Generalized Eigenvalues of a Definite Hermitian Matrix Pair

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Submitted by G. W. Stewart

ABSTRACT

In this note, we study some basic properties of generalized eigenvalues of a definite Hermitian matrix pair. In particular, we prove an interlacing theorem and a minimax theorem. We also obtain upper bounds for the variation of the generalized eigenvalues under perturbation. These results extend and improve those of R.-C. Li, J.-g. Sun, and G. W. Stewart on the topic. © 1998 Elsevier Science Inc.

1. INTRODUCTION

Let \((A, B)\) be a pair of \(n \times n\) Hermitian matrices. We say that it is a definite pair if the Crawford number

\[
c(A, B) = \min \{ |x^* (A + iB)x| : x \in \mathbb{C}^n, x^* x = 1 \} > 0.
\]

We say that \((\alpha, \beta)\) with \(\alpha^2 + \beta^2 = 1\) is a normalized generalized eigenvalue of \((A, B)\) with eigenvector \(x \neq 0\) in \(\mathbb{C}^n\) if

\[
\beta Ax = \alpha Bx.
\]

For a definite pair \((A, B)\), there exists an invertible matrix \(X\) such that

\[
X^*(A + iB)X = \text{diag}(\alpha_1 + i\beta_1, \ldots, \alpha_n + i\beta_n)
\]
where \((\alpha_j, \beta_j) = (\cos \theta_j, \sin \theta_j)\) are the generalized eigenvalues of \((A, B)\) and the \(j\)th column \(x_j\) of \(X\) is the corresponding eigenvector satisfying 
\[ \beta_j Ax_j = \alpha_j Bx_j. \]
This is not the only way to define generalized eigenvalues. One can normalize the eigenvectors rather than the eigenvalues. We do this in a separate paper and show that it can result in much stronger bounds.

Suppose \((A, B) = (A, B) + (E, F)\) for some relatively "small" Hermitian matrices \(E\) and \(F\). For example, one may require that 
\[ |x^*(E + iF)x| < |x^*(A + iB)x| \]
for all unit vectors \(x \in \mathbb{C}^n\), or the stronger (but easier to check) condition that 
\[ r(E + iF) < c(A, B), \]
where 
\[ r(E + iF) := \max\{|x^*(H + iF)x| : x \in \mathbb{C}^n, x^*x = 1\} \]
is the numerical radius of the matrix \(E + iF\) (e.g., see [Li], [St] and [Su]). Then \((\tilde{A}, \tilde{B}) = (A, B) + (E, F)\) will also be definite. We are interested in getting upper bounds on the difference between the generalized eigenvalues \((\tilde{\alpha}, \tilde{\beta})\) of \((\tilde{A}, \tilde{B})\) and those of \((A, B)\). This problem has been considered by several authors, see e.g., [Li], [St], [Su], [SS].

To simplify our discussion, we often assume that the generalized eigenvalues of \((A, B)\) and \((\tilde{A}, \tilde{B})\) are in the same half plane in \(\mathbb{R}^2\). One may replace \((A, B)\) by \((\cos \phi A - \sin \phi B, \sin \phi A + \cos \phi B)\), and \((\tilde{A}, \tilde{B})\) by \((\cos \tilde{\phi} \tilde{A} - \sin \tilde{\phi} \tilde{B}, \sin \tilde{\phi} \tilde{A} + \cos \tilde{\phi} \tilde{B})\), so that all their generalized eigenvalues lie in the upper half plane. Clearly, such replacements will not affect our comparison of the values \((\alpha_j, \beta_j) = (\cos \theta_j, \sin \theta_j)\) and \((\tilde{\alpha}_j, \tilde{\beta}_j) = (\cos \tilde{\theta}_j, \sin \tilde{\theta}_j)\), nor will they change \(r(E + iF)\) and \(c(A, B)\). In fact, we can choose \(\phi\) so that \(B\) is positive definite such that 
\[ c(A, B) = \lambda_n(B) > r(E + iF), \]
where \(\lambda_n(B)\) is the smallest eigenvalue of \(B\). Further, we may arrange \(\alpha_j\) and \(\beta_j\) so that 
\[ -1 < \alpha_1 \leq \cdots \leq \alpha_n < 1, \quad \text{i.e., } \pi > \theta_1 \geq \cdots \geq \theta_n > 0. \]

The numerical range of an \(n \times n\) complex, not necessarily Hermitian matrix \(T\) is the set 
\[ W(T) := \{x^*Tx : x \in \mathbb{C}^n, x^*x = 1\}. \]
We shall not use the numerical range explicitly, but it is useful to think in terms of it. For example, the Crawford number of \((A, B)\) is just the distance from the numerical range \(W(A + iB)\) to the origin; the numerical radius \(r(E + iF)\) if just the furthest point in the numerical range \(W(E + iF)\) from the origin; and a pair \((A, B)\) is definite if and only if \(0 \not\in W(A + iB)\).

We shall also explore other properties of generalized eigenvalues. In particular, we shall prove an interlacing theorem and a minimax theorem. Perturbation theorems will be presented in the last section.

2. BASIC PROPERTIES

In this section we present a generalization of Wielandt's max-min principle and Cauchy's interlace theorem. Notice that in the infimum in (1) we do not require that the \(k\) vectors chosen be orthogonal—they need merely be linearly independent. Because of this the inequality (1) does not immediately imply a normwise perturbation bound on the generalized eigenvalues. We derive a normwise bound in the next section.

**Theorem 2.1.** Suppose \(C = A + iB\) satisfies \(X^*(A + iB)X = \text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})\) with

\[ \pi > \theta_1 \geq \cdots \geq \theta_n > 0 \]

for some invertible matrix \(X\). Then for any sequence \(1 \leq i_1 < \cdots < i_k \leq n\),

\[ \theta_{i_1} + \cdots + \theta_{i_k} = \sup_{W_1 \subseteq \cdots \subseteq W_k} \inf_{y_j \in W_j} \sum_{j=1}^{k} \arg(y_j^* C y_j) \]  

(1)

where \(W_1 \subseteq \cdots \subseteq W_k\) are subspaces of \(\mathbb{C}^n\). The sup inf can actually be attained by vectors in \(W_j\) spanned by the first \(j\) columns of \(X\).

**Proof.** Let \(W_j\) be spanned by the first \(j\) columns of \(X\). Then

\[ \theta_{i_1} + \cdots + \theta_{i_k} = \min_{v_j \in W_j} \sum_{j=1}^{k} \arg(v_j^* C v_j) \]
It remains to prove that the left side of (1) is not less than the right side of (1). To this end, let \( W_1 \subseteq \cdots \subseteq W_k \) be subspaces of \( \mathbb{C}^n \). We shall show that

\[
\theta_{i_1} + \cdots + \theta_{i_k} \geq \inf_{v_j \in W_j} \sum_{j=1}^{k} \arg(v_j^* C v_j).
\]  

(2)

We prove this by induction on \( n \). The result is trivial if \( n = 1 \). Suppose \( n \geq 2 \) and that the result is true for matrices of order \( n - 1 \).

First assume \( k = 1 \) and \( i_1 = p \). Suppose \( W_1 \) has dimension \( p \). Then the subspaces \( W_1 \) and \( \text{span}\{x_{p+1}, \ldots, x_n\} \) and nontrivial intersection, i.e., there exists a nonzero \( y = \sum_{j=p}^{n} \beta_j x_j \in W_1 \) and

\[
y^* Cy = \left( \overline{\beta}_{p+1}, \ldots, \overline{\beta}_n \right) \text{diag}(e^{i\theta_p}, \ldots, e^{i\theta_n}) \left( \beta_{p+1}, \ldots, \beta_n \right)^t,
\]

and thus \( \arg(y^* Cy) \geq \theta_p \).

Notice that a consequence of this special case is the following interlacing result:

**Theorem 2.2.** Suppose \( C = A + iB \) satisfies the hypothesis of Theorem 2.1. If \( Z \in \mathbb{C}^{n \times (n-1)} \) satisfies \( \det(Z^* Z) > 0 \) so that \( Z^* CZ \) is *-congruent to \( \text{diag}(e^{i\phi_1}, \ldots, e^{i\phi_{n-1}}) \), then

\[
\theta_1 \geq \phi_1 \geq \phi_2 \geq \cdots \geq \phi_{n-1} \geq \theta_n.
\]  

(3)

Now return to our proof of Theorem 2.1. Suppose \( k > 1 \) and \( i_k < n \). Let \( W_1 \subseteq \cdots \subseteq W_k \) be subspaces of \( \mathbb{C}^n \) with \( \dim W_j = i_j \). One can construct \( Z \in \mathbb{C}^{n \times (n-1)} \) such that the first \( i_j \) columns of \( Z \) span \( W_j \) for \( j = 1, \ldots, k \), and \( \det(Z^* Z) > 0 \). Suppose \( Z^* CZ \) is *-congruent to \( \text{diag}(e^{i\phi_1}, \ldots, e^{i\phi_{n-1}}) \). Let \( V_j \subseteq \mathbb{C}^{n-1} \) be spanned by the first \( i_j \) standard unit vectors in \( \mathbb{C}^{n-1} \). By the induction assumption that (3), for any \( \delta > 0 \) there exist linearly independent vectors \( v_1, \ldots, v_k \) with \( v_j \in V_j \) such that

\[
-\delta + \sum_{j=1}^{k} \arg(v_j^* Z^* CZ v_j) < \sum_{j=1}^{k} \phi_{i_j} < \sum_{j=1}^{k} \theta_{i_j}
\]

and hence (2) holds.

Finally, consider the case with \( k > 1 \) and \( i_k = n \). Suppose \( W_1 \subseteq \cdots \subseteq W_k \) are such that \( \dim W_j = i_j \). For any \( \delta > 0 \), there exist \( y_j \in W_j \) such that

\[
-\delta + \sum_{j=1}^{k} \arg(y_j^* C y_j) < \inf_{v_j \in W_j} \sum_{j=1}^{k} \arg(v_j^* C v_j).
\]
We may assume that \( \arg(y_j^*C_{y_j}) \geq \arg(y_{j+1}^*C_{y_{j+1}}) \) for \( j = 1, \ldots, k - 1 \). If it is not true, let \( t \) be such that
\[
\arg(y_t^*C_{y_t}) - \arg(y_{t-1}^*C_{y_{t-1}}) = \varepsilon > 0
\]
One can replace \( y_t \) by \( \tilde{y}_t \in W_t \) so that \( \{y_1, \ldots, y_k\} \cup \{\tilde{y}_t\} \setminus \{y_t\} \) is still linearly independent, and \( \|\tilde{y}_t - y_t\| \) is so small that
\[
|\arg(\tilde{y}_t^*C_{y_t}) - \arg(y_t^*C_{y_t})| < \varepsilon.
\]
Then
\[
\sum_{j=1}^{k} \arg(y_j^*C_{y_j}) > \sum_{j=1}^{k} \arg(y_j^*C_{y_j}) + \arg(\tilde{y}_t^*C_{\tilde{y}_t}).
\]
Since \( W_k = \mathbb{C}^n \) contains \( x_n \) where \( \arg(x_n^*C_{x_n}) = \theta_n \), we may assume that
\[
\arg(y_k^*C_{y_k}) = \theta_n.
\]
Otherwise, we can replace \( y_k \) by \( \tilde{y}_k \in \mathbb{C}^n \) with \( \tilde{y}_k \) close to \( x_n \) so that \( \arg(\tilde{y}_k^*C_{\tilde{y}_k}) < \arg(y_k^*C_{y_k}) \). Now, one can apply the induction assumption to the matrix \( Z^*CZ \) where \( Z \in \mathbb{C}^{n \times (n-1)} \) such that the first \( i_j \) columns of \( Z \) span \( W_j \) for \( j = 1, \ldots, k - 1 \), and \( \det(Z^*Z) > 0 \). Consider the chain subspaces \( V_1 \subseteq \cdots \subseteq V_{k-1} \), where \( V_j \) is spanned by the first \( j \) standard unit vectors of \( \mathbb{C}^{n-1} \). Similar to the argument in the previous case, we have
\[
\sum_{j=1}^{k-1} \arg(y_j^*C_{y_j}) \leq \sum_{j=1}^{k-1} \theta_{ij}.
\]
Combining this with the fact that \( \arg(y_k^*C_{y_k}) = \theta_n \), we get inequality (2).

One may relabel the generalized eigenvalues \( \theta_1 \geq \cdots \geq \theta_n \) to
\[
\hat{\theta}_1 \leq \cdots \leq \hat{\theta}_n,
\]
and obtain the following counterpart of Theorem 2.1.

**Theorem 2.3.** Suppose \( C = A + iB \) satisfies \( \hat{X}^*(A + iB)\hat{X} = \text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}) \) with
\[
0 < \hat{\theta}_1 \leq \cdots \leq \hat{\theta}_n < \pi,
\]
for some invertible matrix $\hat{X}$. Then for any sequence $1 \leq i_1 < \cdots < i_k \leq n$,

$$\hat{\theta}_1 + \cdots + \hat{\theta}_k = \inf_{W_1 \subseteq \cdots \subseteq W_k} \sup_{\dim W_j = i_j, y_j \in W_j, \det(y_j^* y_j) > 0} \sum_{j=1}^k \arg(y_j^* C y_j),$$

(4)

where $W_1 \subseteq \cdots \subseteq W_k$ are subspaces of $\mathbb{C}^n$. The inf sup can actually be attained by vectors in $W_j$ spanned by the first $j$ columns of $\hat{X}$.

3. PERTURBATION OF GENERALIZED EIGENVALUES

Let $C = A + iB$ be *-congruent to $\text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_k})$ via $X$, and $\tilde{C} = (A + E) + i(B + F)$ be *-congruent to $\text{diag}(e^{i\tilde{\theta}_1}, \ldots, e^{i\tilde{\theta}_k})$ via $\tilde{X}$. We are interested in obtaining bounds on the vector $|\theta - \tilde{\theta}| = (|\theta_1 - \tilde{\theta}_1|, \ldots, |\theta_n - \tilde{\theta}_n|)$.

First, we use Theorem 2.1 and the technique of Sun [S] to obtain a bound. Let $W_1 \subseteq \cdots \subseteq W_n$ be the chain of subspaces of $\mathbb{C}^n$ such that $W_t$ is spanned by the first $t$ columns of $X$. Similarly, let $\tilde{W}_1 \subseteq \cdots \subseteq \tilde{W}_n$ be the chain of subspaces of $\mathbb{C}^n$ such that $\tilde{W}_t$ is spanned by the first $t$ columns of $\tilde{X}$. If $1 \leq t \leq n$ is such that $\tilde{\theta}_t > \theta_t$, then there exists $y_t \in \tilde{W}_t$ such that

$$\arg(y_t^* C y_t) = \min_{v \in \tilde{W}_t} \arg(v^* C v) < \theta_t < \tilde{\theta}_t \leq \arg(y_t^* \tilde{C} y_t).$$

Hence we have

$$(\tilde{\theta}_t - \theta_t) \leq \max_{v \in \tilde{W}_t} [\arg(v^* \tilde{C} v) - \arg(v^* C v)].$$

Similarly, if $\tilde{\theta}_t < \theta_t$, one can prove

$$(\theta_t - \tilde{\theta}_t) \leq \max_{v \in W_t} [\arg(v^* C v) - \arg(v^* \tilde{C} v)].$$

To estimate $\phi = |\arg(x^* C x) - \arg(x^* \tilde{C} x)|$ for any nonzero $x \in \mathbb{C}^n$, let

$$x^* C x = a + ib \quad \text{and} \quad x^* \tilde{C} x = (a + ib) + (f + ig),$$

where

$$f + ig = x^* (E + iF) x.$$
Then
\[
\sin|\phi| = \frac{|ag - bf|}{\sqrt{a^2 + b^2} \sqrt{(a + f)^2 + (b + g)^2}} \leq \frac{|x^*(E + iF)x|}{|x^*(A + iB)x|},
\]
and
\[
\sin|\phi| = \frac{|(b + g)f - (a + f)g|}{\sqrt{a^2 + b^2} \sqrt{(a + f)^2 + (b + g)^2}} \leq \frac{|x^*(E + iF)x|}{|x^*(A + iB)x|}.
\]

Now, for each \( k = 1, \ldots, n \), let
\[
r_k = \max \left\{ \max_{v \in \mathcal{W}_k} \frac{|v^*(E + iF)v|}{|v^*(A + iB)v|}, \max_{v \in \mathcal{W}_k} \frac{|v^*(E + iF)v|}{|v^*(A + iB)v|} \right\}.
\]
We have \( r_1 \leq \cdots \leq r_n \), and
\[
(|\tilde{\theta}_1 - \theta_1|, \ldots, |\tilde{\theta}_n - \theta_n|) \leq (\sin^{-1}(r_1), \ldots, \sin^{-1}(r_n)).
\]
One may apply the same arguments to \( C = -A + iB \) and obtain
\[
(|\tilde{\theta}_1 - \theta_1|, \ldots, |\tilde{\theta}_n - \theta_n|) \leq (\sin^{-1}(s_1), \ldots, \sin^{-1}(s_n)),
\]
where
\[
s_k = \max \left\{ \max_{v \in \mathcal{V}_k} \frac{|v^*(E + iF)v|}{|v^*(A + iB)v|}, \max_{v \in \mathcal{V}_k} \frac{|v^*(E + iF)v|}{|v^*(A + iB)v|} \right\},
\]
and \( V_1 \supseteq \cdots \supseteq V_n \) (respectively, \( \tilde{V}_1 \supseteq \cdots \supseteq \tilde{V}_n \)) are the chain of subspaces of \( \mathbb{C}^n \) such that \( V_k \) is spanned by the last \( n - k + 1 \) columns of \( X \) (respectively, \( \tilde{X} \)). Clearly, we have \( r_n = s_1 \geq s_2 \cdots \geq s_n \). Combining the above analysis, we have the following result:

**Theorem 3.1.** Let \((A, B)\) and \((\tilde{A}, \tilde{B})\) be definite Hermitian pairs, and let \((E, F) = (A, B) - (\tilde{A}, \tilde{B})\). Suppose \( X^*(A + iB)X = \text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}) \) and \( \tilde{X}^*(A + iB)\tilde{X} = \text{diag}(e^{i\tilde{\theta}_1}, \ldots, e^{i\tilde{\theta}_n}) \) with
\[
0 < \theta_1 \leq \cdots \leq \theta_n < \pi \quad \text{and} \quad 0 < \tilde{\theta}_1 \leq \cdots \leq \tilde{\theta}_n < \pi,
\]
for some invertible matrices \( X \) and \( \tilde{X} \). For \( k = 1, \ldots, n \), define
\[
w_k = \min\{r_k, s_k\},
\]
where \( r_k \) and \( s_k \) satisfy (5) and (6), respectively. Then in the entrywise sense, we have
\[
\left| \bar{\theta}_1 - \theta_1 \right|, \ldots, \left| \bar{\theta}_n - \theta_n \right| \leq \left( \sin^{-1}(w_1), \ldots, \sin^{-1}(w_n) \right).
\]
Consequently, for any absolute norm \( \| \cdot \| \) on \( \mathbb{R}^n \), we have
\[
\| \left| \bar{\theta}_1 - \theta_1 \right|, \ldots, \left| \bar{\theta}_n - \theta_n \right| \| \leq \| \left( \sin^{-1}(w_1), \ldots, \sin^{-1}(w_n) \right) \|.
\]

Note that in applications, \( \tilde{X} \) may not always be available. Nevertheless, if \( X \) and \( \tilde{X} \) are both available then Theorem 3.1 refines the result of Sun [Su] asserting that
\[
\max \{ |\bar{\theta}_k - \theta_k| : 1 \leq k \leq n \} \leq \sin^{-1}(r_n)
\]
and extends it from the sup norm to all absolute norms.

While Theorem 3.1 provides an entrywise bound for \( |\bar{\theta} - \theta| \), it has a weakness, namely, it requires the knowledge of \( X \) and \( \tilde{X} \), and the computation of \( (w_1, \ldots, w_n) \) is rather involved. It is desirable to have a bound for \( |\bar{\theta} - \theta| \) in terms of \( \varepsilon = r(E + iF)/c(A, B) \) only. Li has made such an attempt and proved some majorization inequalities in [Li].

We shall improve the result of Li in the following with shorter proofs. The key idea of our approach is to transform the problem of studying inequalities on matrices to a problem of studying inequalities relating vectors in \( \mathbb{R}^n \), where one can apply the theory of majorization (e.g., see [MO]). Recall that for two vectors \( x, y \in \mathbb{R}^n \), we say that \( y \) weakly majorizes \( x \), denoted by \( x \prec_w y \), if the sum of the \( k \) largest entries of \( x \) is not larger than that of \( y \) for \( k = 1, \ldots, n \); we say that \( y \) majorizes \( x \), denoted by \( x \prec y \), if \( x \prec_w y \) and the sum of the entries of \( x \) is the same as that of \( y \). We shall use \( \sigma(X) = (\sigma_1(X), \ldots, \sigma_n(X)) \) to denote the vector of singular values of a matrix \( X \) with \( \sigma_1(X) \geq \cdots \geq \sigma_n(X) \) and use \( \lambda(Y) = (\lambda_1(Y), \ldots, \lambda_n(Y)) \) to denote the vector of eigenvalues of a Hermitian matrix \( Y \) with \( \lambda_1(Y) \geq \cdots \geq \lambda_n(Y) \).

We first consider the case when the smallest eigenvalue of \( B \) satisfies \( \lambda_n(B) = c(A, B) \). Let
\[
\lambda(B^{-1/2}AB^{-1/2}) = (\lambda_1, \ldots, \lambda_n) = (\cot \theta_1, \ldots, \cot \theta_n),
\]
\[
\lambda(\tilde{B}^{-1/2}AB^{-1/2}) = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n) = (\cot \tilde{\theta}_1, \ldots, \cot \tilde{\theta}_n),
\]
\[
\lambda(\tilde{B}^{-1/2}A\tilde{B}^{-1/2}) = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n) = (\cot \tilde{\theta}_1, \ldots, \cot \tilde{\theta}_n),
\]
Now,
\[
|\theta - \tilde{\theta}| \leq |\cot \theta - \cot \tilde{\theta}|
\]
\[
\langle \omega \sigma (B^{-1/2}(A - \tilde{A})B^{-1/2}) \rangle < \omega \left( \sigma_j(E)/\lambda_{n-j+1}(B) \right)_{1 \leq j \leq n}, \tag{7}
\]

where the first inequality is in the entrywise sense. Next,

\[
|\bar{\theta} - \tilde{\theta}| \leq \tan |\bar{\theta} - \tilde{\theta}|
\]

\[
= \frac{|\tan \theta_j - \tan \tilde{\theta}_j|}{|1 + \tan \theta_j \tan \tilde{\theta}_j|}_{1 \leq j \leq n}
\]

\[
= \frac{|\cot \theta_j - \cot \tilde{\theta}_j|}{|1 + \cot \theta_j \cot \tilde{\theta}_j|}_{1 \leq j \leq n}
\]

\[
= \frac{1}{|1 + \lambda_j \lambda_j|} \left| \frac{1}{\sqrt{\lambda_j \lambda_j} \left( \frac{1}{\sqrt{\lambda_j \lambda_j}} + 1 \right)} \right|_{1 \leq j \leq n}
\]

\[
\leq \frac{1}{2} \left( \frac{1}{\sqrt{\lambda_j \lambda_j}} \right)_{1 \leq j \leq n}
\]

By [1, Corollary 3.3], we have

\[
\left( \frac{|\lambda_j - \tilde{\lambda}_j|}{\sqrt{\lambda_j \lambda_j}} \right)_{1 \leq j \leq n} < \omega \left( \frac{1}{\lambda_j(B^{-1/2}(B + F)B^{-1/2})^{1/2}} \right)_{1 \leq j \leq n}
\]

\[
= \left( \frac{|\lambda_j(B^{-1/2}(B + F)B^{-1/2}) - 1|}{\lambda_j(B^{-1/2}(B + F)B^{-1/2})^{1/2}} \right)_{1 \leq j \leq n}
\]

\[
= \left( \frac{|\lambda_j(B^{-1/2}(B + F)B^{-1/2})|}{\lambda_j(B^{-1/2}(B + F)B^{-1/2})^{1/2}} \right)_{1 \leq j \leq n}
\]

\[
\leq \left( \frac{x_j}{\sqrt{1 - x_j}} \right)_{1 \leq j \leq n}
\]
where $x_j = |\lambda_j(B^{-1/2}FB^{-1/2})| = |\lambda_j(B^{-1}F)|$ for $j = 1, \ldots, n$. Now using Weyl's inequality [MO, Theorem 9.E.1] relating the eigenvalues of a matrix and its singular values for the first majorization and then A. Horn's inequality [MO, Theorem 9.H.1] on the singular values of a product for the second majorization (remembering that $\sigma_i(B^{-1}) = \lambda_{n+1-i}(B)$) we have

$$(\log x_1, \ldots, \log x_n) < (\log \sigma_1(B^{-1}F), \ldots, \log \sigma_n(B^{-1}F))$$

$$< (\log y_1, \ldots, \log y_n),$$

where $y_j = \sigma_j(F)/\lambda_{n-j+1}(B)$ for $j = 1, \ldots, n$. Note that $\log x_j, \log y_j < 0$ for all $j$, and the function $f(t) = e^t/\sqrt{1 - e^t}$ is convex for $t < 0$, we have [MO, Chapter 5, A.1]

$$\left( \begin{array}{c} x_1 \\ \sqrt{1 - x_1} \\ \vdots \\ x_n \\ \sqrt{1 - x_n} \end{array} \right)$$

$$= \left( \begin{array}{c} f(\log x_1) \\ \vdots \\ f(\log x_n) \end{array} \right)$$

$$<_w \left( \begin{array}{c} f(\log y_1) \\ \vdots \\ f(\log y_n) \end{array} \right)$$

$$= \left( \begin{array}{c} y_1 \\ \sqrt{1 - y_1} \\ \vdots \\ y_n \\ \sqrt{1 - y_n} \end{array} \right)$$

$$= \left( \begin{array}{c} \sigma_j(F) \\ \lambda_{n-j+1}(B) \sqrt{1 - (\sigma_j(F)/\lambda_{n-j+1}(B))} \end{array} \right)_{1 \leq j \leq n}.$$

Consequently, we have

$$|\tilde{\theta} - \theta| <_w \frac{1}{2} \left( \begin{array}{c} \sigma_j(F) \\ \lambda_{n-j+1}(B) \sqrt{1 - (\sigma_j(F)/\lambda_{n-j+1}(B))} \end{array} \right)_{1 \leq j \leq n}. \quad (8)$$

To apply the above argument to the general situation when $0 \notin W(A + iB)$ and $c(A, B) > r(E + iF)$ so that both $(A, B)$ and $(\tilde{A}, \tilde{B}) = (A, B) + (E, F)$ are definite pairs, replace $(A, B)$ and $(E, F)$ with $(\cos \phi A - \sin \phi B, \sin \phi A + \cos \phi B)$ and $(\cos \phi E - \sin \phi F, \sin \phi E + \cos \phi F)$, respectively. The effect of such a replacement is just rotating $W(A + iB)$ and $W(E + iF)$ by the same angle $\phi$ in the complex plane, and therefore will not affect $c(A, B)$ and
$r(E + iF)$. Of course, to apply our estimates in (7) and (8), one would choose $\phi$ so that $\lambda_n(B) = c(A, B)$ after the replacement. In such case, we have

$$\left| \tilde{\theta}_1 - \theta_1 \right|, \ldots, \left| \tilde{\theta}_n - \theta_n \right|$$

$$\leq \frac{1}{\lambda_{n-j+1}(B)} \left( \frac{\sigma_j(E)}{2\sqrt{1 - \left( \frac{\sigma_j(F)}{\lambda_{n-j+1}(B)} \right)^2}} \right)_{1 \leq j \leq n}$$

$$\leq \frac{1}{\lambda_n(B)} \left( \frac{\sigma_j(E)}{2\sqrt{1 - \left( \frac{\sigma_j(F)}{\lambda_n(B)} \right)^2}} \right)_{1 \leq j \leq n}$$

$$\leq \frac{1}{\lambda_n(B)} \left( \frac{\sigma_j(E)}{2\sqrt{1 - \left( \frac{r(F)}{\lambda_n(B)} \right)^2}} \right)_{1 \leq j \leq n}.$$  \hspace{1cm} (9)

Consequently, for any symmetric norm (also known as symmetric gauge functions, e.g., see [MO, Chapter 10] for basic definitions and properties) $\| \cdot \|$ on $\mathbb{R}^n$, we have

$$\| (\tilde{\theta}_1 - \theta_1, \ldots, \tilde{\theta}_n - \theta_n) \|$$

$$\leq \left\| \left( \frac{1}{\lambda_{n-j+1}(B)} \left( \frac{\sigma_j(E)}{2\sqrt{1 - \left( \frac{\sigma_j(F)}{\lambda_{n-j+1}(B)} \right)^2}} \right) \right)_{1 \leq j \leq n} \right\|$$

$$\leq \frac{1}{\lambda_n(B)} \left( \frac{\sigma_j(E)}{2\sqrt{1 - \left( \frac{\sigma_j(F)}{\lambda_n(B)} \right)^2}} \right)_{1 \leq j \leq n}$$

$$\leq \frac{1}{\lambda_n(B)} \left( \frac{\sigma_j(E)}{2\sqrt{1 - \left( \frac{r(F)}{\lambda_n(B)} \right)^2}} \right)_{1 \leq j \leq n}.$$  \hspace{1cm} (10)

Clearly, from the above analysis, one sees that sharper bounds for $|\tilde{\theta} - \theta|$ can be computed using more information on $B$. In fact, one might consider bounding $|\theta - \tilde{\theta}|$ and $|\tilde{\theta} - \tilde{\theta}|$ directly instead of the cotangents and tangents of the entries, etc. In any event, if one uses our method, and if only the extreme values $\lambda_n(B)$ and $r(F)$ are known, then the last bounds in (9) and (10), which are simple but less accurate, can be used.
In applications, the value $\phi$ needed to rotate $A + iB$ to achieve $\lambda_n(B) = c(A, B)$ may not be easy to determine, and one may have only the norm bounds for the matrices $E$ and $F$. In such cases, one may need to give up more accuracy. For example, using the facts that

$$\sigma(\cos \phi E + \sin \phi F) \prec_w \sigma([E \mid F]);$$

$$r(\sin \phi E - \cos \phi F) \leq r(E + iF);$$

and combining the estimates for $|\theta - \tilde{\theta}|$ and $|\tilde{\theta} - \hat{\theta}|$ in (7) and (8), we have the following result.

**Theorem 3.2.** Let $(A, B)$ and $(E, F)$ be Hermitian pairs such that $c(A, B) > 0$ and $\varepsilon = r(E + iF)/c(A, B) < 1$ and let $(\tilde{A}, \tilde{B}) = (A, B) + (E, F)$. Then there exists $\phi \in \mathbb{R}$ such that $A + iB$ and $\tilde{A} + i\tilde{B}$ are $*$-congruent to

$$e^{i\phi}\text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}) \quad \text{and} \quad e^{i\phi}\text{diag}(e^{i\tilde{\theta}_1}, \ldots, e^{i\tilde{\theta}_n}),$$

respectively, with

$$0 < \theta_1 \leq \cdots \leq \theta_n < \pi \quad \text{and} \quad 0 < \tilde{\theta}_1 \leq \cdots \leq \tilde{\theta}_n < \pi.$$

Moreover, we have

$$\left| |\tilde{\theta}_1 - \theta_1|, \ldots, |\tilde{\theta}_n - \theta_n| \right| \prec_w \left( 1 + \frac{1}{2\sqrt{1 - \varepsilon}} \right) \sigma([E \mid F]);$$

and hence for any symmetric norm $\| \cdot \|$ on $\mathbb{R}^n$,

$$\left\|(|\tilde{\theta}_1 - \theta_1|, \ldots, |\tilde{\theta}_n - \theta_n|) \right\| \leq \left( 1 + \frac{1}{2\sqrt{1 - \varepsilon}} \right) \frac{1}{c(A, B)} \| \sigma([E \mid F]) \|.$$

The statement of our result is simpler than that of [Theorem 2.2, Li]. It is also stronger in three ways—we bound the difference in the angles themselves not their sines, the constant in our bound is smaller, and finally we specify how to match the eigenvalues while [Li] does not.
REFERENCES


Received 27 May 1997; final manuscript accepted 7 June 1997