Pseudo-linear superposition principle for the Monge–Ampère equation based on generated pseudo-operations

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Abstract

Through this paper, we consider generated pseudo-operations of the following form: $x \oplus y = g^{-1}(g(x) + g(y))$, $x \odot y = g^{-1}(g(x)g(y))$, where $g$ is a continuous generating function. Pseudo-linear superposition principle, i.e., the superposition principle with this type of pseudo-operations in the core, for the Monge–Ampère equation is investigated.

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1. Introduction

One of contemporary problems concerns the way of obtaining new solutions of differential equations. A nonlinear superposition principle (NLSP), i.e., the principle which insures that if $u$ and $v$ are solutions then $u \ast v$ is also a solution for some operation $\ast$, has proved itself to be a useful tool for constructing new solutions of ordinary and partial differential equations (see [3,11,14,33]). This approach had been extended in the direction of noncommutative operations $\ast$ [11,13,17], as well as in the direction of the pseudo-analysis [19,25,26]. The noncommutative operations $\ast$ had also been investigated in the pseudo-analysis’ framework (see [31,32]). Additionally, important connection between NLSP and Lie symmetry algebras [6] for the classical approach had been established in [11].

In 1815 Ampère [2] introduced the Monge–Ampère equation of two variables of the second order and nonlinear. Later on, many authors, including Boillat, Donato, Ramgulam, Rogers and Ruggeri, studied the equation of this type in a more general setting [7–10,34]. Subsequently, Oliveri found a connection between the Monge–Ampère equations and their Lie symmetries [21–23].

The main aim of this paper is to present pseudo-analysis’ approach to the problem of finding new solutions for the homogeneous Monge–Ampère equation of the form $u_{xx}u_{tt} - u_{xt}^2 = 0$. 

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Here, by pseudo-analysis, we consider a generalization of the classical analysis that combines approaches from many different fields and is capable of supplying solutions that were not achieved by the classical tools. The pseudo-linear superposition principle that is being used through this paper is based on the generated pseudo-operations, i.e., operations given by some continuous generating function \( g \) and are in the core of g-calculus \([18,20,24,25,27,28]\). Some of important results concerning pseudo-analysis, both theory and application, can be found in \([4,5,12,15,16,19,25,28,30,31,35]\).

Section 2 contains preliminary notions, such as semiring and pseudo-operations in general, and notation. The problem of pseudo-linear superposition principle for the Monge–Ampère equation is investigated in Section 3. Some concluding remarks are given in Section 4.

2. Preliminary notions

As mentioned before, the basis for the pseudo-superposition principle are generated pseudo-operations. Operations in question belong to the second class of structure called semiring, therefore a short overview of the semiring in general form is given.

Let \([a, b]\) be closed subinterval of \([-\infty, +\infty]\) (in some cases semiclosed subintervals will be considered) and let \(\preceq\) be total order on \([a, b]\). A **semiring** is the structure \(([a, b], \oplus, \oslash)\) when the following hold:

- \(\oplus\) is **pseudo-addition**, i.e., a function \(\oplus : [a, b] \times [a, b] \to [a, b]\) which is commutative, nondecreasing (with respect to \(\preceq\)), associative and with a zero element, denoted by \(0\);
- \(\oslash\) is **pseudo-multiplication**, i.e., a function \(\oslash : [a, b] \times [a, b] \to [a, b]\) which is commutative, positively non-decreasing \((x \preceq y \text{ implies } x \oslash z \preceq y \oslash z, z \in [a, b]^+_0 = \{x : x \in [a, b], 0 \preceq x\})\), associative and for which there exists a unit element denoted by \(1\);
- \(0 \oplus x = 0\);
- \(x \oslash (y \oslash z) = (x \oslash y) \oplus (x \oslash z)\).

There are three basic classes of semirings with continuous (up to some points) pseudo-operations. The first class contains semirings with idempotent pseudo-addition and nonidempotent pseudo-multiplication. Semirings with strict pseudo-operations defined by strictly monotone bijection \(g : [a, b] \to [0, +\infty]\), i.e., g-semirings, form the second class, and semirings with both idempotent operations belong to the third class. More on this structure as well as on corresponding measures and integrals can be found in \([16,19,24,25,27,29]\).

Of the special interest for this paper are operations from the second class of semirings, i.e., strict operations given by the continuous generator \(g : [a, b] \to [0, +\infty]\) in the following manner

\[
x \oplus y = g^{-1}(g(x) + g(y)) \quad \text{and} \quad x \oslash y = g^{-1}(g(x)g(y)).
\]  

(1)

This representation of strict pseudo-operations is based on Aczél’s representation theorem from \([1]\).

If zero element for pseudo-addition is \(a\), i.e., \(0 = a\), generator \(g\) is a strictly increasing function such that \(g(a) = 0\) and \(g(b) = +\infty\). For \(0 = b\) generator \(g\) is a strictly decreasing function such that \(g(b) = 0\) and \(g(a) = +\infty\). Therefore, generator \(g\) is an isomorphism between semigroups \(([a, b], \oplus)\) and \(([0, +\infty], +)\).

Also, since structure of semiring has to be maintained, it is necessary to accept the convention \(0 \cdot (+\infty) = 0\) (follows from \(0 \oslash x = 0\) for all \(x \in [a, b]\)).

Monotonicity of generating function \(g\) is closely connected with the order \(\preceq\) on \([a, b]\), i.e.,

\[x \preceq y \iff g(x) \leq g(y)\]

Additionally, \(x < y\) if and only if \(g(x) \leq g(y)\) and \(x \neq y\).

It should be stressed that operations (1) are base for the g-calculus (see \([18,20,24,25,28]\)).

**Remark 1.** One of representatives for semirings of the first class is semiring \((-\infty, +\infty), \max, +)\). In this case \(\oplus = \max, \oslash = +, 0 = -\infty\) and \(1 = 0\). From \(x \preceq y \iff x \oplus y = y\) follows that for this semiring \(\preceq\) is the usual order. This example can be extended to \((-\infty, +\infty), \max, +)\), however, due to the condition \(0 \oslash x = 0\), the convention \((+\infty) + (-\infty) = -\infty\) has to be accepted.
Semiring of the form \([-\infty, +\infty], \max, \min\) is an example of semirings of the third class. Now \(\oplus = \max, \odot = \min, \mathbf{0} = -\infty, \mathbf{1} = +\infty\) and \(\preceq\) is the usual order. Analogously, semiring \([-\infty, +\infty], \min, \max\) is also an example of semirings of the third class, where \(\oplus = \min, \odot = \max, \mathbf{0} = +\infty, \mathbf{1} = -\infty\) and \(\preceq\) is order opposite to the usual order (see [19, 25]).

**Remark 2.** Further generalization of the pseudo-analysis, known as general pseudo-analysis, had been presented in [31]. In that paper the complete characterization of generalized pseudo-addition and pseudo-multiplication was given, as well as application of generalized pseudo-operations on nonlinear PDE. A special class of generalized pseudo-operations, which was also introduced and applied on nonlinear PDE in [31], represents a generalization of \(g\)-operations of the following form

\[
x \oplus y = g^{-1}(\varepsilon g(x) + g(y)) \quad \text{and} \quad x \odot y = g^{-1}(\varepsilon_2 g(x)g(y)),
\]

where \(\varepsilon\) is an arbitrary but fixed positive real numbers and \(g\) is a positive strictly monotone bijection. Latter on, in [32], this class was broaden to the class of generalized generated pseudo-operations with three parameters of the form

\[
x \oplus y = g^{-1}(\varepsilon_1 g(x) + \varepsilon_2 g(y)) \quad \text{and} \quad x \odot y = g^{-1}(\varepsilon_3 g(x)g(y))
\]

which were used in pseudo-linear superposition principle for Burger’s type equations. It should be stressed that operations (2) and (3) need not be commutative nor associative.

3. **Pseudo-linear superposition principle for the Monge–Ampère equation**

Let us consider the homogeneous Monge–Ampère equation

\[
u_{xx}u_{tt} = u_{x}^2,
\]

such that \(u = u(x, t)\) is a function of two real variables with values in \([a, b]\), where

\[
u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_{tt} = \frac{\partial^2 u}{\partial t^2} \quad \text{and} \quad u_{xt} = \frac{\partial^2 u}{\partial x \partial t}.
\]

Let \((a, b], \oplus, \odot)\) be a semiring of the second class given by some strictly monotone bijection \(g : [a, b] \to [0, +\infty]\), where operations \(\oplus\) and \(\odot\) are (1). Partial pseudo-linear superposition principle based on \((a, b], \oplus, \odot)\) for nonlinear partial differential equation (4) is given by the following theorem. Complete pseudo-linear superposition principle, however for the more restrictive class of solutions, will also be consider in due course of this paper.

**Theorem 3.** Let \(u = u(x, t)\) be a solution of Eq. (4) and let \(g : [a, b] \to [0, +\infty]\) be a generating function for pseudo-operations (1) such that

\[
g(\alpha) \cdot \frac{g''(\alpha \odot u)}{(g'(\alpha \odot u))^2} = \frac{g''(u)}{(g'(u))^2}
\]

holds for \(\alpha \in [a, b]\). Then, \(\alpha \odot u\) is a solution of (4).

**Proof.** Let \(u = u(x, t)\) be a solution of Eq. (4), \(\alpha \in [a, b]\) and \(g\) generator for pseudo-multiplication (1). Since,

\[
\alpha \odot u = g^{-1}(g(\alpha)g(u)),
\]

we have

\[
(\alpha \odot u)_x = \frac{\partial (\alpha \odot u)}{\partial x} = g(\alpha)g'(u)u_x, \quad (\alpha \odot u)_t = \frac{\partial (\alpha \odot u)}{\partial t} = g(\alpha)g'(u)u_t,
\]

\[
(\alpha \odot u)_{xx} = \frac{g'(\alpha)}{g'(\alpha \odot u)}(g''(u)u_x^2 + g'(u)u_{xx}) - \frac{g''(\alpha)g''(\alpha \odot u)}{(g'(\alpha \odot u))^3}(g'(u))^2u_x^2,
\]

\[
(\alpha \odot u)_{tt} = \frac{g'(\alpha)}{g'(\alpha \odot u)}(g''(u)u_t^2 + g'(u)u_{tt}) - \frac{g''(\alpha)g''(\alpha \odot u)}{(g'(\alpha \odot u))^3}(g'(u))^2u_t^2.
\]
and
\[(\alpha \odot u)_{xt} = \frac{g(\alpha)}{g'(\alpha \odot u)} (g''(u)u_{xx}u_t + g'(u)u_{xt}) - \frac{g^2(\alpha)g''(\alpha \odot u)}{(g'(\alpha \odot u))^3} (g'(u))^2u_xu_t.\]

Now, the assumption that \(u\) is a solution of (4), i.e., \(u_{xx}u_{tt} = u_{xx}u_{tt}^2\), yields
\[
(\alpha \odot u)_{xx} (\alpha \odot u)_{tt} - ((\alpha \odot u)_{xt})^2 = g^2(\alpha)g'(u) - (g''(u))^2 \frac{g''(\alpha \odot u)}{(g'(\alpha \odot u))^2} \left(u_{xx}u_t^2 + u_{tt}u_x^2 - 2u_xu_tu_{xt}\right)
\]
which is equal to 0 if (5) holds for the generating function. Therefore, \(\alpha \odot u\) is a solution of Eq. (4) for \(\alpha \in [a, b]\).

If for some generator \(g\) the condition (5) holds, than \(g\) has to be either exponential or power function. This result is given by the following theorem.

**Theorem 4.** Let \(g : [a, b] \to [0, +\infty]\) be a strictly increasing function from the class \(C^2([a, b])\). Then, \(g\) is a solution of the equation
\[
g(\alpha) \cdot \frac{g''(\alpha \odot z)}{(g'(\alpha \odot z))^2} = \frac{g''(z)}{(g'(z))^2},
\]
where \(\alpha\) is an arbitrary but fixed real parameter from \([a, b]\) and \(z \in [a, b]\), if and only if it is of the form
\[
g(z) = e^{Az+B}, \quad \text{while } A > 0 \text{ and } B \in \mathbb{R},
\]
or
\[
g(z) = (Az + B)^p, \quad \text{while } A > 0 \text{ and } B, p \in \mathbb{R}.
\]

**Proof.** It is easy to check that for functions (8) and (9), Eq. (7) holds, therefore it remains to prove the other direction of our claim, i.e., if \(g\) fulfills (7) then it has to be of the form (8) or (9).

Since \(g : [a, b] \to [0, +\infty]\) is a strictly increasing function from the class \(C^2([a, b])\), it is insured that \(g', g''\) and \(g^{-1}\) exist and that \(g'(z) > 0\) for all \(z \in [a, b]\). If we introduce another function \(h\) in the following manner
\[
h(z) = \frac{1}{g'(z)} = \frac{d}{dz} \left(\frac{1}{g'(z)}\right), \quad z \in [a, b],
\]
from
\[
\frac{d}{dz} \left(\frac{1}{g'(z)}\right) = -\frac{g''(z)}{(g'(z))^2}
\]
and (7) follows
\[
h(z) = g(\alpha)h(\alpha \odot z),
\]
where \(\alpha\) is some real parameter from \([a, b]\), i.e.,
\[
\alpha \odot z = h^{-1}(h(z)/g(\alpha))
\]
holds for all \(z \in [a, b]\). Based on properties of \(g\), each \(z \in [a, b]\) can be written as \(z = g^{-1}(z^*)\) for some \(z^* \in [0, +\infty]\), therefore (12) and the new notation
\[
f(z^*) = h \circ g^{-1}(z^*), \quad z^* \in [0, +\infty],
\]
will give us
\[
f(g(\alpha)z^*) = f(z^*)/g(\alpha),
\]
which is the classical Cauchy functional equation (\(g(\alpha)\) is some positive real parameter). Now, the solution of the classical Cauchy functional equation (see [1]) will give us
\[
f(z^*) = \beta/z^*,
\]
where $\beta$ is some real parameter. This leads to $h(z) = \beta / g(z)$, where $z = g^{-1}(z^*)$ and, finally, based on (10) and (11), to the equation

$$-\frac{g''(z)}{g'(z)} = \beta \frac{g'(z)}{g(z)},$$

with solution

$$g'(z) = e^{-\gamma} \cdot g^{-\beta}(z), \quad \beta, \gamma \in \mathbb{R}.$$  

Now, for $\beta = -1$ we obtain generator of the form

$$g(z) = e^{Az+B},$$

where $A > 0$ and $B \in \mathbb{R}$, while for $\beta \neq -1$ our generator is

$$g(z) = (Az+B)^p,$$

where $A > 0$, $B \in \mathbb{R}$ and $p = 1/(\beta + 1) \in \mathbb{R}$.  

**Remark 5.** Similar result can be obtained for strictly decreasing function $g$ from $C^2([a,b])$. In that case $g$ is a solution of (7) if and only if it is of the form

$$g(z) = e^{Az+B}, \quad \text{while } A < 0 \text{ and } B \in \mathbb{R},$$

or

$$g(z) = (Az+B)^p, \quad \text{while } A < 0 \text{ and } B, p \in \mathbb{R}. \quad (14)$$

**Remark 6.** Generators of the form (8) (or (14)) are solutions of the well-known Bateman equation $dg/dz = Ag(z)$, $A \neq 0$. Even more, having in mind results from Theorem 4, it can be easily shown that Theorem 3 partially holds for the Bateman equation

$$dg/dz = Cg(z), \quad C \neq 0, \quad (16)$$

i.e., if $u$ is a solution of (16), than $\alpha \odot u$ is again solution of (16) where $\alpha$ is some real parameter from $[a,b]$. However, parameter $\alpha$ cannot be chosen freely, it depends on the choice of the generator $g$. If the chosen generator is of the form (8) (or (14)) than $\alpha = -B/A$, and if the generator is (9) (or (15)) than $\alpha = (1-B)/A$, providing that such $\alpha$ belongs to $[a,b]$.

If we continue to consider the whole class of solutions of Eq. (4), the pseudo-linear superposition principle with pseudo-addition $\oplus$ given by (1) in its core is just a special case of Theorem 3. This part of the puzzle is presented by the following corollary.

**Corollary 7.** If $u = u(x,t)$ is a solution of Eq. (4), then for all $n \in \mathbb{N}$, $\underbrace{u \oplus u \oplus \cdots \oplus u}_n$ is a solution of (4).

**Proof.** Since for all $n \in \mathbb{N}$ we have

$$\underbrace{u \oplus u \oplus \cdots \oplus u}_n = g^{-1}(n) \odot u$$

(see [25]), this claim follows directly from Theorem 3.  

As already mentioned, the complete pseudo-linear superposition principle can be obtained for the more restrictive class of solutions, i.e., for subclass of the following form

$$F_1 = \{ u \mid u = \varphi(x+t) \},$$

where $\varphi$ is some real function with values in $[a,b]$ from the class $C^2$. In this case, restrictions previously imposed on generator $g$ can be omitted.
Remark 8. It can be easily verified that functions from $\mathcal{F}_1$ are solutions of Eq. (4) and that $\mathcal{F}_1 \subset \mathcal{F}$, where $\mathcal{F}$ is the family of all solutions of Eq. (4). Even more, a solution $u = u(x, y)$ belongs to the class $\mathcal{F}_1$ if and only if $u_x = u_t$.

Theorem 9. Let $u$ and $v$ be solutions of (4) from $\mathcal{F}_1$, let $\oplus$ and $\odot$ be pseudo-operations (1) given by some generator $g$ and let $\alpha \in [a, b]$. Then

(a) $u \oplus v \in \mathcal{F}_1$,
(b) $\alpha \odot u \in \mathcal{F}_1$.

Proof. (a) Let $u$ and $v$ be solutions of (4) from $\mathcal{F}_1$ and $u \oplus v = g^{-1}(g(u) + g(v))$ for some generator $g$. Required partial derivatives in this case are

$$
(u \oplus v)_{xx} = \frac{1}{g'(u \oplus v)} \left[ g'(u)u_{xx} + g''(u)u_x^2 + g'(v)v_{xx} + g''(v)v_x^2 \right]
$$

and

$$
(u \oplus v)_{tt} = \frac{1}{g'(u \oplus v)} \left[ g'(u)u_{tt} + g''(u)u_t^2 + g'(v)v_{tt} + g''(v)v_t^2 \right]
$$

and

$$
(u \oplus v)_{xt} = \frac{1}{g'(u \oplus v)} \left[ g'(u)u_{xt} + g''(u)u_xu_t + g'(v)v_{xt} + g''(v)v_xv_t \right]
$$

Since $u$ and $v$ are solutions of (4), we have the following

$$
(u \oplus v)_{xx} (u \oplus v)_{tt} - (u \oplus v)_{xt}^2 = \frac{1}{(g'(u \oplus v))^3} \left[ g'(u)g''(u)(u_{xx}u_t^2 + u_{tt}u_x^2 - 2u_xu_tu_{xt}) + g'(v)g''(v)(v_{xx}v_t^2 + v_{tt}v_x^2 - 2v_xv_tv_{xt}) \right]
$$

and

$$
= \frac{1}{(g'(u \oplus v))^3} \left[ g'(u)g''(u)(u_{xx}u_t^2 + u_{tt}u_x^2 - 2u_xu_tu_{xt}) + g'(v)g''(v)(v_{xx}v_t^2 + v_{tt}v_x^2 - 2v_xv_tv_{xt}) \right]
$$

Now, due to the assumption $u, v \in \mathcal{F}_1$, hold

$$
u_x = u_t, \quad v_x = v_t, \quad u_{xx} = u_{tt} = u_{xt} \quad \text{and} \quad v_{xx} = v_{tt} = v_{xt}.
$$

Therefore,

$$
(u \oplus v)_{xx} (u \oplus v)_{tt} - (u \oplus v)_{xt}^2 = 0,
$$

i.e., $u \oplus v$ is a solution of (4) for all generators $g$. Even more, it can be easily verified that $u \oplus v \in \mathcal{F}_1$.

(b) If $u \in \mathcal{F}_1$, i.e., $u$ is a solution of (4) and $u_x = u_t$, Eq. (6) will imply

$$
(\alpha \odot u)_{xx} (\alpha \odot u)_{tt} - (\alpha \odot u)_{xt}^2 = 0
$$
regardless to the choice of the generator $g$. Therefore, $\alpha \odot u$ is a solution of Eq. (4) for $\alpha \in [a, b]$. Additionally, since $u \in F_1$, we have
\[ (\alpha \odot u)_x = \frac{g(\alpha)g'(u)}{g'(\alpha \odot u)} u_x = \frac{g(\alpha)g'(u)}{g'(\alpha \odot u)} u_x = (\alpha \odot u)_t, \]
this solution belongs to the subclass $F_1$. □

**Corollary 10.** If $u$ and $v$ are solutions of Eq. (4) from the class $F_1$ and $\alpha_1$ and $\alpha_2$ are real parameters from $[a, b]$, then pseudo-linear combination
\[ \alpha_1 \odot u \oplus \alpha_2 \odot v \]
is a solution of Eq. (4), also from the class $F_1$.

**Proof.** Follows directly from the previous theorem. □

Based on the previous results, it is possible to formulate the opposite direction for Theorem 3.

**Proposition 11.** If $u \in F$ and $\alpha \odot u \in F$, where $\odot$ is given by some generator $g$ and $\alpha \in [a, b]$, then generator $g$ fulfills the condition (5) or $u \in F_1$.

**Proof.** Proof is an immediate consequence of (6) and the definition of family $F_1$. □

### 3.1. Pseudo-linear superposition principle based on generated pseudo-operations with three parameters

As mentioned earlier, pseudo-linear superposition principle based on generalized pseudo-operations had been investigated in [31,32]. Specially, operations with three parameters that need not be commutative nor associative were applied on Burger’s type equations. In this section we will consider application of this special class of generalized pseudo-operations on the Monge–Ampère equation. First, let us recall the definition of generated pseudo-operations with three parameters from [32]. In order to avoid possible confusion, this operations will be denoted with $\oplus'$ and $\odot'$.

Let $g : [a, b] \rightarrow [0, +\infty]$ be a strictly monotone bijection and let $\varepsilon_1, \varepsilon_2$ and $\delta$ be arbitrary but fixed positive real numbers. Generated pseudo-operations with three parameters are
\[ u \oplus' v = g^{-1}(\varepsilon_1 g(u) + \varepsilon_2 g(v)) \] (17)
and
\[ u \odot' v = g^{-1}(g(g(u))g(v)). \] (18)

Similarly to the pseudo-linear superposition principle based on (1), we can observe family of all solutions of Eq. (4) and impose restriction on the generator $g$. This result is given by the next theorem and it follows from Theorem 3.

**Theorem 12.** Let $u = u(x, t)$ be a solution of Eq. (4) and let $g : [a, b] \rightarrow [0, +\infty]$ be a generating function for (18) such that
\[ g''(\alpha) \cdot \frac{g''(\alpha \odot' u)}{(g'(\alpha \odot' u))^2} = \frac{g''(u)}{(g'(u))^2} \] (19)
holds for a real parameter $\alpha \in [a, b]$. Then $\alpha \odot' u$ is a solution of (4).

**Proof.** Since for all real parameters $\alpha \in [a, b]$ and all solutions $u$ holds
\[ \alpha \odot' u = g^{-1}(g^\delta(\alpha)) \odot u, \]
where $\odot$ is the pseudo-multiplication from (1), this claim follows from Theorem 3. □

If we consider family of solutions $F_1$, complete pseudo-linear superposition principle is obtained. As in the case of pseudo-linear superposition principle based on (1), now no restrictions for generator $g$ are requested.
Theorem 13. Let \( u \) be a solutions of (4) from \( \mathcal{F}_1 \), let \( \odot' \) be a pseudo-multiplication of the form (18) given by some generator \( g \) and let \( \alpha \in [a, b] \). Then \( \alpha \odot' u \in \mathcal{F}_1 \).

Proof. As already seen, for all parameters \( \alpha \in [a, b] \) and all solutions \( u \) holds \( \alpha \odot' u = g^{-1}(g^\delta(\alpha)) \odot u \), where \( \odot \) is the pseudo-multiplication from (1), so this claim follows from Theorem 9(b).

It should be stressed that previous two theorems can be also considered as generalizations of Theorems 3 and 9(b), since for \( \delta = 1 \) we have \( \odot' = \odot \). Even more, since for \( \varepsilon_1 = \varepsilon_2 = 1 \) holds \( \odot' = \odot \), we can present the further generalization of Theorem 9. Although the proof is similar to one of Theorem 9, it is not omitted in order to illustrate the difference between the commutative and noncommutative approach for the Monge–Ampère equation.

Theorem 14. Let \( u \) and \( v \) be solutions of (4) from \( \mathcal{F}_1 \), let \( \oplus' \) and \( \odot' \) be pseudo-operations (17) and (18) given by some generator \( g \) and let \( \alpha \in [a, b] \). Then

\[(a) \ u \oplus' v \in \mathcal{F}_1, \quad (b) \ u \odot' \alpha \in \mathcal{F}_1.\]

Proof. (a) Let \( u \) and \( v \) be solutions of Eq. (4) from \( \mathcal{F}_1 \) and let \( u \oplus' v = g^{-1}(\varepsilon_1 g(v) + \varepsilon_2 g(u)) \). Now, partial derivatives of the first order are

\[
(u \oplus' v)_x = \frac{\varepsilon_1 g'(u)u_x + \varepsilon_2 g'(v)v_x}{g'(u \oplus' v)} \quad \text{and} \quad (u \oplus' v)_t = \frac{\varepsilon_1 g'(u)u_t + \varepsilon_2 g'(v)v_t}{g'(u \oplus' v)}.
\]

Partial derivatives of the second order are of the following forms

\[
(u \oplus' v)_{xx} = \frac{1}{g'(u \oplus' v)} \left[ \varepsilon_1 g''(u)u_{xx} + \varepsilon_2 g''(v)v_{xx} + 2 \varepsilon_1 \varepsilon_2 g'(u)g'(v)u_xv_x \right],
\]

\[
(u \oplus' v)_{tt} = \frac{1}{g'(u \oplus' v)} \left[ \varepsilon_1 g''(u)u_{tt} + \varepsilon_2 g''(v)v_{tt} + 2 \varepsilon_1 \varepsilon_2 g'(u)g'(v)u_tv_t \right],
\]

\[
(u \oplus' v)_{xt} = \frac{1}{g'(u \oplus' v)} \left[ \varepsilon_1 g''(u)u_{xt} + \varepsilon_2 g''(v)v_{xt} + 2 \varepsilon_1 \varepsilon_2 g'(u)g'(v)(u_xv_t + u_tv_x) \right].
\]

From assumption that \( u \) and \( v \) are solutions of Eq. (4), i.e., from

\[ u_{xx}u_{tt} = u_{xt}^2 \quad \text{and} \quad v_{xx}v_{tt} = v_{xt}^2, \]

follows

\[
(u \oplus' v)_{xx} (u \oplus' v)_{tt} - (u \oplus' v)_{xt}^2 = \frac{1}{g'(u \oplus' v)^2} \left[ \varepsilon_1 g''(u)g''(v)(u_{xx}u_{tt}^2 + u_{tt}u_{xx}^2 - 2u_{xx}u_{tt}u_{xt}) + \varepsilon_1 \varepsilon_2 g'(u)g'(v)\left(u_{xx}v_{tt}^2 + v_{tt}u_{xx}^2 - 2v_{xx}u_{tt} \right) + \varepsilon_1 \varepsilon_2 g'(u)g'(v)\left(u_{tt}v_{xx}^2 + v_{xx}u_{tt}^2 - 2v_{tt}u_{xx} \right) + g''(u)g'(v)\left(v_{xx}u_{tt}^2 + v_{tt}u_{xx}^2 - 2v_{xx}u_{tt} \right) + \varepsilon_1 \varepsilon_2 g''(u)g''(v)\left(u_{xx}^2v_{tt}^2 + u_{tt}^2v_{xx}^2 - 2u_{xx}u_{tt}v_{xx} \right) \right].
\]
are solutions of Eq. (4) and 
\[ u \]
Also, from (20) follows that for this choice of 
\[ u \]
Proof. Corollary 15. If \( u \) and \( v \) are solutions of Eq. (4) from the class \( \mathcal{F}_1 \) and \( \alpha_1 \) and \( \alpha_2 \) are parameters from \([a,b] \), then pseudo-linear combinations
\[ \alpha_1 \odot' u \oplus' \alpha_2 \odot' v \quad \text{and} \quad u \odot' \alpha_1 \oplus' v \odot' \alpha_2 \]
are solutions of Eq. (4) that belong to the class \( \mathcal{F}_1 \).

Proof. Follows directly from the previous theorem.
4. Conclusion

In this paper we have applied nonlinear superposition principle on the homogeneous Monge–Ampère equation. Presented approach was based on generated pseudo-operations given by some generating function. Both commutative and noncommutative pseudo-operations were considered. In both cases, for the family of all solutions of the equation in question, we obtained partial pseudo-linear superposition principle for some specific generators. However, for solutions from subfamily $\mathcal{F}_1$, the pseudo-linear superposition principle holds regardless to the choice of the generating function.

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