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On the disc-structure of perfect graphs II. The co-C_4-structure ☆

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Abstract

Let $\mathscr{F}$ be any family of graphs. Two graphs $G_1=(V_1,E_1), G_2=(V_2,E_2)$ are said to have the same $\mathscr{F}$-structure if there is a bijection $f:V_1\to V_2$ such that a subset S induces a graph belonging to $\mathscr{F}$ in G_1 iff its image $f(S)$ induces a graph belonging to $\mathscr{F}$ in G_2. We prove that if a C_5-free graph H has the $\{2K_2,C_4\}$-structure of a perfect graph G then H is perfect.

Keywords: Perfect graphs; Graph colouring

1. Introduction

A graph G is *perfect* if for each induced subgraph H of G, the chromatic number of H equals the number of vertices in a largest clique of H. A *hole* is a chordless cycle with at least four vertices. An *anti-hole* is the complement of a hole. Berge [1] proposed the conjecture that a graph is perfect iff it does not contain an odd hole or its complement as an induced subgraph. This conjecture is known as the Strong Perfect Graph Conjecture (SPGC for short) and is still open. A weaker conjecture, also proposed by Berge, states that a graph is perfect if and only if its complement is. This conjecture was proved by Lovász [10] and this result is known nowadays as the Perfect Graph Theorem (PGT for short). We shall call a graph *Berge* if it contains no odd hole and no odd anti-hole as induced subgraphs.

We would like to propose a generalization of the PGT. For this purpose, we need a few definitions. Let $\mathscr{F}$ be a family of graphs. Two graphs $G_1=(V_1,E_1), G_2=(V_2,E_2)$ are said to have the same $\mathscr{F}$-structure if there is a bijection $f:V_1\to V_2$ such that a subset S induces a graph belonging to $\mathscr{F}$ in G_1 iff its image $f(S)$ induces a graph

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belonging to $\mathscr{F}$ in G_2. A graph is called a *disc* if it is isomorphic to a hole with at least five vertices or the complement of such a hole. Holes and discs play special roles in perfect graph theory. For example, it is well known that hole-free (triangulated) graphs are perfect, and a theorem of Hayward showed that disc-free graphs are perfect [4]. Define the *disc-structure* to be the $\mathscr{F}$-structure with $\mathscr{F}$ being the set of discs of all lengths.

Chvátal [2] conjectured and Reed [13] proved that perfection of a graph depends only on its $\{P_4\}$-structure. In other words, Reed's theorem states that a graph H is perfect iff it has the $\{P_4\}$-structure of some perfect graph G. Since the P_4 is an induced subgraph of a disc, the disc-structure, in some sense, generalizes the P_4-structure. And it is natural to conjecture that perfection of a graph depends only on its disc-structure.

Conjecture 1 (*Disc Conjecture*). *If a graph H has the disc-structure of a perfect graph G then H is perfect.*

Both Reed's theorem and the Disc Conjecture are semi-strong perfect graph statements, in the sense that they imply the PGT and are implied by the SPGC (however, the known proofs of all semi-strong perfect graph theorems rely on the PGT). Also, one can restate the SPGC as a statement on the odd disc-structure. At the moment, a proof of the Disc Conjecture seems hard to find. Even a seemingly much simpler statement, that in a minimal imperfect graph every P_4 extends into a disc, has not been established. By observing that every disc contains a hole, we propose studying the hole- and co-hole-structure of a graph. Define the *hole-structure* (respectively, *co-hole-structure*) to be the $\mathscr{F}$-structure with $\mathscr{F}$ being the set of holes of all lengths (respectively, set of holes and anti-holes of all lengths). The hole-structure is not invariant under complementation, but the co-hole-structure is (we shall discuss the hole-structure later). Thus the following theorem is also a semi-strong perfect graph theorem.

Theorem 1. *If a graph H has the co-hole structure of a perfect graph G then H is perfect.*

We find it more convenient to prove a stronger statement. It is customary to let $2K_2$ denote the complement of C_4. Obviously, if two graphs have the same co-hole-structure then they have the same $\{C_4, 2K_2\}$-structure; and, if H has the co-hole-structure of a perfect graph G, then H must be C_5-free. Thus, Theorem 1 is implied by the following theorem, to provide its proof is the purpose of this paper.

Theorem 2. *If a C_5-free graph H has the $\{2K_2, C_4\}$-structure of a perfect graph G then H is perfect.*

Every disc, except for the C_5, contains a $2K_2$ or C_4 as an induced subgraph. In this sense, Theorem 2 is a weakening of the Disc Conjecture. Conceivably, one could prove that if two graphs H and G have the same disc-structure but not the same

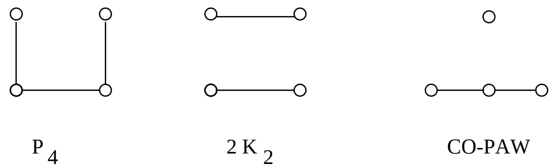

Fig. 1.

$\{2K_2, C_4\}$-structure then they must have a certain property P (like having a star-cutset or an even pair) that minimal imperfect graphs cannot have. Then, Theorem 2 could contribute to a proof of the Disc Conjecture in the following way. Let H and G be two graphs with the same disc-structure. Without loss of generality, we may assume that H is minimal imperfect and G is perfect. Since H cannot have property P (assuming we have a proof for P), H and G must have the same $\{2K_2, C_4\}$-structure, contradicting Theorem 2. The same approach was used in [8] to prove a theorem on the sibling-structure of perfect graphs using Reed's theorem on the P_4-structure (see Fig. 1).

In Section 2, we shall give definitions and background results. We shall give a proof of the above theorem in Sections 3–8. In the last section (Section 9), we shall discuss a conjecture (The Hole Conjecture) related to the Disc Conjecture.

2. Background

In this section, we introduce definitions and background results needed to prove Theorem 2. Let G be a graph. Then $\bar{G}$ denotes the complement of G. Let S be a set of vertices of G. $G[S]$ shall denote the subgraph of G induced by S. S is *homogeneous* if $2 \leqslant |S| < |V(G)|$ and every vertex outside S is adjacent to all or to no vertices of S. S is called a *star-cutset* if $G - S$ is not connected and in S there is a vertex adjacent to all other vertices of S. Let x be a vertex of G, then $N_G(x)$ denotes the set of vertices adjacent to x in G. A vertex x is said to *dominate* a vertex y in G in $N_G(y) \subseteq N_G(x) \cup \{x\}$. Vertices x and y are *comparable* if x dominates y or vice versa. By C_k, we denote the chordless cycle of length k. By $v_1v_2\ldots v_k$ we denote the chordless cycle with vertices $v_1, v_2, \ldots, v_k$ and edges v_iv_{i+1} for $1 \leqslant i \leqslant k$ with $v_{k+1} = v_1$. An *even-pair* is a set of two vertices x, y such that every induced path joining x to y has an even number of edges. If $xy \in E(G)$ then we say that x *sees* y in G, else we say that x *misses* y in G.

A graph is *minimal imperfect* if it is not perfect but each of its proper induced subgraphs is. The PGT implies that a graph is minimal imperfect iff its complement is. We shall rely on a number of results on minimal imperfect graphs. In the remainder of this section, we let H be a minimal imperfect graph.

Chvátal [3] proved that

$$H \text{ cannot contain a star-cutset.} \tag{1}$$

From (1), it is easy to see that

no vertex in H can dominate another vertex. (2)

Using (1), Hoàng [7] showed that

in H, each P_3 extends into a hole. (3)

(1) also implies the following statement which was first established by Lovász [10]:

H cannot contain a homogeneous set. (4)

Meyniel [11] proved that

H cannot contain an even-pair. (5)

Let $\alpha(H)$ denote the number of vertices of a largest stable set in H. A theorem of Padberg [12] showed that

each vertex in H belongs to $\alpha(H)$ stable sets of size $\alpha(H)$. (6)

Finally, Hayward [4] proved that

H must contain a disc. (7)

3. Proof of Theorem 2

Let G be a perfect graph and suppose that a C_5-free graph H has the $\{2K_2, C_4\}$-structure of G. We may assume that G and H are defined on the same set of vertices so that a subset S of $V(H)$ $(=V(G))$ induces a C_4 or a $2K_2$ in H iff S induces a C_4 or a $2K_2$ in G. We may assume that H is imperfect and so it contains, as induced subgraph, a minimal imperfect graph. Thus we may assume that H is minimal imperfect. By (7), we may assume that H contains a disc D and by replacing H by its complement if necessary we may assume that $H[D]$ is a hole of length at least 6.

By $Q_k[v_1, v_2, \ldots, v_k]$, we denote the graph with vertices $v_1, v_2, \ldots, v_k$ such that, for any i, j with $i < j$, the edge v_iv_j is present iff $j - i \geqslant 3$ and $|j - i|$ is odd. By $Z_k[v_1, v_2, \ldots, v_k]$ $(k \geqslant 6)$ for an even integer k, we denote the graph $Q_k[v_1, v_2, \ldots, v_k] - v_1v_k$ (Fig. 2 shows the graphs Q_8 and Z_8). Note that $Z_k - x$ is isomorphic to Q_{k-1} for any vertex $x \in Z_k$. A *domino* is the graph obtained from a C_6 by adding a chord that does not form a triangle with two edges of the C_6. A *co-domino* is the complement of a domino (see Fig. 3). The following lemma shows that H must be Berge.

Lemma 1. *Let H' and G' be two graphs defined on the same set of vertices such that a subset S of $V(H') = V(G')$ induces a $2K_2$ or a C_4 in H' iff S induces a $2K_2$ or a C_4 in G'. Suppose that H' is a hole $v_1v_2 \ldots v_k$. If k is odd and at least 7 then G' or $\overline{G}'$ is the same hole $v_1v_2 \ldots v_k$; if k is even and at least 8 then G' or $\overline{G}'$ is the same hole $v_1v_2 \ldots v_k$ or the graph $Z_k[v_1, v_2, \ldots, v_k]$; and if $k = 6$ then G' or $\overline{G}'$ is one*

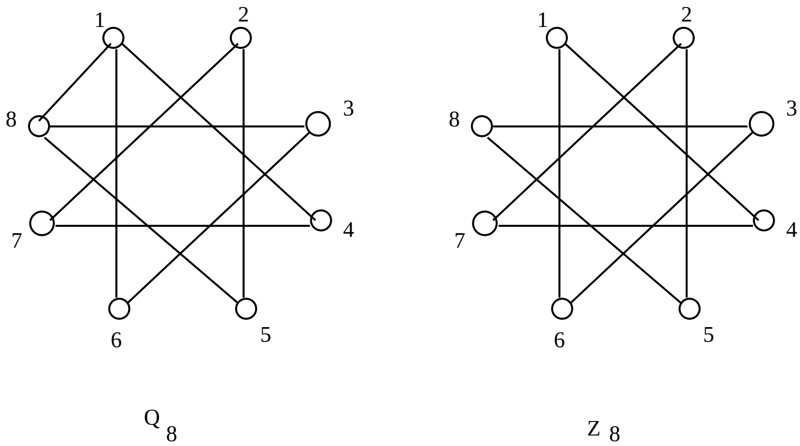

Fig. 2.

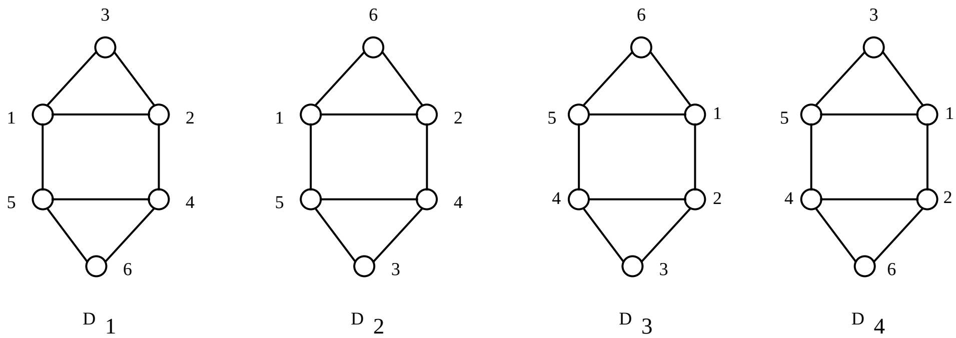

Fig. 3.

of the following graphs:

(i) $Z_6[v_1, v_2, v_3, v_4, v_5, v_6]$,
(ii) *the co-domino*,
(iii) *the* C_6*'s* $v_1v_2v_6v_4v_5v_3$, $v_1v_5v_3v_4v_2v_6$, $v_1v_3v_2v_4v_6v_5$,
(iv) *the* C_6 $v_1v_2v_3v_4v_5v_6$.

The Theorem follows from the following four Lemmata.

Lemma 2. *If there is a set* D *of vertices such that* $H[D]$ *is the hole* $v_1v_2 \ldots v_k$, *and* $G[D]$ *or* $\bar{G}[D]$ *is the graph* $Z_k[v_1, v_2, \ldots, v_k]$ *then* H *has a homogeneous set*, *or an even-pair*.

Lemma 3. *If* $H[D]$ *is a* C_6, *and* $G[D]$ *is a domino or co-domino then* H *contains an even-pair*.

Lemma 4. *If $H[D]$ is the C_6 $v_1v_2v_3v_4v_5v_6$, and $G[D]$ or $\bar{G}[D]$ is one of the C_6's $v_1v_2v_6v_4v_5v_3$, $v_1v_5v_3v_4v_2v_6$, $v_1v_3v_2v_4v_6v_5$ then some v_i belongs to precisely one stable set of size $\alpha(H)$ in H, or H has a star-cutset. In particular, H is not minimal imperfect.*

Lemmata 2, 3, and 4 imply that whenever $H[D]$ is a hole of length at least six, $G[D]$ must be the same hole with the same cyclic order, or its complement. The following lemma shall complete the proof of the theorem.

Lemma 5. *Suppose that, for any set D of vertices, whenever $H[D]$ is the hole $v_1v_2\dots v_k$ ($k \geqslant 6$), $G[D]$ or $\bar{G}[D]$ is the hole $v_1v_2\dots v_k$. Then G or $\bar{G}$ is isomorphic to H.*

We shall prove Lemmata 1–5 in the remainder of this paper.

4. Proof of Lemma 1

Observation 1. *Suppose a graph G' has the $\{2K_2, C_4\}$-structure of the induced path P_k with $k \geqslant 6$. Then, either* (i) *G' contains a $2K_2$ and no C_4 and has the $\{2K_2\}$-structure of P_k, or* (ii) *G' contains a C_4 and no $2K_2$ and has the $\{C_4\}$-structure of $\overline{P_k}$.*

Proof. Let $S_1, S_2, \dots$ be the four-vertex sets of the P_k ($k \geqslant 6$) that induce a $2K_2$. Let $f(S_i)$ denote the image of S_i in G'. It is easy to prove (by induction on k) that for any two sets S_i, S_j, there is a sequence $S_i, S_{i_1}, S_{i_2}, \dots, S_j$ such that any two consecutive sets in the sequence intersect at three vertices. Thus if $f(S_i)$ induces a $2K_2$ (respectively, C_4) in G', then the image of every set in this sequence induces a $2K_2$ (respectively, C_4) in G'. So, if some $f(S_i)$ induces a $2K_2$ (respectively, C_4) in G', then G' contains no C_4 (respectively, no $2K_2$) and has the $\{2K_2\}$ -structure (respectively, $\{C_4\}$-structure) of a P_k (respectively, $\overline{P_k}$). □

The following observation can be proved by a routine case analysis and so we omit the proof.

Observation 2. *The only graph with the $\{2K_2\}$-structure of the P_6 $v_1v_2\dots v_6$ is the P_6 $v_1v_2\dots v_6$ itself, or the P_6 $v_1v_5v_3v_4v_2v_6$, or the graph $Q_6[v_1, v_2, \dots, v_6]$.*

Observation 3. *Let H' be the P_k $v_1v_2\dots v_k$ with $k \geqslant 6$ and G' be a graph with the $\{2K_2\}$ -structure of H'. If $G'[\{v_1, v_2, \dots, v_{k-1}\}]$ is the graph $Q_{k-1}[v_1, v_2, \dots, v_{k-1}]$, then G' is the graph $Q_k[v_1, v_2, \dots, v_k]$. If $G'[\{v_1, v_2, \dots, v_{k-1}\}]$ is the P_{k-1} $v_1v_2\dots v_{k-1}$, then G' is the graph P_k $v_1v_2\dots v_k$.*

Proof. By induction on k. It is easy to see that the Observation is true for $k = 6$. Assume that the Observation is true for k (we shall show that it is true for $k + 1$).

Consider the graph $G'[\{v_1, v_2, \ldots, v_k, v_{k+1}\}]$. Suppose that $G'[\{v_1, v_2, \ldots, v_k\}]$ is the graph $Q_k[v_1, v_2, \ldots, v_k]$. For $i = 1, 2, \ldots, k-3$, the $2K_2$ $\{v_i, v_{i+1}, v_k, v_{k+1}\}$ implies that $v_i v_{k+1} \in E(G')$ if and only if $k - i$ is even and at least two. Thus, $G'[\{v_1, v_2, \ldots, v_{k+1}\}]$ is the graph $Q_{k+1}[v_1, v_2, \ldots, v_{k+1}]$. A similar argument shows that if $G'[\{v_1, v_2, \ldots, v_k\}]$ is the graph P_k $v_1 \ldots v_k$ then $G'[\{v_1, v_2, \ldots, v_k, v_{k+1}\}]$ is the graph P_{k+1} $v_1 \ldots v_k v_{k+1}$. □

Observation 4. *The only graphs with the $\{2K_2\}$-structure of the P_k $v_1 v_2 \ldots v_k$ $(k \geqslant 7)$ is the P_k $v_1 v_2 \ldots v_k$ itself and the $Q_k[v_1, v_2, \ldots, v_k]$.*

Proof. Using Observations 2 and 3, one can show (see Appendix, Fact 1) that the Observation is true for $k = 7$. Now, using induction on k and Observation 3, we can see that the Observation must hold. □

Observation 5. *Let H' be the hole C_k $v_1 v_2 \ldots v_k$ $(k \geqslant 7)$, and G' be a graph having the $\{2K_2\}$-structure of H'. Then*

(i) *if k is odd then G' is the hole C_k $v_1 v_2 \ldots v_k$,*
(ii) *if k is even then G' is the hole C_k $v_1 v_2 \ldots v_k$ or the graph $Z_k[v_1, v_2, \ldots, v_k]$.*

Proof. Using Observation 2 and a simple case analysis, one can show that the Observation is true for $k = 7$. Suppose that $k > 7$. Write $F = G'[\{v_1, v_2, \ldots, v_{k-1}\}]$. If F is the P_{k-1} $v_1 \ldots v_{k-1}$ then it is easy to see that G' is the hole $v_1 \ldots v_k$. By Observation 4, we may now assume that F is the graph $Q_{k-1}[v_1, v_2, \ldots, v_{k-1}]$. Since $G'[\{v_{k-2}, v_{k-3}, \ldots, v_1, v_k\}]$ is a Q_{k-1}, by Observation 3, we know that $G'[\{v_{k-2}, v_{k-3}, \ldots, v_1, v_k\}]$ is the graph $Q_k[v_{k-2}, v_{k-3}, \ldots, v_1, v_k]$. In particular, we have $v_k v_1, v_k v_2 \notin E(G')$. If k is odd then by definition of F, we have $v_2 v_{k-1}, v_2 v_3 \notin E(G')$ and so $\{v_2, v_3, v_k, v_{k-1}\}$ does not induce a $2K_2$ or C_4 in G', a contradiction. So we know that k is even. The $2K_2\{v_2, v_3, v_k, v_{k-1}\}$ of H' implies that $v_k v_{k-1} \notin E(G')$. Thus G' is the graph $Z_k[v_1, v_2, \ldots, v_k]$. □

Now we can complete the proof of Lemma 1.

Let H' be the hole $v_1 v_2 \ldots v_k$. By replacing G' by its complement if necessary, we may assume that some set S of G' induces a $2K_2$. Suppose that $k \geqslant 7$. It follows from Observation 1 that G' has the $\{2K_2\}$-structure of H', and the Lemma follows from Observation 5. Now, suppose $k = 6$. If G' has the $\{2K_2\}$-structure of H', then a simple case analysis (see Appendix A, Fact 2) shows that G' is the graph $Z_6[v_1, v_2, v_3, v_4, v_5, v_6]$, or one of the C_6's $v_1 v_2 v_6 v_4 v_5 v_3$, $v_1 v_5 v_3 v_4 v_2 v_6$, $v_1 v_3 v_2 v_4 v_6 v_5$, $v_1 v_2 v_3 v_4 v_5 v_6$. We may now suppose that some set X different from S induces a C_4 in G'. Let H' induce the C_6 $v_1 v_2 v_3 v_4 v_5 v_6$. Write $S_1 = \{v_1, v_2, v_4, v_5\}$, $S_2 = \{v_2, v_3, v_5, v_6\}$, $S_3 = \{v_1, v_6, v_3, v_4\}$. Then each S_j induces a $2K_2$ in H'. By interchanging G' with its complement if necessary, we may assume that one of these three sets induces a C_4 in G' and the remaining two sets induce a $2K_2$ in G'. A case analysis (see Appendix A, Fact 3) shows that G' must be a co-domino. Actually, we can describe the labelling of the co-domino in more detail. Without loss of generality, we may assume that, in G', S_1 induces a C_4, and S_2, S_3 induce a $2K_2$. It is a routine matter to verify that the co-domino must be one of the four (labelled) graphs shown in Fig. 3. □

5. Proof of Lemma 2

Define $P_H(D)$ (respectively, $U_H(D), R_H(D)$) to be the set of vertices outside D that see some but not all (respectively, all, no) vertices of D in H. We shall say that a set S of vertices is *bad* if $G[S]$ is a $2K_2$ or C_4 and $H[S]$ is not, or vice versa.

Let D be the smallest set of vertices such that $H[D]$ is the hole $v_1v_2\dots v_k$, and $G[D]$ or its complement is the graph $Z_k[v_1, v_2, \dots, v_k]$. By replacing G by its complement if necessary, we may assume that $G[D]$ is the graph $Z_k[v_1, v_2, \dots, v_k]$.

We may assume that $P_H(D)$ is not empty for otherwise D is a homogeneous set or $D = V(H)$ implying that H has an even-pair. We claim that

$$\text{no vertex } x \in P_H(D) \text{ forms a } C_r \text{ with a path } P_{r-1} \text{ of } D,\ r \geqslant 4. \tag{8}$$

Suppose that, for some i, some $x \in P_H(D)$ forms in H a C_r with a path P_{r-1} $v_iv_{i+1}\dots v_{i+r-2}$ of D, $r \geqslant 4$. We must have $r \neq 4$, for otherwise $\{x, v_i, v_{i+1}, v_{i+2}\}$ is a bad C_4 in H. Since H contains no odd hole, we have that r is even and at least six. Lemma 1 implies that $X = G[\{x, v_i, v_{i+1}, \dots, v_{i+r-2}\}]$ is a hole, or anti-hole, or Z_r, or $\bar{Z}_r$. But it is easy to see that X cannot be a hole, or anti-hole, or $\bar{Z}_r$. Thus X is the graph $Z_r[x, v_i, v_{i+1}, \dots, v_{i+r-2}]$. In particular, we have $xv_i, xv_{i+1} \notin E(G)$ and $xv_{i+2} \in E(G)$. We must have $r = k$ for otherwise the hole $xv_i \dots v_{i+r-2}$ contradicts the choice of D. Let y be the vertex in D that sees v_i and v_{i+r-2} in H. If $xy \notin E(H)$ then the C_4 xv_iyv_{i+r-2} is bad in H, if $xy \in E(H)$ then the $2K_2$ $\{x, y, v_{i+1}, v_{i+2}\}$ is bad in H (we have $yv_{i+1} \notin E(G)$, $yv_{i+2} \in E(G)$ by definition of Z_k). Thus (8) is justified.

We claim that

$$\text{in } H, \text{ each vertex } x \in P_H(D) \text{ has precisely one neighbour in } D. \tag{9}$$

Suppose there is a vertex x in $P_H(D)$ that has at least two neighbours in D. By (8), it is easy to see that, in H, x has precisely two neigbours in D, and furthermore, these two neighbours are consecutive vertices v_i, v_{i+1} of D. We shall implicitly refer to Observations 1–4 many times. The induced path $v_{i+2}v_{i+3}\dots v_{i-1}v_ix$ of H implies that its corresponding image in G is the graph $Q_k[v_{i+2}, v_{i+3}, \dots, v_{i-1}, v_i, x]$. Since k is even, we have $xv_{i+2} \in E(G)$. Now the induced path $xv_{i+1}v_{i+2}\dots v_{i-1}$ of H implies that its image in G is the graph $Q_k[x, v_{i+1}, v_{i+2}, \dots, v_{i-1}]$; in particular, we have $xv_{i+2} \notin E(G)$, a contradiction. Thus, (9) is justified.

We shall show that

$$\text{in } H, \text{ each vertex in } D \text{ has at most one neighbour in } P_H(D). \tag{10}$$

Suppose a vertex v_i has two neighbours x, y in $P_H(D)$. Write $S_1 = \{x, v_i, v_{i+1}, \dots, v_{i-2}\}$, $S_2 = \{y, v_i, v_{i+1}, \dots, v_{i-2}\}$. By (9), S_j is a chordless path for $j = 1, 2$. By Observations

1, 2 and 4, we know that $G[S_j]$ is the graph $Q_k[z, v_i, v_{i+1}, \dots, v_{i-2}]$ with $z = x$ if $j = 1$, and $z = y$ if $j = 2$. In particular, in G, x and y see v_{i+2}, v_{i+4} and miss v_i, v_{i+1}, v_{i+3}. If $xy \in E(G)$ then $\{x, y, v_i, v_{i+3}\}$ is a bad $2K_2$ in G, if $xy \notin E(G)$ then $\{x, y, v_{i+2}, v_{i+4}\}$ is a bad C_4 in G. Thus (10) is justified.

Next, we shall show that

$$\text{in } H, \text{ if } v_i \text{ has a neighbour in } P_H(D) \text{ then } v_{i+1} \text{ has no neighbour in } P_H(D). \tag{11}$$

Suppose that, in H, there is a vertex v_i such that v_i sees a vertex $x \in P_H(D)$ and v_{i+1} sees a vertex $y \in P_H(D)$. By Observations 2 and 4, the image in G of the induced path $xv_iv_{i-1} \dots v_{i+2}$ of H is the graph $Q_k[x, v_i, v_{i-1}, \dots, v_{i+2}]$. This implies $xv_i, xv_{i-1} \notin E(G)$. Similarly, the image in G of the induced path $xv_iv_{i+1} \dots v_{i-2}$ of H is the graph $Q_k[x, v_i, v_{i+1}, \dots, v_{i-2}]$. This implies $xv_{i+1}, xv_{i+3} \notin E(G)$ and $xv_{i+2} \in E(G)$. The image in G of the induced path $yv_{i+1}v_{i+2} \dots v_{i-1}$ is the graph $Q_k[y, v_{i+1}, v_{i+2}, \dots, v_{i-1}]$. This implies $yv_{i+1}, yv_{i+2} \notin E(G)$ and $yv_{i+3} \in E(G)$. The image in G of the induced path $yv_{i+1}v_iv_{i-1} \dots v_{i+3}$ is the graph $Q_k[y, v_{i+1}, v_i, v_{i-1}, \dots, v_{i+3}]$. This implies $yv_i \notin E(G)$ and $yv_{i-1} \in E(G)$. We must have $xy \notin E(H)$ for otherwise xyv_iv_{i+1} is a bad C_4 in H, and $xy \in E(G)$ for otherwise $\{x, v_{i+2}, y, v_{i+3}\}$ is a bad $2K_2$ in G. Now, $xyv_{i-1}v_{i+2}$ is a bad C_4 in G. Thus (11) holds.

Let v_i be a vertex in D that has a neighbour in $P_H(D)$. To conclude the proof, we claim that

$$\{v_i, v_{i+2}\} \text{ is an even pair of } H. \tag{12}$$

Suppose there is in H an odd induced path P with endpoints v_i and v_{i+2}. The interior vertices of P must belong to $R_H(D) \cup P_H(D) \cup (D - \{v_i, v_{i+1}, v_{i+2}\})$. By (11), v_{i+1} sees no interior vertex of P in H. But then P and v_{i+1} form an odd hole of H, a contradiction. □

6. Proof of Lemma 3

Let $H[D]$ induce the C_6 $v_1v_2v_3v_4v_5v_6$. The proof of Lemma 1 allows us to assume that $G[D]$ is one of the four (labelled) graphs shown in Fig. 3. In particular, in G, S_1 must be the C_4 $v_1v_2v_4v_5$.

First, we shall show that

$$\text{in } H, \text{ there is no vertex } y \text{ outside } D \text{ that sees } v_2 \text{ and misses } v_4. \tag{13}$$

Suppose that such a vertex y exists. We shall show that

$$y \text{ sees } v_5 \text{ in } H. \tag{14}$$

Suppose that y misses v_5 in H. The $2K_2$ $\{y, v_2, v_4, v_5\}$ of H implies that, in G, y sees v_2, v_5 and misses v_4. Next, we have

$yv_1 \in E(G)$, for otherwise G contains the bad C_4 $v_1v_5yv_2$,
$yv_6 \in E(H)$, for otherwise H contains the bad $2K_2$ $\{y, v_2, v_5, v_6\}$,
$yv_1 \in E(H)$, for otherwise H contains the bad C_4 $yv_2v_1v_6$.

Now, H contains the bad $2K_2$ $\{y, v_1, v_4, v_5\}$. Thus, (14) is justified.

We continue the proof of (13). We may suppose such a vertex y exists and that $yv_5 \in E(H)$. We must have $yv_3 \in E(H)$, for otherwise H contains the C_5 $yv_2v_3v_4v_5$, a contradiction. We shall argue using the four graphs in Fig. 3.

First, suppose that $G[D]$ is the graph D_1. The C_4 $yv_3v_4v_5$ of H implies that y sees v_3 and misses v_4, v_5 in G. We must have

$yv_6 \in E(G)$, for otherwise G contains the bad $2K_2$ $\{v_3, y, v_4, v_6\}$,
$yv_2 \notin E(G)$, for otherwise G contains the bad C_4 $yv_2v_4v_6$.

But now, G contains the C_5 $v_3v_2v_4v_6y$. So $G[D]$ cannot be the graph D_1.

Second, suppose that $G[D]$ is the graph D_2. Then, H contains the bad C_4 $yv_3v_4v_5$. So $G[D]$ cannot be the graph D_2.

Third, suppose that $G[D]$ is the graph D_3. The C_4 $yv_3v_4v_5$ of H implies that $yv_3, yv_5 \in E(G)$, $yv_4 \notin E(G)$. We must have

$yv_2 \notin E(G)$, for otherwise G contains the bad C_4 $v_2v_4v_5y$,
$yv_1 \notin E(G)$, for otherwise G contains the bad C_4 $yv_1v_2v_3$.

But now, G contains the C_5 $v_5v_1v_2v_3y$. So $G[D]$ cannot be the graph D_3.

Fourth and last, suppose that $G[D]$ is the graph D_4. The C_4 $yv_3v_4v_5$ of H implies that $yv_3, yv_4 \in E(G)$, $yv_5 \notin E(G)$. We must have

$yv_1 \notin E(G)$, for otherwise G contains the bad C_4 $v_1yv_4v_5$,
$yv_2 \notin E(G)$, for otherwise G contains the bad C_4 $yv_2v_1v_3$.

But now, G contains the C_5 $v_3v_1v_2v_4y$. So $G[D]$ cannot be the graph D_4.

Since all the eventualities are covered, (13) is proved. We claim that

in H, there is no vertex z outside D that sees v_4 and misses v_2. (15)

To see this, write $v_1 = v_{5'}$, $v_2 = v_{4'}$, $v_3 = v_{3'}$, $v_4 = v_{2'}$, $v_5 = v_{1'}$, $v_6 = v_{6'}$. Then $H[D]$ is the hole $v_{1'}v_{2'}v_{3'}v_{4'}v_{5'}v_{6'}$ and $G[D]$ is one of the four graphs in Fig. 3 with subscript i replaced by i' for $i = 1, \ldots, 6$. Now, (15) follows from (13) with v_2 (respectively, v_4) replaced by $v_{4'}$ (respectively, $v_{2'}$).

We may assume that $\{v_2, v_4\}$ is not an even-pair of H, for otherwise we are done. Thus there is an odd induced path $v_2x_1x_2 \ldots x_rv_4$ in H. By (13) and (15), we have $x_1, x_r \in D$ (i.e. $x_1 = v_1$, $x_r = v_5$). Since $H[D]$ is a C_6, we have $x_2, x_{r-1} \notin D$. Since the path is odd, we have $x_2 \neq x_{r-1}$. The $2K_2$ $\{x_2, v_1, v_4, v_5\}$ of H implies that

$x_2v_1, x_2v_4 \in E(G)$, $x_2v_5 \notin E(G)$. We must have

$v_2x_2 \in E(G)$, for otherwise G contains the bad C_4 $x_2v_1v_2v_4$,
$v_3x_2 \in E(H)$, for otherwise H contains the bad $2K_2$ $\{x_2, v_1, v_3, v_4\}$.

Now, H contains the bad C_4 $x_2v_1v_2v_3$. □

7. Proof of Lemma 4

By replacing G by its complement if necessary, we may assume that $G[D]$ is a C_6. A certain symmetry allows us to assume that $G[D]$ is $v_1v_2v_6v_4v_5v_3$. We can justify this in the following way. If $G[D]$ is the C_6 $v_1v_5v_3v_4v_2v_6$ then consider the mapping $f(v_1) = v_{1'}$, $f(v_2) = v_{6'}$, $f(v_3) = v_{5'}$, $f(v_4) = v_{4'}$, $f(v_5) = v_{3'}$, $f(v_6) = v_{2'}$. Then $H[D]$ is of the form $v_{1'}v_{2'}v_{3'}v_{4'}v_{5'}v_{6'}$ and $G[D]$ is of the form $v_{1'}v_{2'}v_{6'}v_{4'}v_{5'}v_{3'}$. Similarly, if $G[D]$ is the C_6 $v_1v_3v_2v_4v_6v_5$ then the mapping $f(v_1) = v_{3'}$, $f(v_2) = v_{2'}$, $f(v_3) = v_{1'}$, $f(v_4) = v_{6'}$, $f(v_5) = v_{5'}$, $f(v_6) = v_{4'}$ gives the desired conclusion.

We claim that for any two vertices $y, z \notin D$,

$$\text{if } yz \in E(H) \text{ and } v_6y, v_6z \notin E(H) \text{ then } yv_5 \in E(H) \text{ or } zv_5 \in E(H), \text{ or both.} \tag{16}$$

Suppose that (16) is false for some two vertices $y, z \notin D$. Write $S = \{y, z, v_6, v_5\}$. Since $H[S]$ is a $2K_2$, $G[S]$ must be a $2K_2$ or a C_4.

Suppose $G[S]$ is a C_4. We may assume that $G[S]$ is the C_4 v_6zv_5y. We have $zv_4 \in E(G)$ (for otherwise, G contains the bad C_4 $zv_6v_4v_5$) and $yv_4 \in E(G)$ (for otherwise, G contains the bad C_4 $yv_6v_4v_5$). We must have $v_4y \in E(H)$ or $v_4z \in E(H)$, for otherwise H contains the bad $2K_2$ $\{v_4, v_5, y, z\}$. Without loss of generality, we may assume $v_4y \in E(H)$. If $yv_1 \in E(H)$ then H contains the C_5 $yv_4v_5v_6v_1$; if $yv_1 \notin E(H)$ then H contains the bad $2K_2$ $\{y, v_4, v_6, v_1\}$.

We may now suppose that $G[S]$ is a $2K_2$. Without loss of generality, we may assume that $yv_6, zv_5 \in E(G)$ and $yv_5, zv_6, yz \notin E(G)$. We must have

$yv_2 \notin E(H)$, for otherwise H contains the bad $2K_2$ $\{y, v_2, v_5, v_6\}$,
$zv_3 \notin E(H)$, for otherwise H contains the bad $2K_2$ $\{z, v_3, v_5, v_6\}$,
$yv_3 \notin E(G)$, for otherwise we have $yv_4 \in E(G)$ (else G contains the C_5 $yv_3v_5v_4v_6$), and so G contains the bad C_4 $yv_3v_5v_4$,
$zv_2 \notin E(G)$, for otherwise we have $zv_4 \in E(G)$ (else G contains the C_5 $zv_2v_6v_4v_5$), and so G contains the bad C_4 $zv_2v_6v_4$,
$zv_2 \in E(H)$, for otherwise the $2K_2$ $\{z, v_5, v_6, v_2\}$ is bad in G,
$yv_3 \in E(H)$, for otherwise the $2K_2$ $\{y, v_6, v_5, v_3\}$ is bad in G,
$yv_2, zv_3 \in E(G)$, for otherwise the C_4 yzv_2v_3 is bad in H.

But now the $2K_2$ $\{y, v_2, z, v_5\}$ is bad in G. Thus, (16) holds.

By symmetry, for any $y, z \notin D$, we have

$$\text{if } yz \in E(H) \text{ and } v_6y, v_6z \notin E(H) \text{ then } yv_1 \in E(H) \text{ or } zv_1 \in E(H), \text{ or both.} \tag{17}$$

Now, we claim that

$$\text{there are no vertices } y, z \text{ outside } D \text{ with } yz \in E(H) \text{ and } v_6y, v_6z \notin E(H). \tag{18}$$

Suppose that there are vertices y, z outside D with $yz \in E(H)$ and $v_6y, v_6z \notin E(H)$. By (16), we may assume that $v_5y \in E(H)$. We have $v_1y \notin E(H)$, for otherwise H contains the bad C_4 $v_1yv_5v_6$. By (17), we have $v_1z \in E(H)$. We have $v_5z \notin E(H)$, for otherwise H contains the bad C_4 $v_5zv_1v_6$. But now, H contains the C_5 $yzv_1v_6v_5$. Thus, (18) is proved.

We shall show that

$$\text{there is no vertex } x \text{ outside } D \text{ with } xv_2 \in E(H) \text{ and } xv_6 \notin E(H). \tag{19}$$

Suppose there is a vertex x outside D with $xv_2 \in E(H)$ and $xv_6 \notin E(H)$. We have

$xv_5 \notin E(H)$, for otherwise we have $xv_1 \in E(H)$ (else H contains the C_5 $xv_5v_6v_1v_2$) and so $xv_5v_6v_1$ is a bad C_4 in H,
$xv_5 \in E(G)$ and $xv_2, xv_6 \notin E(G)$, for otherwise the $2K_2$ $\{x, v_2, v_5, v_6\}$ is bad in H,
$xv_4 \in E(H)$, for otherwise H contains the bad $2K_2$ $\{x, v_2, v_4, v_5\}$,
$xv_3 \in E(H)$, for otherwise H contains the bad C_4 $xv_2v_3v_4$.

Now, $\{x, v_3, v_5, v_6\}$ is a bad $2K_2$ in H. (19) is proved.

By symmetry we know that

$$\text{there is no vertex } x \text{ outside } D \text{ with } xv_4 \in E(H) \text{ and } xv_6 \notin E(H). \tag{20}$$

We are now ready to complete the proof of Lemma 4. Write $M = H - N_H(v_6) - \{v_6\}$. By (18), (19) and (20), we see that $M - \{v_3\}$ is a stable set. M is connected, for otherwise $\{v_6\} \cup N_H(v_6)$ is a star-cutset of H. So, every vertex in M is adjacent to v_3. Thus, there is a unique maximal stable set of size at least three that contains v_6. It is easy to see that if H is minimal imperfect and does not contain an odd disc, then $\alpha(H) \geqslant 3$. Thus, the fact, that v_6 belongs to a unique maximal stable set of size at least three, is a contradiction to (6). □

8. Proof of Lemma 5

Let $H[D]$ be the hole $v_1v_2\ldots v_k$ $(k \geqslant 6)$. By replacing G by its complement if necessary, we may assume that $G[D]$ is the hole $v_1v_2\ldots v_k$. Define $P_H(D)$ (respectively, $U_H(D)$, $R_H(D)$) to be the set of vertices outside D that see some but not all (respectively, all, no) vertices of D in H. Define the sets $P_G(D)$, $U_G(D)$, $R_G(D)$ of G in the same way. Two vertices x, y are called a *variant pair* if x sees y in H but misses it in G, or vice versa.

We shall need a number of Observations.

The hypothesis of Lemma 5 implies the following

Observation 6. *If* $u_1u_2\dots u_t$ *is a hole of* H *and for some* i, u_{i+1} *sees* u_i *and* u_{i+2} *in* G *then* $u_1u_2\dots u_t$ *is a hole of* G.

Observation 7. *For every vertex* x, $x\in P_H(D)$ *iff* $x\in P_G(D)$; *furthermore for any vertex* $y\in D$ *we have* $xy\in E(H)$ *iff* $xy\in E(G)$.

Proof. We shall show that

$$\text{if } x\in P_H(D),\ xv_i\in E(H),\ xv_{i-1}\notin E(H) \text{ for some } i, \text{ then } xv_i\in E(G). \tag{21}$$

Suppose that $x\in P_H(D)$, $xv_i\in E(H)$, $xv_{i-1}\notin E(H)$ for some i. If $xv_{i-2}\in E(H)$ then the C_4 $xv_iv_{i-1}v_{i-2}$ of H implies that $xv_i\in E(G)$. Suppose that $xv_{i-2}\notin E(H)$, then $xv_{i-3}\notin E(H)$ (for otherwise H contains the C_5 $xv_iv_{i-1}v_{i-2}v_{i-3}$) and now the $2K_2$ $\{x,v_i,v_{i-2},v_{i-3}\}$ implies $xv_i\in E(G)$. Thus (21) holds. A similar argument shows that

$$\text{if } x\in P_G(D),\ xv_i\in E(G),\ xv_{i-1}\notin E(G) \text{ for some } i, \text{ then } xv_i\in E(H). \tag{22}$$

We remark that (21) and (22) also hold with v_{i-1} replaced by v_{i+1}.

Next, we claim that

$$\text{if } x\in P_H(D),\ xv_i\in E(H) \text{ for some } i, \text{ then } xv_i\in E(G). \tag{23}$$

Suppose that $xv_i\in E(H)$ but $xv_i\notin E(G)$. By (21), we must have $xv_{i-1},xv_{i+1}\in E(H)$. Since $x\in P_H(D)$, there are subscripts j,k with $j<k$ such that $xv_r\in E(H)$ for $r=j,j+1,\dots,i,\dots,k-1,k$ and $xv_{j-1},xv_{k+1}\notin E(H)$ (by shifting the vertices of D cyclically if necessary, we may assume that $j<k$). By (21), we have $xv_j,xv_k\in E(G)$. Let j',k' be two subscripts with $j'<k'$, $j'\geqslant j$, $k'\leqslant k$ such that $xv_{j'},xv_{k'}\in E(G)$ and $xv_r\notin E(G)$ for $r=j'+1,\dots,i,\dots,k'-1$. It is now easy to see that either x belongs to a C_5 in G, or x forms a bad C_4 or $2K_2$ with some three vertices in $\{v_{j'},v_{j'+1},\dots,v_i,\dots,v_{k'}\}$. Thus (23) is proved. A similar argument shows that

$$\text{if } x\in P_G(D),\ xv_i\in E(G) \text{ for some } i, \text{ then } xv_i\in E(H).\qquad\square \tag{24}$$

Observation 7 implies the following

Observation 8. $U_H(D)\cup R_H(D)=U_G(D)\cup R_G(D)$.

Observation 9. *Let* x,y *be two vertices of* H. *Then we have*

(i) $x,y\in U_H(D)$ *and* $xy\notin E(H)$ *iff* $x,y\in U_G(D)$ *and* $xy\notin E(G)$, *and*
(ii) $x,y\in R_H(D)$ *and* $xy\in E(H)$ *iff* $x,y\in R_G(D)$ *and* $xy\in E(G)$.

Proof. Suppose $x,y\in U_H(D)$ and $xy\notin E(H)$. Let S be any C_4 (in H) containing x,y and some two vertices of D. By Observation 8, we have $x,y\in U_G(D)\cup R_G(D)$. If

$x, y \in U_G(D)$ then $xy \notin E(G)$, for otherwise S is a bad C_4 of H. Thus, without loss of generality, we may assume that $x \in R_G(D)$. But now S is a bad C_4 of H. The 'only if' part of (i) is proved, the "if" part follows by interchanging H and G. A similar argument establishes (ii). □

As we shall see, the Lemma follows from the following three Claims.

Claim 1. *Let $z \in U_H(D) \cup P_H(D)$, $y \in R_H(D)$, $zy \in E(H)$. Then we have $y \in R_G(D)$, $zy \in E(G)$. Furthermore, if $z \in U_H(D)$ then $z \in U_G(D)$ and if $z \in P_H(D)$ then $z \in P_G(D)$.*

Claim 2. *$P_H(D) = P_G(D)$, $U_H(D) = U_G(D)$ and $R_H(D) = R_G(D)$.*

Claim 3. *For any two vertices $x, y \in P_H(D) \cup U_H(D)$, we have $xy \in E(H)$ iff $xy \in E(G)$.*

We are going to show that Lemma 5 follows from the above three claims. Suppose that there is a variant pair z, y. By Claim 2, we must have

$$z, y \in V(H) - D \ (= V(G) - D) \text{ for any variant pair } z, y.$$

By Observation 9(ii) and Claim 3, we may assume $y \in R_H(G)$ and $z \in P_H(D) \cup U_H(D)$. By Claim 1, z misses y in H and sees it in G.

In H, there must be a path joining y to a vertex in D lying entirely in $V(H) - (\{z\} \cup (N_H(z) - D))$; for otherwise $\{z\} \cup (N_H(z) - D)$ is a star cutset separating y and D. Consider such a shortest path P and let v be the vertex in P that sees y in H. We must have $v \in P_H(D) \cup U_H(D)$ for otherwise, by Observation 9, $\{y, v, z, d\}$ is a bad $2K_2$ in H for some vertex $d \in D$. By Claim 1, y sees v in G. Let d be a neighbour of z in D (in both H and G). Then d misses v (in both H and G), for otherwise G contains the bad C_4 $yvdz$. Now, $\{y, v, z, d\}$ is a bad $2K_2$ of H. □

In the remainder of this section, we shall prove the above three claims. First, we shall need the following

Claim 4. *Suppose in H there are vertices u,z such that $uz \notin E(H)$, $u,z \in P_H(D) \cup U_H(D)$, and there is a chordless path $x_1x_2 \ldots x_t$ between u and z (with $u = x_t, z = x_1$) whose interior vertices belong to $R_H(D)$, then $G[\{x_1, x_2, \ldots, x_t\}]$ is the chordless path $x_1x_2 \ldots x_t$ and, with $v \in \{u,z\}$,*

$v \in P_G(D)$ *if* $v \in P_H(D)$,
$v \in U_G(D)$ *if* $v \in U_H(D)$,
$x_i \in R_G(D)$ *for* $i = 2, \ldots, t-1$.

Proof. Let $u, z, x_1, x_2, \ldots, x_t$ be as in the Claim.

Suppose we have $u,z \in U_H(D)$. Then by Observation 9 we have $u,z \in U_G(D)$, $uz \notin E(G)$. Consider any vertex $d \in D$. The hole $dx_1x_2 \ldots x_t$ of H implies that $G[\{d, x_1, x_2, \ldots, x_t\}]$ is the same hole $dx_1x_2 \ldots x_t$ by Observation 6 (since $du, dz \in E(G)$).

In particular, we have $x_i d \notin E(G)$ for $i = 2, \ldots, t-1$; and so $x_i \notin U_G(D)$. By Observation 8, we have $x_i \in R_G(D)$ and the Claim is proved.

Suppose we have $u, z \in P_H(D)$. In H, there is a chordless path $a_1 a_2 \ldots a_r$ such that $a_1 = u$, $a_r = z$, $a_i \in D$ for $i = 2, \ldots, r-1$ $(r \geqslant 3)$. By Observation 7, we have $u, z \in P_G(D)$ and $G[\{a_1, a_2, \ldots, a_r\}] - a_1 a_r$ is the same chordless path $a_1 a_2 \ldots a_r$. Now, H has the hole $a_1 a_2 \ldots a_r x_2 \ldots x_{t-1}$ and by Observation 6, $G[\{a_1, a_2, \ldots, a_r, x_2, \ldots, x_{t-1}\}]$ is the same hole $a_1 a_2 \ldots a_r x_2 \ldots x_{t-1}$. In particular, we have $x_i a_2 \notin E(G)$ for $i = 2, \ldots, t-1$; and so $x_i \notin U_G(D)$. The claim now follows from Observation 8.

We may assume that one of the two vertices u, z belong to $P_H(D)$ and the other vertex belongs to $U_H(D)$. Without loss of generality, we may assume $u \in P_H(D)$, $z \in U_H(D)$. By Observation 7, we have $u \in P_G(D)$.

Let us first suppose that $t = 3$. In H, consider a neighbour $v_i \in D$ of u. Then H contains the C_4 $x_2 u v_i z$. If $z v_i \in E(G)$ then we have $u x_2, z x_2 \in E(G)$ (for otherwise the C_4 $x_2 u v_i z$ of H is bad; note that we have $u v_i \in E(G)$ by Observation 7), and we are done by Observation 8. So we know that $z v_i \notin E(G)$ and it follows from Observation 8 that $z \in R_G(D)$. Now we have $u x_2 \notin E(G)$ for otherwise the C_4 $x_2 u v_i z$ of H is bad. In H, if u misses both v_{i+2} and v_{i+3} then $H[\{x_2, u, v_{i+2}, v_{i+3}\}]$ is a bad $2K_2$ by Observation 7. Thus, we have $u v_j \in E(H)$ for $j = i+2$ or $i+3$. Now, $H[\{u, v_i, z, v_j\}]$ is a bad C_4.

Now, we may assume that $t \geqslant 4$. In H, let $v_i \in D$ be a neighbour of u and let F denote the graph $H[\{x_1, x_2, \ldots, x_t, v_i\}]$. Note that F is the hole of the form $x_1 x_2 \ldots x_t v_i$. If $t = 4$ then F is a C_5, a contradiction. So we have $t \geqslant 5$. By Observation 9, we have that $G[\{x_2, x_3, \ldots x_{t-1}\}]$ is the chordless path of the form $x_2 x_3 \ldots x_{t-1}$ and $\{x_2, x_3, \ldots x_{t-1}\} \subseteq R_G(D)$. By Observation 6, the graph $G[\{x_1, x_2, \ldots, x_t, v_i\}]$ is a hole in the same cyclic order as F, i.e. $x_1 x_2 \ldots x_t v_i$. Now, the Claim follows from Observation 8. □

Proof of Claim 1. There must be a nonempty set $S(d)$ of paths $P(d)$ joining y to a vertex $d \in D$ in the graph $H - (\{z\} \cup N_H(z) - D)$ for otherwise $\{z\} \cup N_H(z) - D$ is a star-cutset separating y and D. Consider such a shortest path $P(d)$ (over all choices of d and all lengths in $S(d)$). This path can be written as $p_1 p_2 \ldots p_t$ with $p_1 = y$, $p_t = d$. Clearly, the choice of the path implies that $p_{t-1} \in P_H(D) \cup U_H(D)$ and $p_i \in R_H(D)$ for $i = 2, \ldots, t-2$. Now, Claim 1 follows from Claim 4 (with $u = p_{t-1}$). □

Claim 5. $U_H(D) \subseteq U_G(D)$.

Proof. Let z be a vertex in $U_H(D)$. We shall prove that $z \in U_G(D)$. By Observation 9, we may assume that

$$\text{in } H,\ z \text{ sees all vertices of } U_H(D) - \{z\}, \tag{25}$$

for otherwise $z \in U_G(D)$ and we are done.

Consider a P_3 $v_i z v_{i+2}$ for any i. By (3), this P_3 extends into a hole $x_1 x_2 \ldots x_r$ where $x_1 = z$, $x_2 = v_{i+2}$, $x_r = v_i$. Clearly we have $x_3, x_{r-1} \in P_H(D)$ (by (25)) and $x_j \in P_H(D) \cup R_H(D)$ for $j = 4, \ldots, r-2$.

If $r=4$, then $x_1x_2x_3x_4$ is a C_4 of H. By Observation 7, we have $x_3v_i, x_3v_{i+2} \in E(G)$. This implies that in G, z sees v_i, v_{i+2} and misses x_3 (for otherwise $x_1x_2x_3x_4$ is a bad C_4 of H). This implies $z \in U_G(D)$ by Observation 8.

Now, we may assume that $r \geqslant 6$. Suppose that $r > 6$. The $2K_2$ $\{x_3, x_4, x_r, x_{r-1}\}$ in H implies that $x_3x_4 \in E(G)$ (note that $x_rx_{r-1} \in E(G)$ by Observation 7). The $2K_2$ $\{x_3, x_4, z, x_r\}$ in H implies $zx_r \in E(G)$. Thus we have $z \in U_G(D)$ by Observation 8.

Now we may assume that $r=6$. Furthermore, we may assume $z \in R_G(D)$ for otherwise we are done. If $x_4x_2 \notin E(G)$ then $\{x_4, x_5, z, x_2\}$ is a bad $2K_2$ in H (note that $x_2x_5 \notin E(G)$ because $x_5 \in P_H(D)$). So we have $x_4x_2 \in E(G)$, and by Observations 7 and 8, we have $x_4 \in U_G(D)$. By Observation 8, we have $x_4 \in R_H(D)$. But now, by Claim 1 (with $y = x_4$, $z = x_5$), we have $x_4 \in R_G(D)$, a contradiction.

Proof of Claim 2. By Observation 7, we have $P_H(D) = P_G(D)$. If $R_H(D) = \emptyset$ then the Claim follows from Claim 5.

Now, suppose that $R_H(D) \neq \emptyset$. Since H must be connected, each vertex $y \in R_H(D)$ must see another vertex, say z, in H. Clearly, z belongs to $R_H(D) \cup P_H(D) \cup U_H(D)$. By Observation 9(ii) and Claim 1, we have $y \in R_G(D)$; and so it follows that $R_H(D) \subseteq R_G(D)$. Since $U_H(D) \cup R_H(D) = U_G(D) \cup R_G(D)$, the desired conclusion follows from Claim 5. □

Proof of Claim 3. By contradiction. We shall often refer to Claim 2 implicitly. Consider a variant pair x, y with $x, y \in P_H(D) \cup U_H(D)$. In D, x and y must be comparable for otherwise there are vertices x', y' in D such that x (respectively, y) sees x' (respectively, y') and misses y' (respectively, x') in both G and H (recall Claim 2); if $x'y' \notin E(H)$ then $\{x, y, x', y'\}$ induces a bad $2K_2$ in H or in G, if $x'y' \in E(H)$ then $\{x, y, x', y'\}$ induces a bad C_4 in H or in G. Now, by interchanging x and y if necessary we may assume that x dominates y in D, i.e. $N_H(y) \cap D \subseteq N_H(x) \cap D$.

We see that

$$\text{there are no nonadjacent vertices } a, b \text{ in } D \text{ that see both } x \text{ and } y. \tag{26}$$

If (26) is false then $\{a, x, b, y\}$ would be a bad C_4 in H or in G.

Let v_i be a neighbour of y in D. If x misses v_{i+2} and v_{i+3} then $\{y, x, v_{i+2}, v_{i+3}\}$ would be a bad $2K_2$ in H or in G. So we can let v_j be a vertex in $\{v_{i+2}, v_{i+3}\}$ that sees x. By (26), y must miss v_j.

We shall show that

$$\text{if } x, y \text{ is a variant pair with } x, y \in P_H(D) \cup U_H(D), \text{ then } xy \notin E(H). \tag{27}$$

Suppose that $xy \in E(H)$ (and therefore $xy \notin E(G)$). By (3), the P_3 yxv_j of H extends into a hole $H[C]$. This hole cannot have length at least six, for otherwise $G[C]$ and $\bar{G}[C]$ cannot be the same hole (with the same cyclic order), a contradiction to the hypothesis of Lemma 5. Let the hole be v_jxyu for some vertex u. Since $u \in V(H) - R_H(D)$, we have $v_ju \in E(G)$. But now $G[\{v_j, x, y, u\}]$ cannot be a $2K_2$ or C_4 in G, a contradiction. (27) is proved.

We may suppose now that x misses y in H but sees it in G. If x dominates y in H then we would have a contradiction to (2). So we may assume that there is a vertex z that sees y and misses x in H. The choice of x, y implies that $z \notin D$. If $zv_j \notin E(H)$ then $zv_j \notin E(G)$ by Claim 2 and so $\{z, y, x, v_j\}$ would be a bad $2K_2$ in H. We may suppose that $zv_j \in E(H)$. Suppose now that $xz \notin E(G)$. Then we must have $yz \notin E(G)$ for otherwise G contains the bad C_4 v_jxyz; but the variant pair y, z contradicts (27). So we have $xz \in E(G)$. By (3), the P_3 xv_jz of H extends into a hole $H[C]$. If $H[C]$ has length four then it is a bad C_4 of H, if it has length at least six then $G[C]$ and $\bar{G}[C]$ cannot be the same hole (with the same cyclic order), a contradiction to the hypothesis of Lemma 5. □

9. The hole-structure

Recall that two graphs G_1, G_2, defined on the same vertex set, are said to have the same *hole-structure* if a set C induces a hole in G_1 iff C induces a hole in G_2. In [6], the following conjecture was proposed.

Conjecture 2 (*Hole Conjecture*). *If a graph H has the hole-structure of a perfect graph G, then H is perfect.*

We shall prove

Theorem 3. *The Strong Perfect Graph Conjecture implies the Hole Conjecture.*

Proof. We only need show that if H has the hole-structure of a Berge graph G then H is Berge. Let H and G have the same hole-structure and let G be Berge. Suppose that H contains an odd disc $H[D]$. If $H[D]$ is an odd hole then by definition of hole-structure, $G[D]$ is an odd hole, a contradiction. Thus $H[D]$ is an odd anti-hole of length at least seven (note that the C_5 is self-complementary). This means that $H[D]$ has the $\{C_4\}$-structure of $G[D]$. By Observation 5, $G[D]$ is an odd anti-hole, a contradiction. □

The SPGC can be restated in the following way.

Conjecture 3 (*The* F-*Conjecture*). *Let $\mathscr{F}$ be any family of graphs. Then a Berge graph H is perfect iff it has the $\mathscr{F}$-structure of a perfect graph G.*

It is not difficult to show that the $\mathscr{F}$-Conjecture is equivalent to the SPGC [6]. Note that the $\mathscr{F}$-Conjecture generalizes the PGT whenever $\mathscr{F}$ has the property that a graph G belongs to $\mathscr{F}$ iff its complement $\bar{G}$ does. Reed's theorem and Theorem 2 fit into this frame work.

Besides Theorem 2 and Reed's theorem, it is known that the $\mathscr{F}$-Conjecture holds for $\mathscr{F} = \{\text{paw}, \text{copaw}\}$ [6]. We note that the P_4, $2K_2$, and co-paw (See Fig. 1) are induced subgraphs of the P_5 and we think that it would be interesting to prove the $\mathscr{F}$-Conjecture for $\mathscr{F}=\{P_5,\bar{P}_5\}$. It had also been established that the $\mathscr{F}$-Conjecture holds for $\mathscr{F}=\{P_3\}$ (Hougardy [9]), $\mathscr{F}=\{P_3,\bar{P}_3\}$, and $\mathscr{F}=\{K_3,\bar{K}_3\}$ ([5], the last two results are actually equivalent). We note that the results described here are independent of each other.

Appendix A.

Fact 1. *Let H' be the P_7 $v_1v_2\dots v_7$ and let G' be the graph defined on the same vertex-set with the same $\{2K_2\}$-structure as G'. Then G' is the P_7 $v_1v_2\dots v_7$ or the graph $Q_7[v_1,v_2,\dots,v_7]$.*

Proof. By Observation 2, the graph $G'[\{v_1,v_2,\dots,v_6\}]$ is (i) the P_6 $v_1v_2\dots v_6$ itself, or (ii) the P_6 $v_1v_5v_3v_4v_2v_6$, or the graph (iii) $Q_6[v_1,v_2,\dots,v_6]$. Note that in G', the sets $\{v_1,v_2,v_6,v_7\}$, $\{v_2,v_3,v_6,v_7\}$, $\{v_3,v_4,v_6,v_7\}$ must induce a C_4 or $2K_2$. A simple case analysis shows that in case (i), v_7 must see v_6 and miss $v_1,v_2,\dots,v_5$ in G'; and so G' is the P_7 $v_1v_2\dots v_7$. Similarly, one can show that case (ii) cannot occur, and in case (iii) G' is the graph $Q_7[v_1,v_2,\dots,v_7]$. □

Fact 2. *Let H' be the C_6 $v_1v_2\dots v_6$. Let G' be a graph defined on the same vertex-set as H' and suppose that G' has the $\{2K_2\}$-structure of H'. Then G' is the graph $Z_6[v_1,v_2,v_3,v_4,v_5,v_6]$, or one of the C_6's $v_1v_2v_6v_4v_5v_3$, $v_1v_5v_3v_4v_2v_6$, $v_1v_3v_2v_4v_6v_5$, $v_1v_2v_3v_4v_5v_6$.*

Proof. Write $S_1=\{v_1,v_2,v_4,v_5\}$, $S_2=\{v_2,v_3,v_5,v_6\}$, $S_3=\{v_1,v_6,v_3,v_4\}$. Then each S_j must induce a $2K_2$ in G'. There are three different ways the set S_1 induces a $2K_2$ in G'.

Case 1: $v_1v_2,v_4v_5\in E(G')$. By considering the set S_2, we know that $v_2v_3\in E(G')$ or $v_2v_6\in E(G')$ but not both. In the former case, a routine case analysis shows that G' is the C_6 $v_1v_2v_3v_4v_5v_6$; in the latter case G' is the C_6 $v_1v_2v_6v_4v_5v_3$.

Case 2: $v_1v_4,v_2v_5\in E(G')$. Clearly, G' must be the graph $Z_6[v_1,v_2,\dots,v_6]$.

Case 3: $v_1v_5,v_2v_4\in E(G')$. Similar to Case 1, one can show that G' is the C_6 $v_1v_5v_3v_4v_2v_6$ or the C_6 $v_1v_5v_6v_4v_2v_3$. □

Fact 3. *Let H' be the C_6 $v_1v_2\dots v_6$. Let G' be a graph defined on the same vertex-set as H' and suppose that G' has the $\{2K_2,C_4\}$-structure of H' and that G' contains a $2K_2$ and a C_4. Then G' or its complement is the domino.*

Proof. Write $S_1=\{v_1,v_2,v_4,v_5\}$, $S_2=\{v_2,v_3,v_5,v_6\}$, $S_3=\{v_1,v_6,v_3,v_4\}$. Without loss of generality, we may assume that in G', S_1 induces a C_4 and S_2,S_3 induces a $2K_2$. We are going to show that G' is a co-domino that must be one of the four labelled graphs shown in Fig. 3. There are six ways we can label the vertices of the C_4 induced by S_1 in G'.

Case 1: S_1 *is the* C_4 $v_1v_2v_4v_5$. Consider the $2K_2$ induced by S_2. We must have $v_3v_2 \in E(G')$ or $v_3v_5 \in E(G')$ but not both. If $v_3v_2 \in E(G')$ then G' is the graph D_1 or D_3 depending on how S_3 induces a $2K_2$; if $v_3v_5 \in E(G')$ then G' is the graph D_2 or D_4 depending on how S_3 induces a $2K_2$.

Case 2: S_1 *is the* C_4 $v_1v_2v_5v_4$. Since S_2 induces a $2K_2$ we must have $v_3v_6 \in E(G')$ and $v_2v_3, v_2v_6, v_5v_3, v_5v_6 \notin E(G')$. The $2K_2$ induced by S_3 implies that $v_3v_1 \notin E(G')$ but now $\{v_1, v_2, v_3, v_6\}$ induces a bad $2K_2$ in G', a contradiction. This case cannot occur.

Case 3: S_1 *is the* C_4 $v_1v_4v_2v_5$. Since S_2 induces a $2K_2$ we must have $v_3v_6 \in E(G')$ and $v_2v_3, v_2v_6, v_5v_3, v_5v_6 \notin E(G')$. The $2K_2$ induced by S_3 implies that $v_3v_1 \notin E(G')$ but now $\{v_1, v_5, v_3, v_6\}$ induces a bad $2K_2$ in G', a contradiction. This case cannot occur.

Case 4: S_1 *is the* C_4 $v_1v_4v_5v_2$. We can relabel the cycle as $v_1v_2v_5v_4$ and use the argument of Case 2.

Case 5: S_1 *is the* C_4 $v_1v_5v_2v_4$. We can relabel the cycle as $v_1v_4v_2v_5$ and use the argument of Case 3.

Case 6: S_1 *is the* C_4 $v_1v_5v_4v_2$. We can relabel the cycle as $v_1v_2v_4v_5$ and use the argument of Case 1. □

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