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The Canonical Module of a Stanley–Reisner Ring

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INTRODUCTION

According to works of Stanley, Reisner, Hochster, Baclawski and others it is well known that methods of commutative algebra are powerful tools for investigating simplicial complexes and related questions.

The main goal of this paper is to calculate the local cohomology modules of the Stanley–Reisner ring $k[\Delta]$ of an (arbitrary) simplicial complex (Section 2).

From this result we derive an explicit description of a dualizing complex of $k[\Delta]$ and its canonical module K_Δ (Section 3). Furthermore we construct a homogeneous map of degree $d = \dim k[\Delta]$ embedding the canonical module in $k[\Delta]$. The obtained canonical ideal generalizes a result of Baclawski [1].

We conclude with some remarks on the number of generators of K_Δ, the type of $k[\Delta]$ in the Cohen–Macaulay case.

Most of our results generalize without or with slight corrections to the case $k = \mathbb{Z}$.

1. SIMPLICIAL COMPLEXES AND STANLEY–REISNER RINGS

1.1. A finite set V and a (nonempty) set Δ of subsets of V satisfying the property

$$\sigma \in \Delta, \qquad \tau \subset \sigma \Rightarrow \tau \in \Delta$$

is called a simplicial complex on the vertex set V.

Note that the empty set $\varnothing$ is a face of every simplicial complex but $\{v\}$, $v \in V$, need not be a simplex.

Throughout this work fix a total order on V and set

$$a(A, B) := \operatorname{card}\{a < b : a \in A, b \in B\}$$

if $A, B \subset V$.

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$|\Delta|$ denotes the geometric realization of Δ, $\hat{\sigma}$ the barycenter of $\varnothing \neq \sigma \in \Delta$. For further definitions and notations cf. [8].

1.2. For a (possibly empty) simplex $\tau \in \Delta$ we denote by

$$\operatorname{link}_\Delta \tau := \{\sigma \in \Delta : \sigma \cap \tau = \varnothing; \sigma \cup \tau \in \Delta\} \text{ the link of } \tau$$

$$\operatorname{st}_\Delta \tau := \{\sigma \in \Delta : \sigma \cup \tau \in \Delta\} \text{ the star of } \tau$$

$$\operatorname{cost}_\Delta \tau := \{\sigma \in \Delta : \sigma \not\supset \tau\} \text{ the contrastar of } \tau.$$

Set

$$\Delta_U := \{\sigma \in \Delta : \sigma \subset U\} \text{ if } U \subset V.$$

1.3. Let k be an arbitrary field (fixed throughout the work).

LEMMA. *For simplicial (co)homologies with coefficient group k and $\sigma \in \Delta$ there are natural k-vectorspace isomorphisms*

(a) $\tilde{H}_{i-|\sigma|}(\mathit{link}_\Delta \sigma) \simeq H_i(\Delta, \operatorname{cost}_\Delta \sigma)$
$H^i(\Delta, \operatorname{cost}_\Delta \sigma) \simeq \tilde{H}^{i-|\sigma|}(\mathit{link}_\Delta \sigma)$

and

(b) $H_i(\Delta, \operatorname{cost}_\Delta \sigma) \cong H_i(|\Delta|, |\Delta| - \hat{\sigma})$
$H^i(\Delta, \operatorname{cost}_\Delta \sigma) \cong H^i(|\Delta|, |\Delta| - \hat{\sigma}) \qquad (\varnothing \neq \sigma)$

where on the right-hand side of (b) *singular (co)homologies are meant.*

(a) follows immediately from the complex isomorphism

$$\tilde{C}_{i-|\sigma|}(\operatorname{link}_\Delta \sigma) \to \tilde{C}_i(\Delta)/\tilde{C}_i(\operatorname{cost}_\Delta \sigma)$$

via

$$\tau \mapsto (-1)^{a(\sigma,\tau)}(\tau \cup \sigma) + \tilde{C}_i(\operatorname{cost}_\Delta \sigma),$$

and (b) from the fact that one can construct a deformation retract from $|\Delta| - \hat{\sigma}$ to $|\operatorname{cost}_\Delta \sigma|$.

1.4. Let $S(V) := k[x_v : v \in V]$ denote the polynomial ring in card (V) variables over the field k. This ring has a natural $\mathbb{N}^V$-multigrading. Write

$$x^\alpha := \prod_{v \in V} x_v^{\alpha_v} \qquad \text{if} \quad \alpha = (\alpha_v) \in \mathbb{Z}^V$$

$$s(\alpha) := \operatorname{supp} x^\alpha = \{v \in V : \alpha_v \neq 0\}$$

$$n(\alpha) := \{v \in V : \alpha_v < 0\}$$

and $x_{s(\alpha)}$ for short if $\alpha \in \{0, 1\}^V \subset \mathbb{Z}^V$.

Let $I(\Delta)$ be the ideal of $S(V)$ generated by all "non-face monomials"

$$I(\Delta) := (x_\sigma : \sigma \subset V; \sigma \notin \Delta) \qquad \text{and} \qquad k[\Delta] := S(V)/I(\Delta).$$

$k[\Delta]$ is called the Stanley–Reisner ring of the simplicial complex Δ and will be the main object of the following investigations.

Because $I(\Delta)$ is multihomogeneous the $\mathbb{N}^V$-multigrading on $S(V)$ can be transferred to $k[\Delta]$.

2. Local Cohomology Modules

Following an unpublished idea of M. Hochster [4] we now calculate the local cohomology modules of $R = k[\Delta]$ with support in the irrelevant ideal m of $S(V)$. For definitions and notations cf. [3].

2.1. Let Δ be a simplicial complex on the vertex set $V = \{1, ..., n\}$ and $m := (x_v : v \in V)$ the irrelevant ideal of $S(V)$.

Let $K^\cdot(\mathbf{x}^t; R) = \otimes_{i=1}^n (0 \to R \to^{x_i^t} R \to 0)$ be the cokoszul complex of R relative to $\mathbf{x}^t := (x_1^t, x_2^t, ..., x_n^t)$. Then we have [3, 4.7]

$$H_m^i(R) \cong \varinjlim_t H^i(\mathbf{x}^t; R) \cong H^i(\varinjlim_t K^\cdot(\mathbf{x}^t; R))$$

and

$$\varinjlim_t \left(\bigotimes_{i=1}^n (0 \to R \xrightarrow{x_i^t} R \to 0) \right) \cong \bigotimes_{i=1}^n \varinjlim_t (0 \to R \xrightarrow{x_i^t} R \to 0)$$

because direct limits commute with cohomologies and tensor products. An easy calculation shows that

$$\varinjlim_t (0 \to R \xrightarrow{x^t} R \to 0) \cong (0 \to R \xrightarrow{i} R_x \to 0)$$

where R_x denotes the localisation of R at the multiplicative set $\{x^n : n \in \mathbb{N}\}$ of powers of x and i the natural localisation map. Therefore local cohomologies of R can be calculated by calculating cohomologies of the complex

$$0 \to C^0 \to C^1 \to \cdots \to C^n \to 0 \tag{1}$$

with components $C^i := \oplus_{\sigma \subset V, |\sigma| = i} R_{x_\sigma}$ and boundary maps induced by

$$f_{\tau\sigma} : R_{x_\tau} \to R_{x_\sigma} \qquad (|\tau| = i - 1, |\sigma| = i)$$

via

$$f_{\tau\sigma} = \begin{cases} \text{0-map} & \text{if } \tau \not\subset \sigma \\ (-1)^{a(\tau,\sigma-\tau)} i & \text{if } \tau \subset \sigma. \end{cases}$$

2.2. The complex (1) admits a natural $\mathbb{Z}^V$-multigrading induced by the $\mathbb{N}^V$-multigrading of R since the boundary maps preserve it. Therefore we have

$$H_m^{\cdot}(R) = \bigoplus_{U \in \mathbb{Z}^V} [H_m^{\cdot}(R)]_U$$

as k-vectorspaces where $[\ ]_U$ denotes the Uth graded part. Write $R_\sigma := R_{x_\sigma}$ for short.

$$[R_\sigma]_U := \begin{cases} x^U \cdot k & \text{if } n(U) \subset \sigma \text{ and } s(U) \cup \sigma \in \Delta \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

This is easy to verify because x^U is the only monomial that can be contained in $[R_\sigma]_U$. Only in the first case x^U is not 0 in R_σ and contained in R_σ.

Following (2) $[C^i]_U$ as k-vectorspace is generated in a natural way by $\operatorname{card}\{\sigma \subset V: |\sigma| = i,\ n(U) \subset \sigma,\ \sigma(U) \cup \sigma \in \Delta\}$ basic elements. Examining the boundary maps we get an isomorphism of complexes

$$[C^{i+1}]_U \simeq \tilde{C}^i(\sigma \in \Delta: n(U) \subset \sigma, s(U) \cup \sigma \in \Delta). \tag{3}$$

Removing $n(U)$ we get the isomorphism of complexes

$$\tilde{C}^i(\sigma \in \Delta: n(U) \subset \sigma, s(U) \cup \sigma \in \Delta) \simeq \tilde{C}^{i-|n(U)|}(st_{\operatorname{link}_\Delta n(U)}(s(U) - n(U)))$$

via

$$\sigma \mapsto (-1)^{a(n(U), \sigma - n(U))}(\sigma - n(U)). \tag{4}$$

It follows

$$[H_m^{i+1}(R)]_U \cong \tilde{H}^{i-|n(U)|}(\operatorname{st}_{link_\Delta n(U)}(s(U) - n(U))). \tag{5}$$

If U is "partly positive," i.e., $s(U) - n(U) \neq \emptyset$, the star in (5) is contractible and therefore acyclic. This means

$$[H_m^{i+1}(R)]_U = 0 \qquad \text{if } s(U) \neq n(U) \text{ or } s(U) \notin \Delta. \tag{6}$$

On the other hand, if $s(U) = n(U)$ we get

$$\operatorname{st}_{\operatorname{link}_\Delta n(U)}(\emptyset) = \operatorname{link}_\Delta s(U).$$

Composing (4) and (1.3a) we get the isomorphism of complexes

$$\tilde{C}^i(\sigma \in \Delta: s(U) \subset \sigma) \simeq \tilde{C}^i(\Delta)/\tilde{C}^i(\mathrm{cost}_\Delta s(U)) \tag{7}$$

via

$$\sigma \simeq \sigma + \tilde{C}^i(\mathrm{cost}_\Delta s(U)).$$

It follows

THEOREM 1. *As k-vectorspaces* $H_m^{i+1}(k[\Delta])$ *and*

$$\bigoplus_{-U \in \mathbb{N}^V} H^i(\Delta, \mathrm{cost}_\Delta s(U)) = \tilde{H}^i(\Delta) \oplus \left(\bigoplus_{\varnothing \neq \sigma \in \Delta} (H^i(\Delta, \mathrm{cost}_\Delta \sigma))^{\mathbb{N}^\sigma} \right)$$

are isomorphic.

2.3. (1) COROLLARY [5, 2.1].

$$\begin{aligned} \textit{depth } k[\Delta] - 1 &= \min(i: H^i(\Delta, \mathrm{cost}_\Delta \sigma) \neq 0 \textit{ for some } \sigma \in \Delta) \\ &= \min(i: \tilde{H}^{i-|\sigma|}(\textit{link}_\Delta \sigma) \neq 0 \textit{ for some } \sigma \in \Delta) \\ &= \min(i: H^i(|\Delta|, |\Delta| - p) \neq 0 \textit{ for some } p \in |\Delta| \textit{ or } \tilde{H}^i(\Delta) \neq 0). \end{aligned}$$

This follows from the characterisation of depth by local cohomology [3, 4.10] *depth* $R = \min(i: H_m^i(R) \neq 0)$.

(2) COROLLARY.

$$\begin{aligned} \mathrm{endim}\, k[\Delta] - 1 &= \min(i: H^i(\Delta, \mathrm{cost}_\Delta \sigma) \neq 0 \textit{ for some } \varnothing \neq \sigma \in \Delta) \\ &= \min(i: \tilde{H}^{i-|\sigma|}(\textit{link}_\Delta \sigma) \neq 0 \textit{ for some } \varnothing \neq \sigma \in \Delta) \\ &= \min(i: H^i(|\Delta|, |\Delta| - p) \neq 0 \textit{ for some } p \in |\Delta|). \end{aligned}$$

Here $\mathrm{endim}\, R := \min(i: H_m^i(R)$ is not fin. gen.) denotes the finiteness dimension of R (cf. [7]). The corollary follows from the fact that $H_m^i(R)$ is artinian and so of finite length if it is finitely generated.

We recall that $k[\Delta]$ is Cohen–Macaulay iff depth $k[\Delta] = \dim k[\Delta]$ $(= \dim \Delta + 1)$ and quasi-Cohen–Macaulay [7] iff $\mathrm{endim}\, k[\Delta] = \dim k[\Delta]$. For the special nature of $k[\Delta]$ the later in fact means that $k[\Delta]$ is Buchsbaum.

According to these definitions the underlying complex Δ is also called Cohen–Macaulay resp. Buchsbaum.

2.4. To transfer the S-module structure from $H^i_{\mathfrak{m}}(R)$ to the right-hand side of (2.2.8) it suffices to follow up the multiplication maps

$$\cdot x^W\colon [H^{i+1}_m]_U \to [H^{i+1}_m]_{U+W} \qquad W \in \mathbb{N}^V$$

under the isomorphisms of (2.2).

If $\tau \subset \sigma$ we get $\operatorname{cost}_\Delta \tau \subset \operatorname{cost}_\Delta \sigma \subset \Delta$. The exact sequence of this triple gives rise to define

$$a^i\colon \tilde{C}^i(\Delta)/\tilde{C}^i(\operatorname{cost}_\Delta \sigma) \to \tilde{C}^i(\Delta)/\tilde{C}^i(\operatorname{cost}_\Delta \tau)$$

$$a_i\colon \tilde{C}_i(\Delta)/\tilde{C}_i(\operatorname{cost}_\Delta \tau) \to \tilde{C}_i(\Delta)/\tilde{C}_i(\operatorname{cost}_\Delta \sigma)$$

and the coresponding maps for (co)homologies which we will also denote by a^i and a_i.

If we wind up the isomorphisms (2.2.3–7) we get the map corresponding to $\cdot x^W$ on the right-hand side of (2.2.8) as

$$\cdot x^W = \begin{cases} \text{0-map} & \text{if } U \text{ or } U+W \text{ are partly positive or } s(U) \notin \Delta \\ a^i\colon H^i(\Delta, \operatorname{cost}_\Delta s(U)) \to H^i(\Delta, \operatorname{cost}_\Delta s(U+W)) \end{cases} \tag{2}$$

if $s(U) \in \Delta$ and $0 \leqslant W \leqslant -U$ (in the componentwise partial order on $\mathbb{Z}^V$).

Comments. The first case follows from (2.2.6) by $[H^i_m]_U = 0$ or $[H^i_m]_{U+W} = 0$. The contraposition of the assumptions of the first case is: $s(U) \in \Delta$ and $U \leqslant 0$, $U + W \leqslant 0$. This is equivalent to $s(U) \in \Delta$ and $0 \leqslant W \leqslant -U$ ($W \geqslant 0$ by definition). Furthermore we get $s(U) \supset s(U+W)$ and a^i makes sense.

We have proved the following

THEOREM 2. (2.2.8) *describes an S-module isomorphism*

$$H^{i+1}_m(k[\Delta]) \cong \bigoplus_{\substack{-U \in \mathbb{N}^V \\ s(U) \in \Delta}} H^i(\Delta, \operatorname{cost}_\Delta s(U))$$

giving the right-hand side an S-module structure via (2).

3. Dualizing Complex and Canonical Module

3.1. DEFINITION [2, 8.1]. A dualizing complex over the (local) ring (R, m) is a complex

$$I_\cdot\colon 0 \to I_0 \to I_1 \to \cdots \to I_n \to 0 \tag{1}$$

of injective R-modules with finitely generated cohomology modules, such that the canonical morphism

$$R \to \operatorname{Hom}_R(I_., I_.)$$

is a quasi-isomorphism (i.e., induces an isomorphism in cohomologies).

PROPOSITION [2, 8.5]. *A complex* (1) *of injective R-modules with finitely generated cohomologies is a dualizing complex over R if* $\operatorname{Hom}_R(k, I_.) \cong k[-m]$ *for some* $m \in \mathbb{Z}$.

Here modules are viewed as complexes concentrated in the zero component and by $k[-m]$ the complex concentrated in the mth component is meant.

Analogous results are valid in the category of homogeneous R-modules.

3.2. Let Hom be the Hom-functor in the category of multihomogeneous R-modules. Let C^i be as in (2.1.1)

$$C_i := \operatorname{Hom}_k(C^i, k) = \bigoplus_U [\operatorname{Hom}_k(C^i, k)]_U = \bigoplus_U \operatorname{Hom}_k([C^i]_U, k)$$

with the natural $\mathbb{Z}^V$-grading and

$$D_i := \bigoplus_{U \geqslant 0} [C_i]_U \cong \bigoplus_{U \geqslant 0} C_{i-1}(\Delta, \operatorname{cost}_\Delta s(U))$$

a submodule of the R-module C_i (analogous to (2.4)).

THEOREM 3.

$$C_. : 0 \to C_{N+1} \to C_N \to \cdots \to C_0 \to 0$$

is a dualizing complex of $R = k[\Delta]$ (in the category of homogeneous R-modules). The complex

$$D_. : 0 \to D_{N+1} \to D_N \to \cdots \to D_0 \to 0$$

is quasi-isomorphic to $C_.$, i.e., the embedding $D_. \subset C_.$ induces an isomorphism in homologies ($N = \dim \Delta$).

Proof. (1) All C^i are flat R-modules: Let M be an (hom.) R-module: $M \otimes_R C^i = \oplus_{|\sigma| = i} M \otimes_R R_{x_\sigma} = \oplus M_{x_\sigma}$ by (2.1). So $\cdot \otimes_R C^i$ is an exact functor because localisation is.

(2) All C_i are injective modules:

$$\operatorname{Hom}_R(\cdot, C_i) = \operatorname{Hom}_R(\cdot, \operatorname{Hom}_k(C^i, k)) = \operatorname{Hom}_k(\cdot \otimes_R C^i, k)$$

is exact because $\cdot \otimes_R C^i$ by (1) and $\mathrm{Hom}_k(\cdot, k)$ as dualizing vector spaces are.

(3) $C_.$ and $D_.$ are quasi-isomorphic: In accordance with the multihomogeneous structure of $C_.$ it suffices to show that $[C_.]_U$ and $[D_.]_U$ have the same homologies. But this follows from the exactness of $\mathrm{Hom}_k(\cdot, k)$ and (2.6.6).

(4) $D_.$ is finitely generated and therefore its homologies are too. To verify this one should remember the definition of the multiplicative structure on $\oplus_{U \geqslant 0} C_{i-1}(\Delta, \mathrm{cost}_\Delta s(U)) \cong D_i$.

(5) $\mathrm{Hom}_R(k, C_.) \cong \mathrm{Hom}_k(k \otimes_R C^{\cdot}, k)$ by (2). By (1) we have

$$k \otimes_R C^i \cong \bigoplus_{|\sigma| = i} k_{x_\sigma}.$$

If $\sigma \neq \varnothing$, $x_\sigma \in \mathfrak{m}$ and therefore $k_{x_\sigma} = 0$. It follows

$$k \otimes_R C^{\cdot} \cong (0 \to k \to 0 \to \cdots \to 0)$$

and $\mathrm{Hom}_R(k, C_.) \cong k$.

3.3. $H_{N+1}(C_.) =: K_\Delta$ is called the canonical module of $R = k[\Delta]$. Theorem 2 implies

THEOREM 4.

$$K_\Delta \cong \bigoplus_{\substack{U \in \mathbb{N}^V \\ s(U) \in \Delta}} H_N(\Delta, \mathrm{cost}_\Delta s(U)) \qquad (N = \dim \Delta)$$

with the S-module structure on the RHS defined by (2.4).

3.4. Let's describe a homogeneous map of degree $N+1$ embedding K_Δ in R.

Map $H_N(\Delta, \mathrm{cost}_\Delta \sigma)$ into R in the following way:

$$\sum a_\tau \tau + \tilde{C}_N(\mathrm{cost}_\Delta \sigma) \in H_N(\Delta, \mathrm{cost}_\Delta \sigma) \mapsto \sum a_\tau x_\tau x_\sigma \in R.$$

This map is well defined (independent of the choise of the representative) because $\tau \in \tilde{C}_N(\mathrm{cost}_\Delta \sigma)$ is maximal and so $x_\tau x_\sigma = 0$ in R.

Let $K(\Delta)$ be the ideal of R generated by the images of all $H_N(\Delta, \mathrm{cost}_\Delta \sigma)$ $(\sigma \in \Delta)$.

THEOREM 5.

$$\varphi: K_\Delta \to R = k[\Delta]$$

via

$$\sum_{\sigma \supset s(U)} a_\sigma \sigma + \tilde{C}_N(\text{cost}_\Delta s(U)) \mapsto \sum a_\sigma x_\sigma x^U$$

defines an embedding of K_Δ in R which is homogeneous of degree $N+1$ ($N = \dim \Delta$) and $\varphi(K_\Delta) = K(\Delta)$.

Proof. All assertions follow immediately from Theorem 4 and the above discussion. For example, let's prove the injectivity of φ:

Let be $a \in \text{Ker}\,\varphi$ and $a = \sum a_U$, $a_U = \sum_{\sigma \supset s(U)} a_\sigma^U \sigma$, its decomposition into multihomogeneous components $a_U \in H_N(\Delta, \text{cost}_\Delta s(U))$, $U \in \mathbb{N}^V$. Then we have

$$0 = \varphi(a) = \sum_U \sum_{\sigma \supset s(U)} a_\sigma^U x_\sigma x^U$$

and by multiplying with x_σ

$$0 = \sum_{U: \sigma \supset s(U)} a_\sigma^U x_\sigma^2 x^U.$$

(All terms $U: \sigma \not\supset s(U)$ vanish because σ is maximal and therefore $x_\sigma x^U = 0$ in R.) But if $U \neq W$, $x_\sigma^2 x^U$ and $x_\sigma^2 x^W$ have distinct degrees and hence $a_\sigma^U = 0$ for all U and σ. So $a = 0$.

Remark. This result extends a result of Baclawski [1] and shows it natural access.

3.5. Now let's point out some information about the generators of the canonical module K_Δ. By (3.3) we get

$$[K_\Delta / mK_\Delta]_U \cong \text{Coker}\left(\bigoplus_{0 \leqslant W < U} H_N(\Delta, \text{cost}_\Delta s(W)) \to H_N(\Delta, \text{cost}_\Delta s(U)) \right). \quad (1)$$

That is, K_Δ is generated by elements of multidegree $U \in \{0, 1\}^V$ and has at least $\dim_k \tilde{H}_N(\Delta)$ generators.

K_Δ is generated exactly by $\tilde{H}_N(\Delta)$ if and only if

$$a_N : \tilde{H}_N(\Delta) \to H_N(\Delta, \text{cost}_\Delta \sigma)$$

is surjective for all $\sigma \in \Delta$.

By the long exact sequence of homologies of the pair $\text{cost}_\Delta \sigma \subset \Delta$ this is valid if $\tilde{H}_{N-1}(\text{cost}_\Delta \sigma) = 0$. If $\tilde{H}_{N-1}(\Delta) = 0$, e.g., in the Cohen–Macaulay case, these conditions are even equivalent.

Baclawski [1] calls simplicial complexes Δ $2-\text{CM}$ iff Δ is Cohen–Macaulay and for every vertex $v \in V$ $\text{cost}_\Delta(v)$ has the same dimension as Δ and is CM too.

PROPOSITION [1, Theorem 2]. *Let Δ be CM. $K(\Delta)$ is generated by $\tilde{H}_N(\Delta)$ if and only if Δ is $2-CM$.*

This can also be concluded from the above by a Mayer–Vietoris argument.

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