APPLIED MATHEMATICS IN THE TENTH CENTURY: ABU'L-WAFĀ' CALCULATES THE DISTANCE BAGHDAD-MECCA

BY E. S. KENNEDY
AMERICAN UNIVERSITY OF BEIRUT
BEIRUT, LEBANON

SUMMARIES

In a compact, two-page presentation, an Iranian scientist of the tenth century illustrates the determination of the great circle distance between two points on the earth's surface. His first method employs a technique, well known in his time, for calculating the direction of Muslim prayer. The second method, perhaps of Indian provenance, uses the tangent and versed sine functions. The computations are precise to about one part in a hundred thousand. A solution by an analemma is included.

في بحث مقتضب من صفحتين يعطي عالم ايراني من القرن العاشر طريقة تحديد
المسافة بين نقطتين على سطح الارض . يستعمل في حلّه الاول طريقة كانت
معروفة في زمنه لحساب اتجاه القبلة . وفي حلّه الثاني الذي قد يكون هندي
المنشأ يستعمل دالّة الظل ودالّة الجيب المعكوس . انّ حساباته ذات
دقّة تقارب حوالي جزء واحد من مئة الف . وقد اضفنا حلاًّ ثالثاً بطريقة
الانالمّ .

1. INTRODUCTION

This study is based on a two-page section of a compilation entitled *Dustūr al-Munajjimīn*, MS (Paris) BN Ar. 5968. This extremely interesting anonymous work was put together by a member of the Ismā'īlī sect, sometime along about A.D. 1110, and contains excerpts from chronological, astronomical, and astrological writings (see Zimmermann (1976]). In many instances the compiler names his source, and some of these are known to the literature.

0315-0860/84 $3.00

Toward the bottom of f.63b is the title "On the Determination of the Distances between Localities," and off to the side an inscription which can be read as *li'l-Būzjānī*, "by al-Būzjānī." From this we infer that it is part of the writings of Muḥammad b. Muḥammad, Abu'l-Wafā' al-Būzjānī (fl. 975), an able astronomer and mathematician from Khurāsān in eastern Iran (see DSB [1970-1976, Vol. 1, 39-42]).

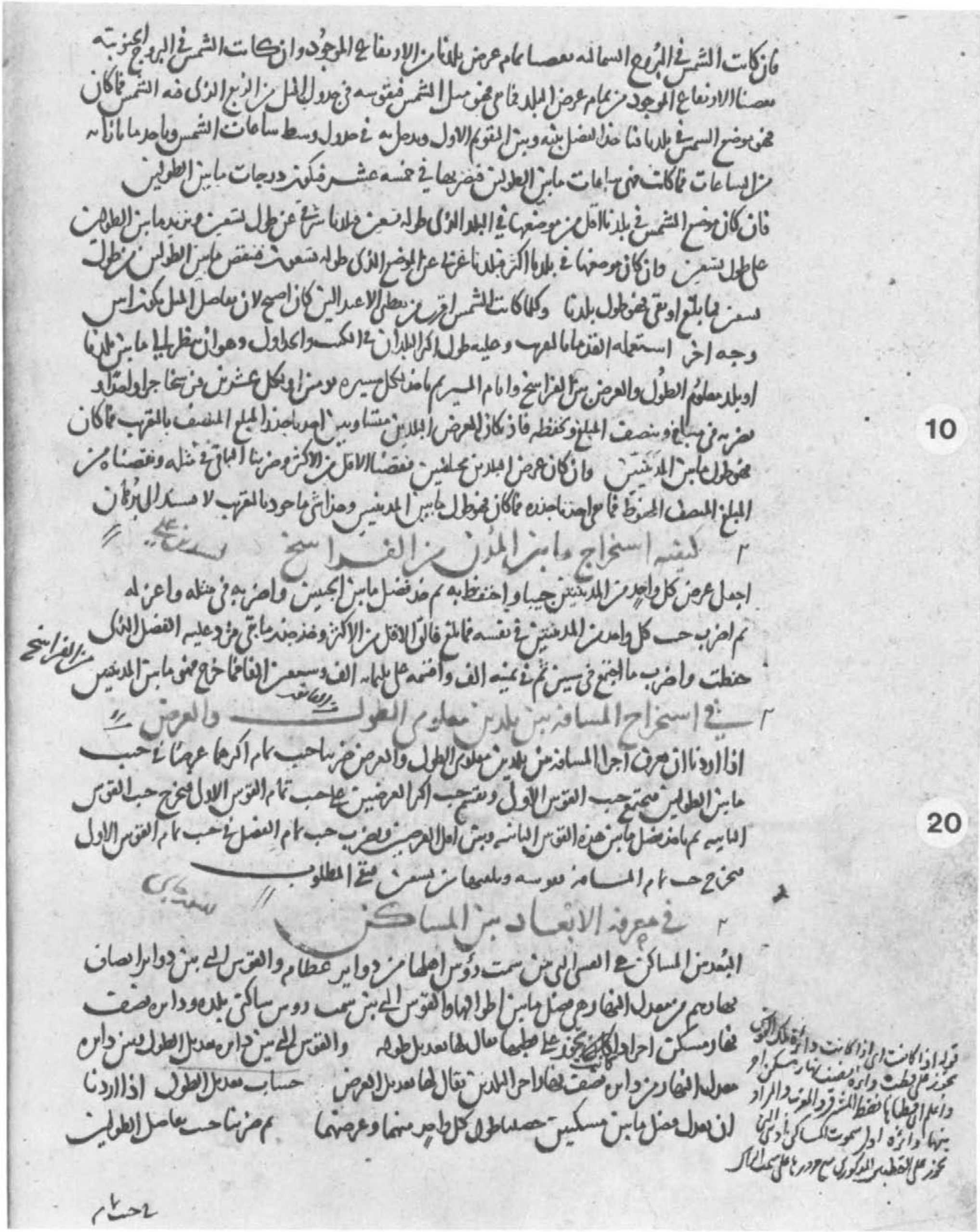
The author gives two rules for calculating the great circle distance between a pair of points on the earth's surface. He applies both to a worked example: given the terrestrial coordinates of Baghdad and Mecca he calculates the distance between them, a matter of some interest to Iraqi Muslims undertaking the pilgrimage.

The first method employs standard medieval spherical trigonometry and can be regarded as a by-product of a common procedure for calculating the *qibla*, the direction of Muslim prayer. It is called by al-Bīrūnī "the method of the *zījes*." Its history is recounted in Berggren [1981].

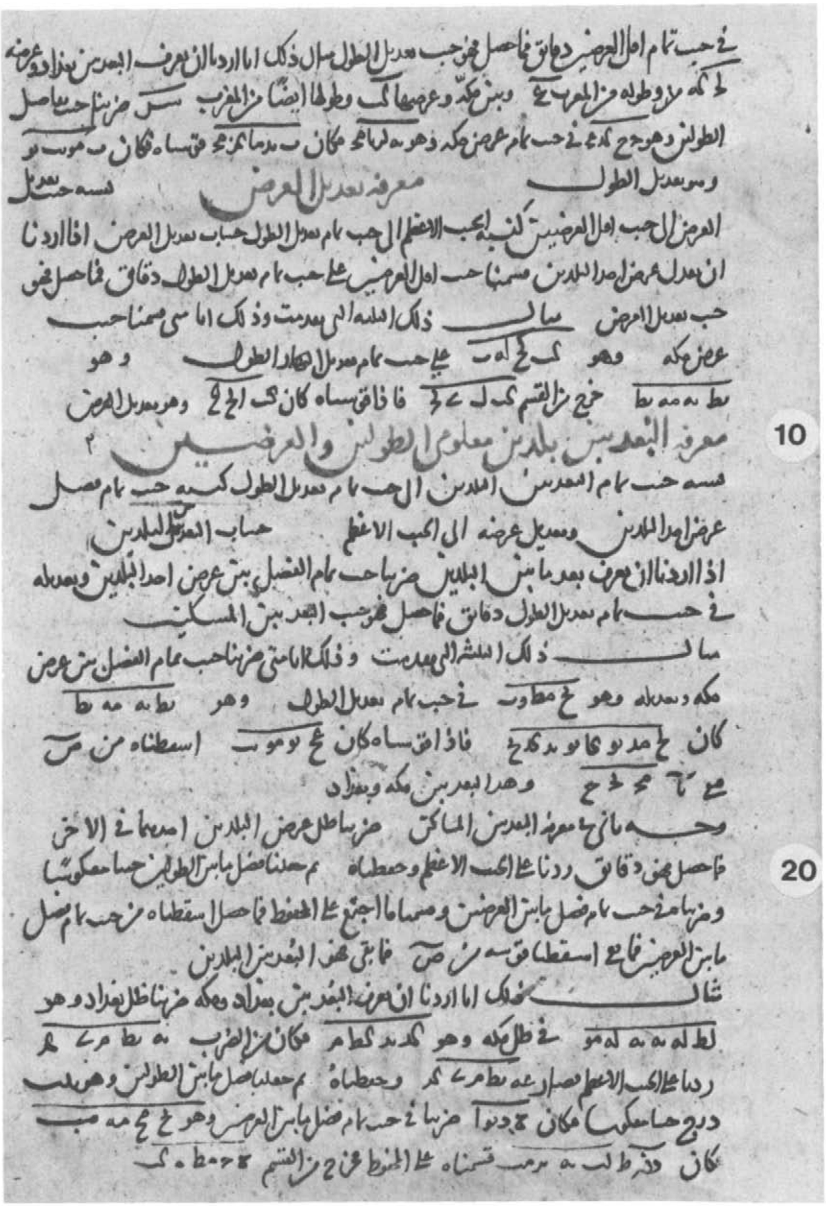
The second method is less ordinary. In addition to the tangent function, it employs the *versed sine* (vers $\theta \equiv 1 - \cos\theta$). Abu'l-Wafā' calls it *al-jayb al-maʿkūs* (the reversed sine), a term he does not use in the treatise studied by Nadir [1960], but which appears frequently in the literature (e.g., Wright [1934, 5], Kennedy [1976, Arabic text 149, 152; 1973, 219, 277, 279, 281]). Also used for the versed sine in other texts is the term *sahm*, Latin *sagitta* (e.g., in Wright [1934, 5], Kennedy [1976, Arabic text 146, 149, 152; 1973, 191]). Occasionally it is called *al-jayb al-mankūs* (the inverted sine), as in Kennedy [1976, Arabic text, 221].

No diagrams or proofs appear in our text, but they are supplied below. The validity of the second method is not obvious. A demonstration has been contributed by Dr. M.-Th. Debarnot, suggested by al-Bīrūnī's "method of the zījes" for calculating a star's hour angle [Debarnot, to appear, 236]. In fact Fig. 1 is very similar to one used by al-Bīrūnī in his second *qibla* calculation in the Taḥdīd [Kennedy 1973, 203]. The "day-triangle" (*muthallath al-nahār*) and the "time triangle" (*muthallath al-waqt*) appear in our Fig. 2, although they are not mentioned in Abu'l-Wafā"s text. They are found frequently in the writings of al-Bīrūnī (e.g., Kennedy [1976, Vol. 2, 79, 97, 130; 1973, 114, 203, 231]). The use of these triangles and the versed sine function may indicate borrowings from Indian astronomy.

Aside from the trigonometry, the text is of interest as an intact example of medieval computational mathematics. Numbers are represented in Arabic alphabetical sexagesimals throughout (see Irani [1955]). The results of the multiplications suggest that all operations were carried out in sexagesimal arithmetic, with none of the very common intermediate transformations into decimal integers (cf. Kennedy [1970, 330-332]). Trigonometric



MS (Paris) BN Ar. 5968, f.63b. Reproduced with permission of the Bibliothèque Nationale, Paris.



MS (Paris) BN Ar. 5968, f.64a. Reproduced with permission of the Bibliothèque Nationale, Paris.

MS (Paris) BN Ar. 5968, f.64b. Reproduced with permission of the Bibliothèque Nationale, Paris.

functions and their inverses are carried to four sexagesimal places. Abu'l-Wafā"s trigonometric tables have not survived, but those of his contemporary and sometime co-worker [al-Bīrūnī 1954-1956] are carried to four places. However, more often than not, the individual determinations are imprecise in the last digit, and for a few they are erroneous by a unit or two in the third digit. The final results are, for the first method,

$$11;43,13,8°,$$

for the second

$$11;43,11,48°,$$

whereas the accurately computed result (based on Abu'l-Wafā's data) is

$$11;43;10,54°.$$

But a maximum error of about one in a hundred thousand, all things considered, seems quite good.

Section 2 is essentially a paraphrase of the text in which mathematical operations have been expressed in modern symbolic form. To indicate the complement of an arc a bar has been placed over the symbol denoting the arc; thus $\bar{\theta} = 90° - \theta$. Sexagesimal digits are separated from each other by the customary commas. Nowhere in the text is there any indication of the denomination of sexagesimal digits; they are to be inferred by the reader. In deference to this the traditional semicolon "sexagesimal point" has been inserted in the transcriptions only for the values of arcs. Pure numbers have been left without these. At some stage in the transmission a careless scribe omitted from the numerals the superscript dot which distinguishes a 50 (the letter *nūn*) from a 10 (*yā'*). The numbers have been transcribed correctly, with restorations enclosed in square brackets.

It is known [Kennedy 1976, Vol. 1, 82; Vol. 2, 33] that Abu'l-Wafā' defined the trigonometric functions with respect to the unit circle, as is the case nowadays, not with the common medieval radius of sixty. However, because of the ambiguity of denomination remarked in the paragraph above, one cannot be sure from the text whether the trigonometric functions there are of the modern or medieval variety (cf. Kennedy [1976, Vol. 2, xv). In transcribing them, the author is given the benefit of the doubt--they are shown as the modern functions.

Numerous critical or explanatory remarks have been interpolated into the text paraphrase below. For instance, the remarks establishing the validity of the first method are such interpolations. References to the manuscript are given in parentheses, the folio and line number, separated by a colon. The authorities of the photographic service of the Bibliothèque Nationale, Paris, have kindly consented to the publication of the text in facsimile.

Section 3 presents the Debarnot proof of the second method.

A common and useful ancient technique for solving spherical astronomical problems was the so-called analemma method. The general idea was to project or rotate elements of the given solid configuration down into a single plane, where the desired magnitude appeared in its true size. The resulting plane configuration was then solved by constructions to scale or by plane trigonometry. The literature abounds with examples of these descriptive geometric methods [Kennedy 1973, 209, 216]. In casting about for the origin of Abu'l-Wafā"s second method, it was reasonable to conjecture that it stemmed from an analemma. Such a solution was indeed found, which exhibits some likeness to the rule. It is presented in Section 4.

2. THE TEXT

The distance between two places is defined (63b:22) as the length of the great circle arc on the celestial sphere between the zeniths of the two localities. Definitions of auxiliary arcs follow, which are illustrated in Fig. 1. On it *M*, for Mecca, is shown east of *B*, for Baghdad, whereas in fact it is to the west. This does not affect the argument or the computations. *N* is the

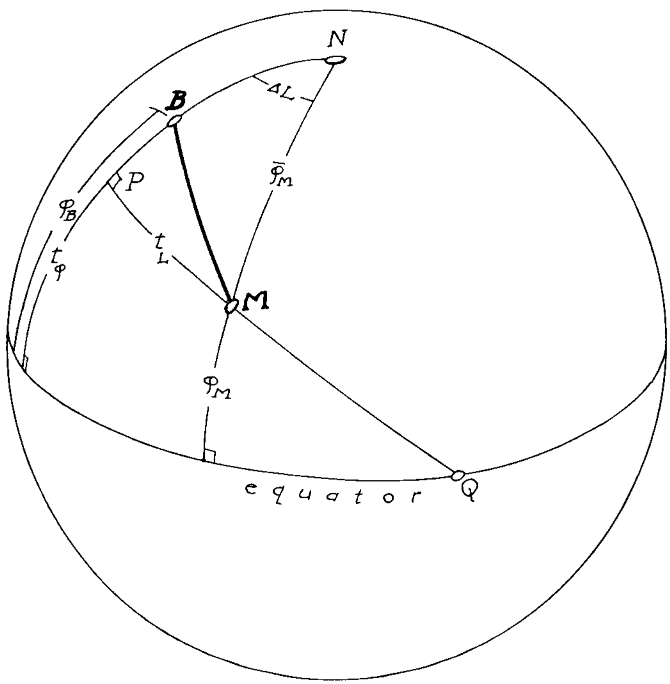


Figure 1

north pole, and Q the pole of meridian NB. Then the "equation of longitude," *ta'dīl al-ṭūl*, is $PM = t_L$, and the "equation of latitude," *ta'dil al-'arḍ*, t_ϕ, is as shown. These are the same terms Abu'l-Wafā' uses in his version of the "method of the zījes" [Berggren 1981, 241]. The equation of longitude t_L is calculated by using the expression

$$\sin t_L = \sin \Delta L \cdot \cos \phi_M, \qquad (64a:1)$$

where M is the locality of lesser latitude. This is valid by application of the sine theorem to right triangle PMN in Fig. 1.

Take as an example the two cities with coordinates (ϕ, terrestrial latitude, and L, longitude) as shown:

(64a:2)

	ϕ	L
Baghdad	33;25°	70°
Mecca	22°	67°

These coordinates for Baghdad are in many sources (cf. Haddad & Kennedy [1971]), for instance, *al-Qānūn al-Mas'ūdī*, by Abu'l-Wafā"s contemporary and occasional collaborator, al-Bīrūnī [1954-1956]. The latitude given for Mecca is that of Ptolemy [Nallino 1899-1907, Vol. 2, 37, 38, No. 103]; we find it among no Muslim sources. The longitude is that used by al-Bīrūnī and others. Accurate values for the latitudes of Baghdad and Mecca are 33;20° and 21;26°, respectively. The accurate value for $\Delta L = L_B - L_M$ is 4;37°, far from Abu'l-Wafā"s 3°. The three-degree difference was obtained from observations made by astronomers of the Caliph al-Ma'mūn.

The calculation is

$$\begin{aligned} \sin t_L &= \sin 3° \cdot \cos 22° \\ &= 3,8,24,43 \times [5]5,37,[5]1,43 \qquad (64a:3) \\ &= 2,[5]4,41,33,13,34. \end{aligned}$$

The last sexagesimal digit in the value of sin 3° should be 34, not 43. The last digit in cos 22° rounds off to 44, not 43, as in the text.

From this

$$t_L = 2;46,[5]2,[5]6°,$$

where the last two digits are wrong. They should be 53,5.

The text proceeds to calculate t_ϕ. Since

$$\sin t_\phi \,/\, \sin \phi_M = 1 \,/\, \cos t_L, \qquad (64a:4)$$

which is valid by application of the sine theorem to right triangle MSQ, hence

$$\begin{aligned} \sin t_\phi &= \sin \phi_M \,/\, \cos t_L \\ (64a{:}8) \qquad &= \sin 22° \,/\, \cos 2;46,52,56° \\ &= 22,28,35,2 \,/\, [5]9,[5]5,49,19 \\ &= 22,30,10,13. \end{aligned}$$

The terminal digit of sin 22° should be 1, not 2; the last two digits of cos t_L should be 45, 33, not 49, 19, and the last two digits of the quotient should be 9,3, not 10,13.

From this

(64a:9) $$t_\phi = 22;1,38,28°,$$

where the terminal digit should be 3.

The concluding step is to utilize the relation

(64a:11) $$\cos BM \,/\, \cos t_L = \cos(\phi_B - t_\phi) \,/\, 1,$$

which is valid by application of the spherical analog of the Pythagorean theorem ($\cos c = \cos a \cdot \cos b$) to the right triangle BMP, to put

$$\begin{aligned} \cos BM &= \cos(\phi_B - t_\phi) \cdot \cos t_L \\ (64a{:}16) \qquad &= \cos(33;25° - 22;1,38,28°) \cos 2;46,52,56° \\ &= [5]8,49,6,2, \times [5]9;[5]5,45,19 \\ (64a{:}17) \qquad &= [5]8,44,[5]6,21,46,[5]4,24,19, \end{aligned}$$

where the last digit in $\cos(\phi_B - t_\phi)$ should be 32, and in cos t_L, 33. Taking the arc sine of the product, the text gives

$$90° - BM = 78;16,46,12°,$$

of which the last two digits should be 45,47, whence

(64a:18) $$BM = 11;43,13,8°,$$

the desired distance.

The text now gives another rule for calculating the same arc. It is to obtain

(64a:19) $$m = \tan \phi_B \cdot \tan \phi_M + 1,$$

then

$$p = \operatorname{vers} \Delta L \cdot \cos \Delta\phi \,/\, m,$$

finally

$$\cos BM = \cos \Delta\phi - p.$$

Applying the rule to the previous example,

(64a:2) m = tan 33;25° · tan 22° + 1

= (39,35,15,[5]5,35,46, X 24,14,29,40) + 60,0,0,...

= 15,[5]9,40,10,24 + 60,0,0,0,0

(64a:25) = 75,[5]9,40,10,24,

where the last two digits of tan 33;25° should be 31,10. Next

(64a:26) p = vers 3° · cos(33;25°−22°) / m

= 0,4,[4]6,1 X [5]8,48,45,42 / 75,59,40,10,24

(64a:27) = 4,50,9,32,15,[5]7,42 / 75,59,40,10,24

= 0,3,49,5,22,

where the last two digits of cos 11;25° should be 46,9.

cos BM = cos 11;25° − p = 58,48,45,42 − 0,3,49,5,22

(64b:1) = [5]8,44,[5]6,36,

where the terminal digit of cos BM should have been rounded up to 37. The arc·sine of this number is reported as

$$90° - BM = 78;16,48,12°,$$

in which the last two digits should be 46,57. The complement of this arc is

$$BM = 11;43,11,48°.$$

3. THE DEBARNOT PROOF OF THE SECOND METHOD

The demonstration uses Fig. 2, in which points B, M, N, and the equator are disposed as in Fig. 1. Now, however, the great circle whose pole is B, the horizon of B, is shown, as are lines dropped from B, M, and K, normal to its plane. The small circle KM is the parallel of latitude through M. DCT is the intersection of the horizon plane of B with the meridian plane through B. MM' and $M'H$ are parallel to the horizon plane of B.

Here

$$KT = KF+FT = KF + FC \tan \phi_B = \cos \phi_M + \sin \phi_M \tan \phi_B$$

$$= \cos \phi_M \ (1+\tan \phi_M \tan \phi_B) = m \cdot \cos \phi_M,$$

where m is the quantity defined in the text at 64a:19.

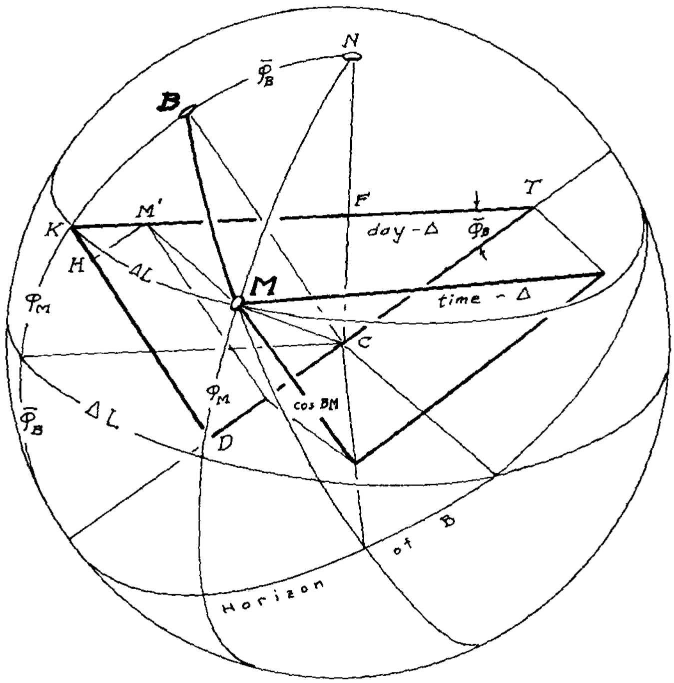


Figure 2

Hence $m = KT / \cos \phi_M$ is what KT would measure if the small circle KM were a great circle.

Now

$$KH/KD = KM'/KT = \text{vers } \Delta L / m,$$

and since

$$KD = \sin(\phi_M + \bar{\Phi}_B) = \cos \Delta\phi,$$

therefore

$$KH = \cos \Delta\phi \text{ vers } \Delta L / m = p,$$

and finally

$$\cos BM = KD - KH = \cos \Delta\phi - p.$$

Thus the auxiliary quantities m and p have been introduced in the order in which they appear in the text, and there can be little doubt but that the originator of the rule used this approach, or something very like it.

The segment KM' in our Figs. 2, 3, and 4 is what al-Bīrūnī calls (in Kennedy [1973, 278 and 281]) the "transformed" (*al-muḥawwal*) versed sine.

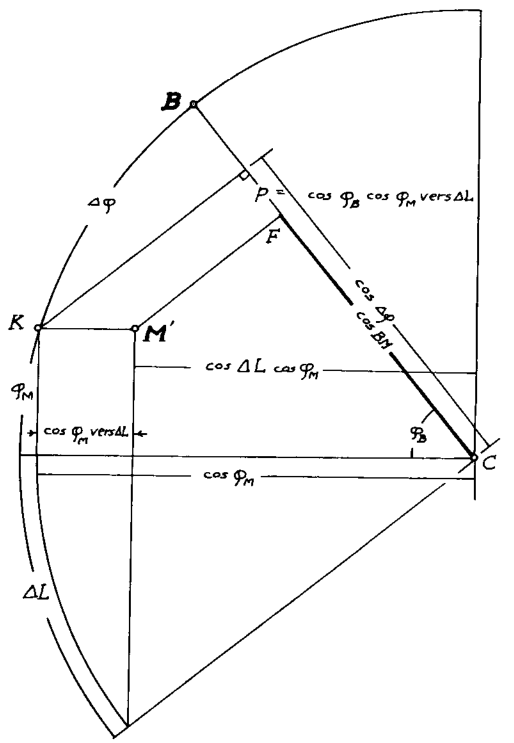


Figure 3

4. AN ANALEMMA SOLUTION

The construction is simple indeed. In the quadrant of the unit circle shown in Fig. 3, lay off from the horizontal the arc ϕ_M terminating at K. Lay off also arc ϕ_B terminating at B. Lay off ΔL downward from the horizontal, and from its endpoint draw a vertical line. Let it intersect at M' with the horizontal line from K. From M' drop $M'F$ perpendicular to BC. Then the distance from C to F, the foot of this perpendicular, is the desired cos BM.

That this construction is valid can be seen from Fig. 4. If the arc ΔL on the equator is rotated down into the meridian plane through B it will assume the position shown on the analemma (Fig. 3). Then M' is seen to be the projection of M on this meridian plane. Now $M'M$ being perpendicular to this plane, and $M'F$ perpendicular to BC, FM must be perpendicular to BC. For if a line is normal to a plane, and a line is drawn from the foot of

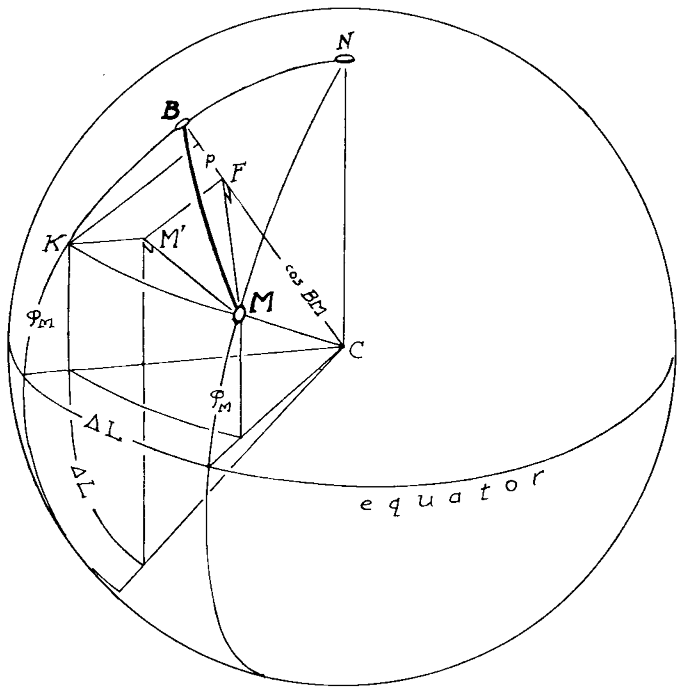


Figure 4

the normal perpendicular to any line in the plane, then any line drawn to the normal from the foot of the latter perpendicular is also perpendicular to the line in the plane (cf. Euclid's *Elements*, XI,8). Hence *FC* is the required cos *BM*.

It is not difficult to see that the segment marked p in Fig. 3 is cos ϕ_B cos ϕ_M vers ΔL. By various trigonometric transformations it can be shown in turn that this expression is equivalent to the definition of p given in the text. So the analemma gives

$$\cos BM = \cos \Delta\phi - p,$$

the last part of the text's second rule.

ACKNOWLEDGMENTS

The author thanks Professors D. A. King and J. L. Berggren, who have read a draft of the paper, and who have pointed out errors and made numerous useful suggestions.

REFERENCES

Berggren, J. L. 1981. On al-Bīrūnī's "Method of the zījes" for the qibla. *Proceedings of the International Congress of the History of Science, Bucharest*, 237-245.

al-Bīrūnī 1954-1956. *Al-Qānūn'l'Mas'ūdī* (Canon Masudicus), 3 vols., Hyderabad-Dn., 1954-1956. See also E. S. Kennedy. 1971. "Al-Bīrūnī's Masudic Canon," *Al-Abhath* 24, 59-81.

Debarnot, M.-Th. *Les clefs d'astronomie d'Abū al-Rayḥān... al-Bīrūnī: La trigonometrie sphérique chez les arabes de l'est à la fin du Xe siècle.* In press.

DSB 1970-1976. *Dictionary of scientific biography*, 16 vols. New York: Scribner's.

Haddad, Fuad I., & Kennedy, E. S. 1971. Geographical tables of medieval Islam. *Al-Abhath* 27, 87-102.

Irani, R. A. K. 1955. Arabic numeral forms. *Centaurus* 4, 1-12.

Kennedy, E. S. 1970. The Arabic heritage in the exact sciences. *Al-Abhath* 23, 327-344.

——— 1973. *A commentary upon Bīrūnī's kitāb taḥdīd al-amākin.* American University of Beirut.

——— 1976. *Al-Bīrūnī. The exhaustive treatise on shadows* (Ifrād al-maqāl...). Translation and commentary, 2 vols. Aleppo: Institute for the History of Arabic Science.

Nadir, N. 1960. Abū al-Wafā' on the solar altitude. *The Mathematics Teacher* 52, 460-463.

Nallino, C. A. 1899-1907. *Al-Battānī sive Albatenii opus astronomicum.* Translation and commentary with Arabic text, 3 vols. Milan.

Wright, R. Ramsay 1934. *The book of instruction in the elements of the art of astrology by Abū'l-Rayḥān...al-Bīrūnī.* An edition and translation of Bīrūnī's *Kitāb al-tafhīm li awā'il ṣanā'at al-tanjīm.* London: Luzac.

Zimmermann, Friedrich 1976. The dustūr al-munajjimīn of MS Paris BN Ar. 5968. *Proceedings of the First International Symposium for the History of Arabic Science, University of Aleppo* 2, 184-192.