lattice K, and there is a planar embedding of K in which a is on the right boundary.*

Proof. Let λ be the left-to-right ordering for the planar embedding of K mentioned in (iv). (See [8] for more details.) There are two linear extensions $C_1^{\#}$ and $C_2^{\#}$ of K such that $x\lambda y$ implies $x<y$ in $C_1^{\#}$ and $y<x$ in $C_2^{\#}$. In particular, $C_1^{\#}$ and $C_2^{\#}$ realize K. Let C_1 and $C_2=(D, a, E)$ be the restrictions of $C_1^{\#}$ and $C_2^{\#}$ to the underlying set of P. Also, let C_3 be a linear extension of $\{x\in P \mid a\not\leq x\}$. We define

$$C_1'=(\langle 0,1\rangle, C_1(0,1)),$$

$$C_2'=(D(0), \langle a,0\rangle, \langle 0,1\rangle, \langle a,1\rangle, E(0,1)),$$

$$C_3'=(C_3(0), \langle 0,1\rangle, \langle b,1\rangle, \langle a,0\rangle).$$

Clearly, each of these chains is a partial linear extension of $(P\cup\{0\})\times\mathbf{2}$. If $x\neq b$ and $x\parallel a$ in P, then by condition (iv), $x\lambda a$, and consequently, $a<x$ in C_2. Therefore *as sets* $D=\{x\in P \mid x<a\}$ and $E=\{x\in P\mid x\not\leq a\}$. Let $Q=P\otimes\mathbf{2}$ and $\bar{Q}=Q-\{\langle 1,0\rangle\}$. We shall show that the restrictions of C_1', C_2', C_3' to the underlying set of $\bar{Q}$ realize $\bar{Q}$.

Condition (i) implies that $\langle x,1\rangle$ occurs in C_2' whenever $\langle x,1\rangle\in Q$ with $x\in P$. Let $s=\langle x,i\rangle\parallel\langle y,j\rangle=t$ in Q, where $x,y\in P$ and $i,j\in\mathbf{2}$. By (N4), $x\parallel y$ in P. If $x\neq b$ or $y\neq a$, then $s<t$ in C_1' or C_2'. Otherwise, $s<t$ in C_3'. Clearly, $\langle a,0\rangle<\langle 0,1\rangle$ in C_2'. Let $\langle x,0\rangle\in Q$ with $x\neq a$. By condition (ii), $a\not\leq x$, and therefore, $\langle x,0\rangle<\langle 0,1\rangle$ in C_3'. Since $\langle 0,1\rangle<\langle x,i\rangle$ in C_1' for any $x\in P$ and $i\in\mathbf{2}$, the proof is complete.

In applying Lemma 6.1, we take advantage of the fact that $L=\mathbf{L}(P)$ is "nearly planar" for most 3-irreducible posets P. In each case, we add one element c to L to form K. We have $a<c<b$ in K. The planar embedding of K is obtained by placing c at the unique crossing in the diagram of L given in [8] or [6], and then removing the (at most one) line that no longer represents a cover. (For the duals, the diagram is reflected top to bottom.)

Immediately following each poset to which Lemma 6.1 applies, we have listed the corresponding a and b (in the notation of [6]).

$C: b_3, c_3$; $C^d: c_3, b_3$; $D: b_2, b_3$; $D^d: b_3, b_2$; $E_n: c, b_{n+3}$;
$E_n^d: b_{n+3}, c$; $F_n: a_{n+2}, d$; $G_n^d: c, b_1$; $H_n: c, d$; $CX_1: a_3, b_3$;
$CX_1^d: b_3, a_3$; $CX_2: a_3, c$; $CX_2^d: c, a_3$; $CX_3: a_3\ b_3$;
$CX_3^d: b_3, a_3$; $EX_1: a_3, b_4$; $EX_1^d: b_4, a_3$; $EX_2: a_3, b_3$;
$FX_1: a_3, b_1$; $FX_1^d: b_1, a_3$; $FX_2: a_3, b_1$; $I_n: c, b_{n+3}$;
$I_n^d: b_{n+3}, c$; $J_n^d: d, b_1$.

We have shown (using duality) that all 3-irreducible posets, except possibly A_n ($n \geq 0$) and B, are regular. In Section 5, we showed that $B = Q^d_{3,2}$ is regular. By the following result, any poset of the form A_n is regular. Thus, all 3-irreducible posets are regular.

Lemma 6.2. *Let P be a normal irreducible poset. Dropping $\langle 1, 0\rangle$ from $P \otimes \mathbf{2}$ lowers the dimension if there is $a \in P$ such that:*

(i) *a is minimal in P.*

(ii) *$a \notin \mathbf{M}^*(P)$.*

(iii) *In P, $x > a$ implies $x \notin \mathbf{J}^*(P)$.*

Proof. Let $J^* = \mathbf{J}^*(P)$, $Q = P \otimes \mathbf{2}$, and $\bar{Q} = Q - \{\langle 1, 0\rangle\}$. Assume that $\dim P = d$. Let $C_1, C_2, \ldots, C_{d-1}$ be linear extensions of $P^{\#} = P - \{a\}$ that realize $P^{\#}$. Let

$$C_1' = (\langle a, 0\rangle, \langle 0, 1\rangle, C_1(0, 1)),$$

and $C_i' = C_i(0, 1)$ for $2 \leq i \leq d-1$. Let A be a linear extension of $\{x \in P \mid a < x\}$, B be a linear extension of $\{x \in P \mid a \nless x\}$ and

$$C_d' = (B(0), \langle 0, 1\rangle, B(1), \langle a, 0\rangle, A(1)).$$

Each of the chains $C_1', C_2', \ldots, C_d'$ is clearly a partial linear extension of $(P \cup \{0\}) \times \mathbf{2}$. We shall show that $\bar{Q}$ is realized by the restrictions of these chains to the underlying set of $\bar{Q}$.

Condition (ii) implies that $\langle a, 1\rangle \notin Q$. The letters x and y denote elements of $P^{\#}$. If $x \parallel y$ in P and $x < y$ in C_i with $1 \leq i \leq d-1$, then $\langle x, j\rangle < \langle y, k\rangle$ in C_i' whenever $j, k \in \mathbf{2}$. If $x < y$ in P and $y \in J^*$, then $\langle x, 1\rangle < \langle y, 0\rangle$ in C_1' and $\langle y, 0\rangle < \langle x, 1\rangle$ in C_d'. (Note that (N4) would imply that $\langle x, 1\rangle \notin Q$.) If $x \in J^*$, $\langle a, 0\rangle < \langle 0, 1\rangle < \langle x, 0\rangle$ in C_1' and $\langle x, 0\rangle < \langle 0, 1\rangle < \langle a, 0\rangle$ in C_d'. If $a \nless x$, then $\langle a, 0\rangle < \langle x, 1\rangle$ in C_1' and $\langle x, 1\rangle < \langle a, 0\rangle$ in C_d'. This completes the proof of the lemma.

Remarks. (1) The dual of Lemma 6.2 could have been applied in the last part of the proof of Theorem 5.2.

(2) Let P be an irreducible poset of length one with $|\mathrm{Irr}(P)| = \emptyset$. Clearly, P is completely normal. It follows from Lemma 6.2 that P is also regular.

7. Irreducible posets

Table 1 gives data for the $(k+3)$-irreducible posets of the form $Q = P \otimes \mathbf{P}(\mathbf{2}^k)$, where P is a 3-irreducible poset and $k \geq 1$. Note that the length of Q is independent of k, and can be given any positive value by suitably choosing P.

Let $d \geq 4$. There are no d-irreducible posets with less than $2d$ elements (Hiraguchi [5]) or with exactly $2d+1$ elements (Kimble [9]). Table 1 contains

Table 1

P	Size of P	$P\otimes\mathbf{P}(2^k)$ Size	Length	Width
A_n	$2n+6$	$2n+6+2k$	1	$n+3+k$
B	7	$10+2k$	2	$4+k$
C	7	$10+2k$	3	$3+k$
D	6	$9+2k$	3	$3+k$
E_n	$2n+7$	$2n+10+2k$	2	$n+4+k$
F_n	$2n+7$	$2n+10+2k$	3	$n+3+k$
G_n	$2n+7$	$4n+12+2k$	$n+3$	$3+k$
H_n	$2n+7$	$4n+12+2k$	$n+3$	$4+k$
CX_1	7	$10+2k$	3	$3+k$
CX_2	7	$10+2k$	2	$4+k$
CX_3	7	$9+2k$	3	$3+k$
EX_1	7	$10+2k$	2	$4+k$
EX_2	7	$10+2k$	3	$3+k$
FX_1	7	$10+2k$	3	$3+k$
FX_2	7	$10+2k$	3	$3+k$
I_n	$2n+8$	$2n+11+2k$	3	$n+4+k$
J_n	$2n+8$	$4n+12+2k$	$n+3$	$3+k$

d-irreducible posets of any cardinality $\geq 2d+2$. In fact, W.T. Trotter, Jr. [15] can construct such posets that are of length one.

$A_n \otimes G_n$ is a 6-irreducible poset whose length = width = $n+3$. By the Embedding Theorem, the following statement holds if $d \geq 6$: *For any n, there is a d-irreducible poset whose length and width both exceed n.*

This statement is false if $d=3$. What happens if $d=4$ or $d=5$?

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