Conditional S-matrices

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Abstract

Conditional S-matrices and conditional S*-matrices (abbreviated CS-matrices and CS*-matrices) were first introduced and studied in [Math. Comput. Modelling 17 (1993) 141; Matrices of Sign-solvable Linear Systems, Cambridge University Press, 1995]. They are generalizations of the well known S-matrices and S*-matrices. CS-matrices and CS*-matrices play an important role in the study of conditionally sign solvable linear systems. We study various properties and recognition criteria of CS-matrices. We also study several special classes of CS-matrices such as square CS-matrices, barely CS-matrices (BCS-matrices) and maximal CS-matrices.

AMS classification: 15A09; 15A48

Keywords: Sign; Matrix; Graph

1. Introduction

The sign pattern of a real matrix $A$, denoted by $\text{sgn} A$, is the $(0, 1, -1)$-matrix obtained from $A$ by replacing each entry by its sign. The set of real matrices with the same sign pattern as $A$ is called the qualitative class of $A$, denoted by $Q(A)$.

A real matrix $A$ is called an L-matrix provided that each matrix with the same sign pattern as $A$ has linearly independent columns. A square L-matrix is called a sign nonsingular matrix (abbreviated SNS matrix).
An $m \times (m + 1)$ matrix $A$ is called an $S^*$-matrix if each submatrix of $A$ of order $m$ is an SNS matrix. An $S^*$-matrix $A$ is called an $S$-matrix if each row of $A$ contains both positive and negative entries.

A linear system of equations $Ax = b$ is sign solvable if it is solvable, and both its solvability and the sign pattern of its solution vector are uniquely determined by the sign patterns of $A$ and $b$. Sign solvable linear systems are motivated by "qualitative economics", and are extensively studied by mathematicians, economists and computer scientists [1–5].

The classes of $L$-matrices and $S^*$-matrices (or $S$-matrices) are fundamental to the study of sign solvable linear systems. In fact, it is shown in [3,5] that the problem of recognizing sign solvable linear systems can be turned to the problems of recognizing $L$-matrices and $S^*$-matrices.

In [2,3], Brualdi and coworkers introduced the classes of $CS^*$-matrices and $CS$-matrices (conditionally $S^*$-matrices and conditionally $S$-matrices) in order to study conditionally sign solvable linear systems. It is shown in [2,3] that $CS^*$-matrices (and $CS$-matrices) are generalizations of $S^*$-matrices (and $S$-matrices), and the problem of recognizing conditionally sign solvable linear systems can be reduced to the problems of recognizing $L$-matrices and $CS^*$-matrices; thus, generalizing the classical recognition results for sign solvable linear systems.

$CS^*$ (and $CS$)-matrices and some special classes of $CS^*$-matrices such as $BCS^*$-matrices were further studied in Section 3 of [2] and in Section 5 of [6]. In [6], a characterization of $CS^*$-matrices in terms of GRSB matrices (also see Theorem 2.A and Theorem 2.B of this paper) is given which aids in the proofs of the main results of this paper.

In this paper we first give some basic properties of $CS$ and $CS^*$-matrices in Section 2. Many of these properties (and some of the main results in Sections 3–5) are generalizations of the corresponding results for $S$- and $S^*$-matrices in [3, Chapter 4]. We also introduce in Section 2 the "sign order" of matrices and the concept of "majorized row" (over a matrix) to generalize the "conformal contraction" operation which preserves the $S^*$ and $S$ property of matrices to the "generalized conformal contraction" operation which preserves the $CS^*$ and $CS$ property of matrices (for $S^*$- and $S$-matrices, the generalized conformal contractions are the same as the usual conformal contractions). Then we study the properties and recognition criteria of $CS$-matrices in Section 3. We also study various properties of some special classes of $CS$-matrices such as $BCS$-matrices (barely $CS$-matrices, which are defined and studied in [2]) in Section 5, maximal $CS$-matrices in Section 5 and square $CS$-matrices in Sections 3 and 4. For $BCS$-matrices, we give sharp lower and upper bounds for the number of rows $m$ and the number of nonzero entries $N(A)$ in terms of the number of columns $n$ of $A$. We also propose a problem about the largest number of nonzero entries of $m \times n$ $BCS$-matrices. For maximal $CS$-matrices, we obtain several necessary conditions and several sufficient conditions for a matrix to be a maximal $CS$-matrix. For square $CS$-matrices, we first notice that each $m \times n$ $CS$-matrices satisfies $n \leq m + 1$, so the most interesting cases in the study
of CS-matrices are $n = m + 1$ and $n = m$. In the case $n = m + 1$, CS-matrices are the same as $S$-matrices. So the remaining case $n = m$ of square CS-matrices are of particular interest. In Section 3, we use graph theoretical methods to give a characterization and a recognition algorithm for square CS-matrices. In Section 4, we also use graph theoretical methods to give sharp lower and upper bounds for the number of nonzero entries of square CS-matrices of order $n$, together with complete characterizations of the extremal CS-matrices whose number of nonzero entries attain these bounds.

2. Some basic properties of CS*-matrices

A strict signing operation of a row (or a column) of a matrix means multiplying that row (or column) by 1 or $-1$. Two $m \times n$ matrices $A$ and $B$ are said to be equivalent (or row equivalent), if $B$ can be obtained from $A$ by suitably permuting and strictly signing the rows and columns (or only rows) of $A$. We use $A \sim B$ to denote that the two matrices $A$ and $B$ are equivalent.

A real matrix $A$ is a row sign balanced matrix (abbreviated RSB matrix), if each row of $A$ contains both positive and negative entries. The matrix $A$ is a generalized row sign balanced matrix (abbreviated GRSB matrix), if some strict column signing matrix of $A$ is an RSB matrix. The matrix $A$ is a nearly $L$-matrix if $A$ is not an $L$-matrix, but each matrix obtained from $A$ by deleting one of its columns is an $L$-matrix.

Definition 2.1 [2]. A real matrix $A$ is called a CS*-matrix if it satisfies the following two conditions:

1. $A$ is a nearly $L$-matrix with no zero rows.
2. For each pair of matrices $A', A'' \in Q(A)$ and nonzero vectors $u'$ and $u''$ with $A' u' = 0$ and $A'' u'' = 0$, we have either $\text{sgn} u' = \text{sgn} u''$ or $\text{sgn} u' = -\text{sgn} u''$.

$A$ is called a CS-matrix if it is both a CS*-matrix and an RSB matrix.

It is obvious from the definition that the property of being a CS*-matrix depends only on the sign pattern of the matrix and is preserved under the permutations and strict signing operations of the rows and columns of the matrix. From condition (1) of Definition 2.1 we also see that if $A$ is an $m \times n$ CS*-matrix, then $n \leq m + 1$.

From [2,6], we know that a CS*-matrix must be a GRSB-matrix, and if $A$ is an $m \times (m + 1)$ real matrix, then $A$ is a CS*-matrix (or CS-matrix) if and only if $A$ is an $S^*$-matrix (or $S$-matrix).

The following characterization of CS*-matrix (in terms of GRSB matrix) is given in [6].
Theorem 2.A [6]. Let $A$ be an $m \times n$ GRSB matrix. Then the following are equivalent:

(1) $A$ is a CS*-matrix.

(2) There exists a $1$ by $n$ real vector $v$ with no zero entries such that the matrix $\begin{pmatrix} A \\ v \end{pmatrix}$ is not a GRSB matrix.

(3) There exists a $1$ by $n$ real vector $v$ with no zero entries such that the matrix $\begin{pmatrix} A \\ v \end{pmatrix}$ is an $L$-matrix.

The CS-form of Theorem 2.A is the following theorem.

Theorem 2.B. Let $A$ be an $m \times n$ RSB matrix and $e_n$ be the $1$ by $n$ vector with all entries equal to $1$. Then the following are equivalent:

(1) $A$ is a CS-matrix.

(2) $\begin{pmatrix} A \\ e_n \end{pmatrix}$ is not a GRSB matrix.

(3) $\begin{pmatrix} A \\ e_n \end{pmatrix}$ is an $L$-matrix.

The following definitions of the sign order “$\preceq$” between two real numbers or two real matrices are introduced in [7].

Definition 2.2. Let $a, b$ be two real numbers, $A = (a_{ij})$ and $B = (b_{ij})$ be two $m \times n$ real matrices.

(1) We say that “$b$ is sign majorized by $a$”, denoted by $b \preceq a$, if $b = 0$ or $\text{sgn } b = \text{sgn } a$.

(2) We say that “$B$ is sign majorized by $A$”, denoted by $B \preceq A$, if $b_{ij} \preceq a_{ij}$ for each $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

(3) An $1$ by $n$ vector $u$ is a majorized row over $A$, if some row of $A$ is sign majorized by $u$.

It is obvious that if $B \preceq A$ and $B$ is a GRSB matrix (or RSB matrix), then $A$ is also a GRSB matrix (or RSB matrix).

Now we use the characterizations of CS*- and CS-matrices given in Theorems 2.A and 2.B to prove the following three theorems.

Theorem 2.1. If $A$ is a CS*-matrix (or CS-matrix) and $B$ is a GRSB matrix (or RSB matrix) with $B \preceq A$, then $B$ is also a CS*-matrix (or CS-matrix).

Proof. We prove the contrapositive. Assume that $B$ is not a CS*-matrix. Then by Theorem 2.A, for each row vector $v$ with no zero entries, $\begin{pmatrix} B \\ v \end{pmatrix}$ is a GRSB matrix. So $\begin{pmatrix} A \\ v \end{pmatrix}$ is also a GRSB matrix and thus $A$ is not a CS*-matrix. □
The following Theorem 2.2 shows that adding a majorized row to a matrix preserves the properties of being an RSB-, GRSB-, CS- and CS*-matrix.

**Theorem 2.2.** Suppose $u$ or $-u$ is a majorized row over a real matrix $A$. Then:

1. $A$ is an RSB matrix if and only if $(A_u)$ is.
2. $A$ is a GRSB-matrix if and only if $(A_u)$ is.
3. $A$ is a CS*-matrix if and only if $(A_u)$ is.
4. $A$ is a CS-matrix if and only if $(A_u)$ is.

**Proof.** (1) is obvious and (2) follows from (1). (3) follows from (2) and Theorem 2.2, and (4) follows from (1) and (3). □

Two $m \times n$ real matrices $A = (a_{ij})$ and $B = (b_{ij})$ are said to be conformal, if $a_{ij}b_{ij} \geq 0$ for each $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

From the definitions we see that each row of an RSB (or GRSB, or CS, or CS*) matrix contains at least two nonzero entries. For convenience, a row containing exactly two nonzero entries will be called a binary row. Now suppose $A$ is a $(0, 1, -1)$-matrix containing a binary row, then $A$ is equivalent to a matrix of the following form:

$$A \sim \begin{pmatrix} 1 & -1 & 0 \\ u_1 & v_1 & A_1 \\ u_2 & v_2 & A_2 \end{pmatrix},$$

where $u_1$, $u_2$, $v_1$, $v_2$ are columns such that $u_2$ and $v_2$ are conformal, and $u_1$, $v_1$ contain no zero entries and $u_1 = -v_1$. Such form (2.1) is called a binormal form of $A$.

Moreover, the matrix $B = (u_2 + v_2, A_2)$ is called a generalized conformal contraction matrix of $A$, and is called a (usual) conformal contraction matrix of $A$ in the special case where $(u_1, v_1, A_1)$ is vacuous (containing no rows).

Note that if the matrix $A$ in (2.1) is an $m \times n$ $S^*$-matrix, then $(u_1, v_1, A_1)$ must be vacuous. Otherwise the $m_1 \times n$ matrix $A'$ obtained from $A$ by deleting $(u_1, v_1, A_1)$ is a CS*-matrix (by Theorem 2.2) with $m_1 < m = n - 1$, a contradiction. So for $S^*$-matrices, the generalized conformal contractions are the same as the usual conformal contractions.

The following Theorem 2.3 is a generalization of Lemma 4.2.1 in [3] for $S^*$-matrices.

**Theorem 2.3.** Let $A$ be a matrix of the binormal form (2.1) and $B = (u_2 + v_2, A_2)$ is a generalized conformal contraction of $A$. Then:
A is an RSB matrix if and only if $B$ is.
(2) $A$ is a GRSB matrix if and only if $B$ is.
(3) $A$ is a CS*-matrix if and only if $B$ is.
(4) $A$ is a CS-matrix if and only if $B$ is.

Proof. By assumption each row of $(u_1, v_1, A_1)$ majorizes over the first row (or its negative) of $A$. So by Theorem 2.2 we may assume that $(u_1, v_1, A_1)$ is vacuous. Thus $A = \begin{pmatrix} 1 & -1 & 0 \\ u_2 & v_2 & A_2 \end{pmatrix}$, and $B = (u_2 + v_2, A_2)$ is a conformal contraction of $A$. The results (1) and (2) are obvious, and (4) follows from (1) and (3). So it suffices for us to prove (3).

Necessity. Since $A$ is a CS*-matrix, by Theorem 2.A there exists a row vector $v$ with no zero entries such that $\begin{pmatrix} A \\ v \end{pmatrix}$ is not a GRSB matrix. Write $v = (a, b, v')$ and

\[
\left( \begin{array}{c}
A \\
v
\end{array} \right) = \begin{pmatrix} 1 & -1 & 0 \\ u_2 & v_2 & A_2 \end{pmatrix}.
\]

Then $ab > 0$ since $A$ is a GRSB matrix but $\begin{pmatrix} A \\ v \end{pmatrix}$ is not. Write

\[
B' = \begin{pmatrix} u_2 + v_2 \\ a + b \\
A_2 \end{pmatrix} = \begin{pmatrix} B \\ v'' \end{pmatrix},
\]

where $v'' = (a + b, v')$ contains no zero entries. Clearly $B'$ is a conformal contraction of $\begin{pmatrix} A \\ v \end{pmatrix}$, so $B'$ is not a GRSB matrix by (2) and thus $B$ is a CS*-matrix.

Sufficiency. Since $B$ is a CS*-matrix, by Theorem 2.A there exists a row vector $u$ with no zero entries such that $\begin{pmatrix} B \\ u \end{pmatrix}$ is not a GRSB matrix. Write

\[
u = (c, u'), \quad \left( \begin{array}{c}
B \\
u
\end{array} \right) = \begin{pmatrix} u_2 + v_2 & A_2 \\ c & u' \end{pmatrix} \quad \text{and} \quad \left( \begin{array}{c}
A \\
v
\end{array} \right) = \begin{pmatrix} 1 & -1 & 0 \\ u_2 & v_2 & A_2 \end{pmatrix},
\]

where $v = (\frac{1}{2} c, \frac{1}{2} c, u')$ contains no zero entries. Then $\begin{pmatrix} B \\ u \end{pmatrix}$ is a conformal contraction of $\begin{pmatrix} A \\ v \end{pmatrix}$. So $\begin{pmatrix} A \\ v \end{pmatrix}$ is not a GRSB matrix by (2) and thus $A$ is a CS*-matrix.

3. CS-matrices and standard RSB matrices

In this section we study the properties and recognition criteria of CS-matrices. For this purpose, we first introduce the concept of standard RSB matrix which has close relationships with those CS-matrices containing no binary rows.
Definition 3.1. A square RSB matrix $A$ is called a standard RSB matrix if all the off-diagonal entries of $A$ are nonnegative.

(It follows that all the diagonal entries of $A$ are negative).

A row vector which is not an RSB matrix is also called an unsigned row.

The following Lemma 3.1 is, in some sense, a generalization of the Theorem 4.1.1 in [3].

Lemma 3.1. Let $A$ be an $m \times n$ CS-matrix each of whose rows contains at least three nonzero entries. Then $m \geq n$ and $A$ is row equivalent to a matrix which contains a standard RSB submatrix of order $n$.

Proof. Let $A_j$ be the matrix obtained from $A$ by deleting its $j$th column ($j = 1, \ldots, n$). Since $A$ is a CS-matrix, $A$ is a nearly $L$-matrix by definition. So each $A_j$ is an $L$-matrix and thus is not an RSB matrix. It follows that each $A_j$ contains an “unsigned row”, say the $m_j$th row. Now we must have $m_i \neq m_j$ for $i \neq j$, for otherwise the $m_j$th row of $A$ would contain at most two nonzero entries, contradicting the hypothesis. Thus we have $m \geq n$ and by row permutations we may assume that $m_j = j$ ($j = 1, \ldots, n$). By strict row signing operations we may further assume that the unsigned $j$th row of $A_j$ contains no negative entry ($j = 1, \ldots, n$). Thus the submatrix of $A$ of order $n$ containing the first $n$ rows is a standard RSB matrix of order $n$. $\square$

Let $A$ be an $m \times n$ $(0, 1, -1)$ RSB matrix. To determine whether or not $A$ is a CS-matrix, we can consider the following two cases:

Case A. $A$ contains a binary row.

Case B. Each row of $A$ contains at least three nonzero entries.

In Case A, we may assume that $A$ is in binormal form (2.1). Let $B = (u_2 + v_2, A_2)$ be a generalized conformal contraction of $A$. Then by Theorem 2.3, $A$ is a CS-matrix if and only if $B$ is. Thus the problem of the determination of a CS-matrix in this case can be reduced to a similar problem for a matrix of smaller size.

In Case B, we first use the process as in the proof of Lemma 3.1 to see if $A$ is row equivalent to a matrix of the form $(B_1, B_2)$, where $B_1$ is a standard RSB matrix. Then we consider when a matrix of such form is a CS-matrix. For this purpose, we first give the following graph theoretical characterization for a standard RSB matrix to be a CS-matrix.

The associated digraph $D(A)$ of a square real matrix $A$ of order $n$ is defined (as usual) to be a digraph with vertex set $V = \{1, \ldots, n\}$ and arc set $E = \{(i, j) \mid a_{ij} \neq 0, i \neq j\}$. 
Theorem 3.1. Let $A$ be a standard RSB matrix of order $n$, and $D(A)$ be the associated digraph of $A$. Then the following are equivalent:

(1) $A$ is a CS-matrix.
(2) $D(A)$ does not contain two vertex disjoint cycles.
(3) $A$ cannot be transformed to a matrix of the form \[
\begin{pmatrix}
A_1 & X \\
Y & A_2
\end{pmatrix},
\]
where $A_1$ and $A_2$ are both standard RSB matrices by simultaneous row and column permutations.

Proof. Since $A$ is a standard RSB matrix, the outdegree of every vertex of $D(A)$ is at least one.

(1) $\Rightarrow$ (2). We prove the contrapositive. Suppose to the contrary that $D(A)$ contains two vertex disjoint cycles, say $C_1$ and $C_2$. For each vertex $v$ of $D(A)$ on neither $C_1$ nor $C_2$ (if any), take one arc $e_v$ with $v$ as its initial vertex. Let $D_1$ be the (spanning) subdigraph of $D(A)$ consisting of these arcs together with the two cycles $C_1$ and $C_2$. Then the outdegree of every vertex of $D_1$ is one. It follows that the undirected component of $D_1$ containing $C_1$ is different from that containing $C_2$. Therefore by suitably simultaneous row and column permutations we may assume that $D_1$ is the associated digraph of the matrix $B = \begin{pmatrix}
A_1 & 0 \\
0 & A_2
\end{pmatrix}$, where $B \preceq A$ and $A_1$ and $A_2$ are both standard RSB matrices. Now it is easy to check that $B$ is still a GRSB matrix. So $B$ is not a CS-matrix by Theorem 2.B. Thus, by Theorem 2.1, $A$ is not a CS-matrix.

(2) $\Rightarrow$ (3). By contrapositive. Assume (3) does not hold. Then $A$ can be transformed to a matrix of the form \[
\begin{pmatrix}
A_1 & X \\
Y & A_2
\end{pmatrix},
\]
where $A_1$ and $A_2$ are both standard RSB matrices by simultaneous row and column permutations. Now each vertex of the digraphs $D(A_1)$ and $D(A_2)$ has outdegree at least one since $A_1$ and $A_2$ are both standard RSB matrices. So both $D(A_1)$ and $D(A_2)$ contain a directed cycle. These two cycles are obviously vertex disjoint cycles of $D(A)$, hence (2) does not hold.

(3) $\Rightarrow$ (1). By contrapositive. Assume that $A$ is not a CS-matrix. Then by Theorem 2.B $A$ is still a GRSB matrix. By suitably simultaneous row and column permutations we may assume that $A = \begin{pmatrix}
A_1 & X \\
Y & A_2
\end{pmatrix}$ where $A_1$ is a square matrix of order $k$ with $1 \leq k \leq (n - 1)$ and

\[
\begin{pmatrix}
-A_1 & X \\
-Y & A_2 \\
-e_k & e_{n-k}
\end{pmatrix}
\]
is an RSB matrix. It follows that $A_1$ and $A_2$ are both standard RSB matrices, and (3) does not hold. □

Now we consider the general situation of Case B in the following Theorem 3.2.
Let \([m] = \{1, \ldots, m\}\) and \([n] = \{1, \ldots, n\}\). Let \(A\) be an \(m \times n\) matrix. If \(S\) is a subset of \([m]\) and \(T\) is a subset of \([n]\), then \(A[S|T]\) denotes the submatrix of \(A\) whose rows have index in \(S\) and whose columns have index in \(T\). If \(T = [n]\), we abbreviate \(A[S|T]\) to \(A[S:]\). The complement of a subset \(T\) of \([n]\) is denoted by \(\bar{T}\).

**Theorem 3.2.** Let \(A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}\) be an \(m \times n\) \((0, 1, -1)\)-matrix with \(m \geq n\), where \(A_1\) is a standard RSB matrix of order \(n\), \(A_2 = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{m-n} \end{pmatrix}\) is an RSB matrix. Then \(A\) is a CS-matrix if and only if for each proper subset \(T\) of \([n]\) (with \(\emptyset \neq T \subset [n]\)), either \(A_1[T|T]\) and \(A_1[\bar{T}|\bar{T}]\) are not both RSB matrices or there exists some index \(i\) with \(1 \leq i \leq m - n\) such that both \(\alpha_i[:|T]\) and \(\alpha_i[:|\bar{T}]\) are unisigned rows.

**Proof.** By assumption \(A\) is an RSB matrix. So by Theorem 2.B, \(A\) is a CS-matrix if and only if \((A\ 0^n)\) is not a GRSB matrix, and if and only if for each proper subset \(T\) of \([n]\), the matrix \(A(T)\) (obtained from \(A\) by changing the signs of those columns with indices in \(T\)) is not an RSB matrix. But

\[
A(T) = \begin{pmatrix} A_1(T) \\ \alpha_1(T) \\ \vdots \\ \alpha_{m-n}(T) \end{pmatrix}
\]

is not an RSB matrix if and only if one of the matrices \(A_1(T), \alpha_1(T), \ldots, \alpha_{m-n}(T)\) is not an RSB matrix. Now \(A_1(T)\) is not an RSB matrix if and only if \(A_1[T|T]\) and \(A_1[\bar{T}|\bar{T}]\) are not both RSB matrix, while \(\alpha_i(T)\) is not an RSB matrix if and only if both \(\alpha_i[:|T]\) and \(\alpha_i[:|\bar{T}]\) are unisigned rows. The theorem now follows from these observations. \(\square\)

**CS-matrix recognition algorithm**

Let \(A\) be an \(m \times n\) real matrix.

**Step 0.** Let \(X = A\).

**Step 1.** Check if \(X\) is an RSB matrix. If “yes”, then go to Step 2. If “no”, then \(X\) is not a CS-matrix and the algorithm stops.

**Step 2.** Check if \(X\) contains a row with exactly two nonzero entries. If “yes”, then go to Step 3. If “no”, then go to Step 4.

**Step 3.** Choose a row \(\alpha\) of \(X\) with exactly two nonzero entries. Delete all the rows of \(X\) which majorizes over \(\alpha\) or \(-\alpha\) (except \(\alpha\) itself) to obtain a matrix \(Y\).
If the resulting matrix $Y$ contains only one row, then the algorithm stops. ($X$ is now a CS-matrix if and only if $Y$ contains only two columns.)

Otherwise, take $B$ to be the conformal contraction matrix of $Y$ on the row $\alpha$. Then replace $X$ by $B$ and go back to Step 2.

**Step 4.** For each $j = 1, \ldots, n$, see if the submatrix $X_j$ (obtained from $X$ by deleting its $j$th column) contains an unsigned row.

If “no”, then $X$ is not a CS-matrix and the algorithm stops. If “yes”, then go to Step 5.

**Step 5.** Suitably permute and strictly sign the rows of $X$ to transform $X$ to a matrix of the form

\[
\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix},
\]

where $Y_1$ is a standard RSB matrix. Then use Theorem 3.2 to determine whether or not $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ is a CS-matrix.

In the special case where $A$ is a square real matrix, the above algorithm is reduced to an algorithm for recognizing square CS-matrices which is simpler than the original algorithm in the following two aspects.

1. If the matrix $Y$ in Step 3 contains fewer rows than $X$, then it suffices to use the $S$-matrix recognition algorithm in [3] to determine whether or not $Y$ is an $S$-matrix (and we need not go to Steps 4 and 5 in this case).

2. If (each time) the matrix $Y$ in Step 3 is equal to $X$, then the matrix $X$ in Step 5 is a square matrix (i.e., the matrix $Y_2$ in Step 5 is vacuous). Thus we can use the graph theoretical condition (2) of Theorem 3.1 (instead of using Theorem 3.2) in Step 5 to determine whether or not the standard RSB matrix $Y_1$ is a CS-matrix.

### 4. Number of nonzero entries of square CS-matrices

In this section we use graph theoretical methods to obtain sharp lower and upper bounds for the number of nonzero entries of square CS-matrices of order $n$, together with the complete characterizations of the CS-matrices which attain these bounds.

The following Lemma 4.1 is a purely graph theoretical result which will be used in the proof of the main result (Theorem 4.1) of this section.

**Lemma 4.1.** Let $D$ be a digraph of order $n \geq 4$ without loops and multiple arcs, $E(D)$ be the arc set of $D$. Suppose $D$ does not contain two vertex disjoint cycles of length 2, then we have

\[
|E(D)| \leq \frac{1}{2}(n - 1)(n + 2).
\]  

**Proof.** If $D$ contains no cycle of length 2, then clearly

\[
|E(D)| \leq \frac{1}{2}n(n - 1) < \frac{1}{2}(n - 1)(n + 2).
\]
Otherwise, let $C$ be a cycle of length 2 of $D$ with two vertices $u$ and $v$. Let $V_1 = V(D) \setminus \{u, v\}$ and $D_1$ be the subdigraph of $D$ induced by the vertex subset $V_1$. Let $E_1 = E(D_1)$ and $E_2$ be the arc subset of $D$ consisting of those arcs with one end vertex in $\{u, v\}$ and another end vertex in $V_1$. Then we have

$$E(D) = E_1 \cup E_2 \cup \{(u, v), (v, u)\}$$

and thus

$$|E(D)| = |E_1| + |E_2| + 2. \quad (4.2)$$

Now $D$ does not contain two vertex disjoint cycles of length 2, so $D_1$ contains no cycle of length 2. Thus we have

$$|E_1| = |E(D_1)| \leq \frac{1}{2}(n-2)(n-3). \quad (4.3)$$

Next we estimate $|E_2|$. If for each vertex $x$ in $V_1$, there are at most three arcs in $D$ with one end vertex $x$ and another end vertex in $\{u, v\}$, then we have

$$|E_2| \leq 3(n-2). \quad (4.4)$$

Otherwise there exists a vertex $x_0$ in $V_1$ such that $(x_0, u)$, $(x_0, v)$, $(u, x_0)$ and $(v, x_0)$ are four arcs in $D$. Then for each vertex $y$ in $V_1$ different from $x_0$, there are at most two arcs in $D$ with one end vertex $y$ and another end vertex in $\{u, v\}$ since $D$ does not contain two vertex disjoint cycles of length 2. So in this case we have

$$|E_2| \leq 4 + 2(n-3) \leq 3(n-2). \quad (4.5)$$

Combining (4.2)–(4.5) we have

$$|E(D)| \leq \frac{1}{2}(n-2)(n-3) + 3(n-2) + 2 = \frac{1}{2}(n-1)(n+2).$$

This completes the proof of the lemma. \qed

Let

$$X = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}. \quad (4.6)$$

Then clearly $X$ is a CS-matrix and a standard RSB matrix of order 3. By Lemma 3.1, every $(1, -1)$ CS-matrix of order 3 is row equivalent to this matrix $X$.

Let $u$ be a column of a matrix $A$. A matrix $B$ is said to be obtained from $A$ by conformally copying the column $u$, if $B$ is obtained from $A$ by inserting a new column $v$ equal to $u$, and then inserting a new row with a 1 and a $-1$ in the columns corresponding to $u$ and $v$, and with 0s elsewhere.

Let $N(A)$ be the number of nonzero entries of $A$. It is shown in [3] that if $A$ is an $n \times (n+1)$ $S$-matrix, then

$$2n \leq N(A) \leq \frac{1}{2}n(n+3). \quad (4.7)$$

Now we will generalize this result to square CS-matrices of order $n$ by showing that (4.7) also gives sharp lower and upper bounds for the number of nonzero entries of square CS-matrices of order $n$. 
Theorem 4.1. Let $A$ be a square $(0, 1, -1)$ CS-matrix of order $n$. Then we have:

1. $2n \leq N(A) \leq \frac{1}{2}n(n+3)$.
2. $N(A) = 2n$ if and only if $A$ is row equivalent to a standard RSB matrix $B$ whose associated (undirected) graph $G(B)$ (which is obtained from the associated digraph $D(B)$ by ignoring the directions of all the arcs of $D(B)$) is a connected graph containing a single cycle.
3. $N(A) = \frac{1}{2}n(n+3)$ if and only if $n \geq 3$ and $A$ is permutation equivalent to a matrix which can be obtained from a $(0, 1, -1)$ CS-matrix of order 3 with no zero entries by a series of conformally copying a column with no zero entries.
4. For each integer $k$ with $2n \leq k \leq \frac{1}{2}n(n+3)$, there exists a square CS-matrix $A$ of order $n$ such that $N(A) = k$.

Proof. (1) $N(A) \geq 2n$ since each row of $A$ (as an RSB matrix) contains at least two nonzero entries. We now use induction on $n$ to prove the upper bound. The result obviously holds for $n \leq 3$, so we assume $n \geq 4$.

Case 1. If each row of $A$ contains at least three nonzero entries.
Then by Lemma 3.1 we may assume $A$ is a standard RSB matrix. By Theorem 3.1 and Lemma 4.1 we have
\[ N(A) = n + |E(D(A))| \leq n + \frac{1}{2}(n-1)(n+2) < \frac{1}{2}n(n+3). \]

Case 2. If some row of $A$ contains only two nonzero entries. Say
\[ A = \begin{pmatrix} 1 & -1 & 0 \\ u & v & A_1 \end{pmatrix} \] (4.8)
where $u = (u_1, \ldots, u_{n-1})^T$ and $v = (v_1, \ldots, v_{n-1})^T$.

Subcase 2.1. If $u$ and $v$ are conformal.
Then by Theorem 2.3 the matrix $B = (u + v, A_1)$ is a square CS-matrix of order $n-1$. By induction we have $N(B) \leq \frac{1}{2}(n-1)(n+2)$ and so
\[ N(A) \leq n + 1 + N(B) \leq \frac{1}{2}n(n+3). \]

Subcase 2.2. If $u$ and $v$ are not conformal, say $u_{n-1} - v_{n-1} < 0$.
Then by Theorem 2.2, the matrix $C$ obtained from $A$ by deleting its last row is an $(n-1) \times n$ CS-matrix, thus an $S$-matrix. By (4.7) we have $N(C) \leq \frac{1}{2}(n-1)(n+2)$, so $N(A) \leq N(C) + n < \frac{1}{2}n(n+3)$.

(2) Sufficiency. If $B$ is a standard RSB matrix such that $G(B)$ is a connected graph with a single cycle, then $B$ is a CS-matrix by Theorem 3.1, so $A$ is a CS-matrix. Also $|E(G(B))| = n$ and all the diagonal entries of $B$ are negative, so
\[ N(A) = N(B) = n + |E(G(B))| = 2n. \]

Necessity. Since $A$ is a CS-matrix, $A$ is a nearly $L$-matrix by definition. So by the well-known König’s Theorem, $A$ has full term rank $\rho(A) = n$. Therefore $A$ is row
equivalent to a matrix $B$ with all diagonal entries negative. Now $B$ is an RSB matrix with $N(A) = N(B) = 2n$, so each row of $B$ contains exactly two nonzero entries and thus each row of $B$ contains exactly one nonzero off-diagonal entry which is positive. It follows that $B$ is a standard RSB matrix of order $n$. Also, the outdegree of every vertex of the associated digraph $D(B)$ is one. So every undirected cycle of $G(B)$ is a directed cycle of $D(B)$, and each pair of distinct cycles of $D(B)$ are vertex disjoint.

Now $B$ is a CS-matrix since $A$ is, so by Theorem 3.1 and the above arguments $G(B)$ contains only one cycle. Also, $N(A) = N(B) = 2n$ imply that $|E(G(B))| = n$ and thus $G(B)$ is connected since $G(B)$ contains only one cycle.

(3) Sufficiency. By Theorem 2.3 $A$ is a CS-matrix. If $n = 3$, then $N(A) = 9 = \frac{1}{2}n(n + 3)$. If $n \geq 4$, then by induction $N(A) = \frac{1}{2}(n - 1)(n + 2) + n + 1 = \frac{1}{2}n(n + 3)$.

Necessity. If each row of $A$ contains at least three nonzero entries, then by Lemma 3.1 we may assume that $A$ is a standard RSB matrix. By Theorem 3.1 and Lemma 4.1 we have that if $n \geq 4$, then $N(A) \leq \frac{1}{2}(n - 1)(n + 2) < \frac{1}{2}n(n + 3)$ contradicting the hypothesis. So if $n \geq 4$, then $A$ must contains some row with exactly two nonzero entries, and we can assume that $A$ has the form as (4.8).

If $u$ and $v$ in (4.8) are not conformal, then by the same proof as the subcase 2.2 of (1) of this theorem, we would have $N(A) < \frac{1}{2}n(n + 3)$, a contradiction. So $u$ and $v$ in (4.8) are conformal. By Theorem 2.3 the matrix $B = (u + v, A_1)$ is a square CS-matrix of order $n - 1$, so $N(B) \leq \frac{1}{2}(n - 1)(n + 2)$ by (1). But

$$N(B) \geq N(A) - (n + 1) = \frac{1}{2}n(n + 3) - (n + 1) = \frac{1}{2}(n - 1)(n + 2).$$

So $N(B) = \frac{1}{2}(n - 1)(n + 2)$. By induction sgn $B$ can be obtained from a $(0, 1, -1)$ CS-matrix of order 3 with no zero entries by a series of conformally copying a column with no zero entries. Also $N(A) = N(B) + n + 1$, so both $u$ and $v$ in (4.8) contains no zero entries and thus $A$ is obtained from sgn $B$ by conformally copying the column sgn$(u + v)$. This proves the necessity part of (3).

(4) Take a square CS-matrix $B$ of order $n$ with $N(B) = \frac{1}{2}n(n + 3)$. Take an RSB matrix $C$ of order $n$ such that $C \preceq B$ and each row of $C$ contains exactly one positive entry and exactly one negative entry. Then $N(C) = 2n$.

Now for each integer $k$ with $2n \leq k \leq \frac{1}{2}n(n + 3)$, we can take a matrix $A$ of order $n$ such that $C \preceq A \preceq B$ and $N(A) = \tilde{k}$. Then $A$ is an RSB matrix since $C$ is. So Theorem 2.1 implies that $A$ is a CS-matrix. □

5. BCS-matrices and maximal CS-matrices

In this section, we study two special classes of CS-matrices: BCS-matrices (barely CS-matrices) and Maximal CS-matrices. BCS-matrices were first introduced and
studied in [2], while maximal CS-matrices are the generalizations of maximal S-matrices. For BCS-matrices, we give sharp lower and upper bounds for the number of rows \( m \) and the number of nonzero entries \( N(A) \) in terms of the number of columns \( n \) of \( A \). For maximal CS-matrices, we obtain several necessary conditions and several sufficient conditions for a matrix to be a maximal CS-matrix.

**Definition 5.1** [2]. An \( m \times n \) real matrix \( A \) is called a BCS-matrix (or BCS*-matrix), if \( A \) is a CS-matrix (or CS*-matrix), and each matrix obtained from \( A \) by deleting a row is not a CS-matrix (or CS*-matrix).

It is obvious from the above definition that each \( S \)-matrix (or \( S^* \)-matrix) is a BCS-matrix (or BCS*-matrix).

Let

\[ \begin{align*}
X &= \{ T \subseteq [n] | T \neq \emptyset, 1 \notin T \}. \tag{5.1}
\end{align*} \]

Then it is easy to see that \(|X| = 2^{n-1} - 1\).

For each index set \( T \subseteq [n] \) and each matrix \( B \) with \( n \) columns, let \( B(T) \) be the matrix obtained from \( B \) by multiplying all the columns with indices in \( T \) by \(-1\).

Thus, if \( e_n \) is the row vector of dimension \( n \) with all coordinates \( 1 \), then \( e_n(T) \) is the \((1, -1)\) row vector of dimension \( n \) whose \( i \)th coordinate equals to \(-1\) if and only if \( i \in T \).

**Theorem 5.1.** Let \( A \) be an \( m \times n \) \((0, 1, -1)\) BCS-matrix. Then we have:

1. \( n - 1 \leq m \leq 2^{n-1} - 1 \). \tag{5.2}
2. \( m = n - 1 \) if and only if \( A \) is an \( S \)-matrix.
3. \( m = 2^{n-1} - 1 \) if and only if \( A \) is equivalent to the matrix

\[
\begin{pmatrix}
e_n(T_1) \\
\vdots \\
e_n(T_m)
\end{pmatrix},
\]

where \( m = 2^{n-1} - 1 \) and \( T_1, \ldots, T_m \) are all the elements of the set \( X \) defined in (5.1).

**Proof.** (1) We have already mentioned after Definition 2.1 that \( n \leq m + 1 \). Now we prove \( m \leq 2^{n-1} - 1 \).

Let \( \alpha_i \) be the \( i \)th row of \( A \) and let \( A_i \) be the matrix obtained from \( A \) by deleting its \( i \)th row \((i = 1, \ldots, m)\). Since \( A \) is a BCS matrix, \( A_i \) is not a CS-matrix for each \( i \in [m] \). So \( \begin{pmatrix} A_i \\ e_n \end{pmatrix} \) is a GRSB matrix. Therefore there exists some index set \( T_i \in X \) such that the matrix \( \begin{pmatrix} A_i(T_i) \\ e_n(T_i) \end{pmatrix} \) is an RSB matrix. On the other hand, \( \begin{pmatrix} A(T_i) \\ e_n(T_i) \end{pmatrix} \) is not an RSB matrix since \( A \) is a CS-matrix and \( \begin{pmatrix} A \\ e_n \end{pmatrix} \) is not a GRSB matrix. So \( \alpha_i(T_i) \) is not
an RSB matrix, but for each \( j \neq i \), \( \alpha_j(T_i) \) is an RSB matrix. It follows that the sets \( T_1, T_2, \ldots, T_m \) in \( X \) are distinct and thus we have \( m \leq |X| = 2^{n-1} - 1 \).

(2) We have mentioned in Section 2 that \( n = m + 1 \) if and only if \( A \) is an S-matrix.

(3) The sufficiency part of the equality \( m = 2^{n-1} - 1 \) is obvious, so we only need to prove the necessity part of the equality \( m = 2^{n-1} - 1 \). Now suppose that \( m = 2^{n-1} - 1 \) and

\[
X = \{ T_1, \ldots, T_m \}
\]

such that \( \alpha_i(T_j) \) is an RSB matrix if and only if \( i \neq j \) (by the above arguments). Then \( A \) must contain no zero entries. For otherwise if some \( a_{ij} = 0 \), then we can find \( T_{k_1} \) and \( T_{k_2} \) in \( X \) with \( k_1 \neq k_2 \) such that both \( \alpha_i(T_{k_1}) \) and \( \alpha_i(T_{k_2}) \) are not RSB matrices, a contradiction. It follows from this that \( \alpha_i = e_n(T_i) \) or \( \alpha_i = -e_n(T_i) \), and so the result is proved. \( \square \)

**Theorem 5.2.** Let \( A \) be an \( m \times n \) (0, 1, -1) BCS-matrix, \( N(A) \) be the number of nonzero entries of \( A \). Then we have:

1. \( 2(n - 1) \leq N(A) \leq n(2^{n-1} - 1) \).
2. \( N(A) = 2(n - 1) \) if and only if \( A \) is an oriented edge-vertex incidence matrix of a tree.
3. \( N(A) = n(2^{n-1} - 1) \) if and only if \( A \) is a matrix satisfying the equality case \( m = 2^{n-1} - 1 \) in Theorem 5.1.

**Proof.** (1) Clearly we have \( 2m \leq N(A) \leq mn \). So (5.4) follows from (5.2).

(2) \( N(A) = 2(n - 1) \) if and only if \( m = (n - 1) \) and \( N(A) = 2m \) if and only if \( A \) is an S-matrix with \( N(A) = 2m \). Thus the left hand equality case of (5.4) follows from [3, Theorem 4.4.1].

(3) \( N(A) = n(2^{n-1} - 1) \) if and only if \( m = 2^{n-1} - 1 \) and \( A \) contains no zero entries. Thus the right hand equality case of (5.4) follows from Theorem 5.1. \( \square \)

Theorem 5.2 gives sharp lower and upper bounds for the number of nonzero entries \( N(A) \) among all the BCS-matrices with \( n \) columns (where the number of rows \( m \) is not fixed). We do not know the sharp lower and upper bounds of \( N(A) \) in terms of the two parameters \( m \) and \( n \). So we propose the following problem:

**Problem.** Let \( b(m, n) \) be the largest number of nonzero entries among all the \( m \times n \) BCS-matrices. Determine \( b(m, n) \).

We know some special values of \( b(m, n) \). For example, if \( m = n - 1 \), then \( b(m, n) = \frac{1}{2}(n - 1)(n + 2) \) by [3, Theorem 4.4.1]. If \( m = 2^{n-1} - 1 \), then \( b(m, n) = mn \) by Theorem 5.2. If \( m = n \), then \( b(m, n) = \frac{1}{2}n(n + 3) \) by Theorem 4.1 (notice that if \( A \) is a CS-matrix of order \( n \) with \( N(A) = \frac{1}{2}n(n + 3) \), then \( A \) must be a BCS-matrix, for otherwise we would have \( N(A) \leq \frac{1}{2}(n - 1)(n + 2) + n < \frac{1}{2}n(n + 3) \).
Let $c(m, n)$ be the largest number of nonzero entries among all the $m \times n$ CS-matrices. Then we know that $c(n -1, n) = \frac{1}{2}(n -1)(n + 2)$ by [3, Theorem 4.4.1], $c(n, n) = \frac{1}{2}n(n + 3)$ by Theorem 4.1, and $c(m, n) = mn$ for all $m \geq 2^{n-1} -1$ (we can take the $(2^{n-1} -1) \times n$ BCS-matrix $B$ with no zero entries as in Theorem 5.1, then add several RSB rows with no zero entries to $B$ to obtain an $m \times n$ CS-matrix with no zero entries). Also we have the following relation between $b(m, n)$ and $c(m, n)$:

$$
c(m, n) = mn - \min_{n-1<k<m} \{ kn - b(k, n) \}. \tag{5.5}
$$

To see this, we notice that each $m \times n$ CS-matrix contains a $k \times n$ BCS-matrix (for some $n-1 \leq k \leq m$) as its submatrix, and each $k \times n$ BCS-matrix $(n-1 \leq k \leq m)$ can be extended to an $m \times n$ CS-matrix by adding $m-k$ arbitrary RSB rows (by Theorem 2.B).

From (5.5) we see that the determination of $c(m, n)$ depends on the determination of $b(k, n)$ for $n-1 \leq k \leq m$.

Now we study maximal CS-matrices.

**Definition 5.2.** A real matrix $A$ is called a maximal CS-matrix, if $A$ satisfies the following two conditions:

1. $A$ is a CS-matrix.
2. Each matrix $B$ with $A \preceq B$ and $\text{sgn} B \neq \text{sgn} A$ is not a CS-matrix.

From Theorem 2.1 we can see that the condition (2) can be replaced by the following condition (2'):

(2') Each matrix obtained from $A$ by replacing a zero entry by a nonzero number is not a CS-matrix.

The following Theorem 5.3 gives several necessary conditions for a matrix $A$ in binormal form (2.1) to be a maximal CS-matrix.

**Theorem 5.3.** Let $A$ be a $(0, 1, -1)$ maximal CS-matrix in binormal form (2.1). Then:

1. $A_1$ contains no zero entries.
2. $u_2 = v_2$.
3. $(u_2, A_2)$ is a maximal CS-matrix.

**Proof.** (1) Suppose $A_1$ contains a zero entry. Let $A'$ be the matrix obtained from $A$ by replacing this zero entry by 1. Let $\alpha$ and $\alpha'$ be the row of $A$ and $A'$ containing this entry. Let $B$ be the submatrix of $A$ obtained from $A$ by deleting the row $\alpha$. 

By the hypothesis on \( u_1 \) and \( v_1 \) we know that both \( \alpha \) (or \( -\alpha \)) and \( \alpha' \) (or \( -\alpha' \)) majorize the first row of \( A \). So by using Theorem 2.2 twice we have

\( A \) is a CS-matrix \( \Rightarrow B \) is a CS-matrix \( \Rightarrow A' \) is a CS-matrix.

Contradicting the hypothesis that \( A \) is a maximal CS-matrix.

(2) Let \( u_2 = (a_1, \ldots, a_k)^T \) and \( v_2 = (b_1, \ldots, b_k)^T \). Suppose \( u_2 \neq v_2 \), then we may assume that \( a_1 = 0 \) and \( b_1 \neq 0 \) (since \( u_2 \) and \( v_2 \) are conformal). Now let \( u'_2 = (b_1, a_2, \ldots, a_k)^T \) and

\[
A' = \begin{pmatrix}
1 & -1 & 0 \\
\ u_1 & v_1 & A_1 \\
\ u'_2 & v_2 & A'_2
\end{pmatrix},
\]

let \( B = (u_2 + v_2, A_2) \) and \( B' = (u'_2 + v_2, A_2) \), then \( \text{sgn } B = \text{sgn } B' \) and \( B \) and \( B' \) are generalized conformal contractions of \( A \) and \( A' \), respectively. Thus by Theorem 2.3 we have:

\( A \) is a CS-matrix \( \Rightarrow B \) is a CS-matrix \( \Rightarrow B' \) is a CS-matrix \( \Rightarrow A' \) is a CS-matrix.

Contradicting the hypothesis that \( A \) is a maximal CS-matrix.

(3) By (2) we have \( \text{sgn}(u_2, A_2) = \text{sgn } B \), where \( B = (u_2 + v_2, A_2) \) is a generalized conformal contraction of \( A \). So \( (u_2, A_2) \) is a CS-matrix by Theorem 2.3.

Now let \( (u'_2, A'_2) \) be an arbitrary matrix obtained from \( (u_2, A_2) \) by replacing a zero entry by 1 or \( -1 \), let

\[
A' = \begin{pmatrix}
1 & -1 & 0 \\
\ u_1 & v_1 & A_1 \\
\ u'_2 & v_2 & A'_2
\end{pmatrix}.
\]

Then \( A' \) is not a CS-matrix since \( A \) is a maximal CS-matrix. Thus \( (u'_2 + v_2, A'_2) \) is not a CS-matrix by Theorem 2.3. But \( \text{sgn}(u'_2 + v_2, A'_2) = \text{sgn}(u'_2, A'_2) \). So \( (u'_2, A'_2) \) is not a CS-matrix. It follows that \( (u_2, A_2) \) is a maximal CS-matrix.

This completes the proof of the theorem. \( \square \)

Notice that the necessary conditions in Theorem 5.3 are not sufficient. For example, take

\[
A = \begin{pmatrix}
-1 & 1 & 0 \\
1 & -1 & 1 \\
1 & 1 & -1
\end{pmatrix} =: \begin{pmatrix}
-1 & 1 & 0 \\
\ u_1 & v_1 & A_1 \\
\ u_2 & v_2 & A_2
\end{pmatrix}.
\]

Then \( A \) is in the binormal form (2.1) and \( A \) satisfies all the necessary conditions in Theorem 5.3. But \( A \) is not a maximal CS-matrix.

The following Theorems 5.4 and 5.5 show that in some special cases, the necessary conditions for maximal CS-matrices given in Theorem 5.3 are also sufficient.

**Theorem 5.4.** Let \( A \) be a \((0, 1, -1)\) CS-matrix containing at least two rows having exactly two nonzero entries. Then \( A \) is a maximal CS-matrix if and only if every binormal form of \( A \) satisfies the three necessary conditions in Theorem 5.3.
Proof. The necessity part is just Theorem 5.3. We now prove the sufficiency part. Suppose to the contrary that $A$ is not a maximal CS-matrix. Let $A'$ be a CS-matrix obtained from $A$ by replacing a zero entry by a nonzero number $b$. Then $A'$ contains at least one row having exactly two nonzero entries, and we may assume that $A'$ is in the following binormal form:

$$A' = \begin{pmatrix} 1 & -1 & 0 \\ u'_1 & v'_1 & A'_1 \\ u'_2 & v'_2 & A'_2 \end{pmatrix},$$

where $u'_1, v'_1$ contains no zero entries with $u'_1 = -v'_1$, and $u'_2$ and $v'_2$ are conformal.

Now the nonzero entry $b$ of $A'$ must be in the $(u'_2, v'_2, A'_2)$ part. For otherwise it will be in $(u'_1, v'_1, A'_1)$ part, and then

$$A = \begin{pmatrix} 1 & -1 & 0 \\ u_1 & v_1 & A_1 \\ u'_2 & v'_2 & A'_2 \end{pmatrix},$$

where $(u_1, v_1, A_1)$ contains a zero entry. If the zero entry is in the $(u_1, v_1)$ part, then $(u_1, v_1)$ contains at least two zero entries by the necessary condition (2) of Theorem 5.3. Contradicting the hypothesis that $u'_1, v'_1$ contains no zero entries. So the zero entry is in the $A_1$ part and $u_1 = u'_1, v_1 = v'_1$, contradicting the necessary condition (1) of Theorem 5.3. Therefore we have

$$A = \begin{pmatrix} 1 & -1 & 0 \\ u_1 & v_1 & A_1 \\ u_2 & v_2 & A_2 \end{pmatrix},$$

where $u_2 = v_2$, $(u_2, A_2)$ is a maximal CS-matrix by hypothesis. Let $C = (u'_2 + v'_2, A'_2)$, then $C$ is also a CS-matrix since it is a generalized conformal contraction of the CS-matrix $A'$. On the other hand, $(u'_2, v'_2, A'_2)$ is obtained from $(u_2, v_2, A_2)$ by replacing a zero entry by a nonzero number $b$, and $u_2 = v_2$, so

$$\text{sgn}(u_2, A_2) = \text{sgn}(u_2 + v_2, A_2) \preceq (u'_2 + v'_2, A'_2) = C$$

but $\text{sgn}(u_2, A_2) \neq \text{sgn}C$. Thus $(u_2, A_2)$ is not a maximal CS-matrix, a contradiction. 

\[\square\]

**Theorem 5.5.** Let $A$ be a $(0, 1, -1)$ RSB matrix in binormal form (2.1). Suppose $A_1$ is an RSB matrix. Then $A$ is a maximal CS-matrix if and only if $A$ satisfies the three necessary conditions in Theorem 5.3.

**Proof.** By Theorem 5.3, we only need to prove the sufficiency part. Firstly, $A$ is a CS-matrix since $(u_2 + v_2, A_2)$ is a CS-matrix which is a generalized conformal contraction of $A$ (note that $\text{sgn}(u_2 + v_2) = \text{sgn}u_2$). Now let $A'$ be a matrix obtained from $A$ by replacing a zero entry by a nonzero number $b$. Then $b$ is not in
the \((u_1, v_1, A_1)\) part by hypothesis. We now prove that \(A'\) is not a CS-matrix by considering the following two cases:

**Case 1.** \(b\) is in the position of part \((u_2, v_2, A_2)\).

Then
\[
A' = \begin{pmatrix}
1 & -1 & 0 \\
u_1 & v_1 & A_1 \\
u'_2 & v'_2 & A'_2
\end{pmatrix}
\]
and we have
\[
\text{sgn}(u_2, A_2) = \text{sgn}(u_2 + v_2, A_2) \leq \text{sgn}(u'_2 + v'_2, A'_2)
\]
but \(\text{sgn}(u_2, A_2) \neq \text{sgn}(u'_2 + v'_2, A'_2)\). So \((u'_2 + v'_2, A'_2)\) is not a CS-matrix since \((u_2, A_2)\) is a maximal CS-matrix. But \(u'_2\) and \(v'_2\) must still be conformal, so \(A'\) is not a CS-matrix by Theorem 2.3.

**Case 2.** \(b\) is in the position of part \((1, -1, 0)\).

We may assume \(b > 0\) and
\[
A' = \begin{pmatrix}
1 & -1 & b & 0 & \cdots & 0 \\
u_1 & v_1 & A_1 \\
u_2 & v_2 & A_2
\end{pmatrix}.
\]
Since \(u_2 = v_2\) and \(A_1\) is an RSB matrix, the matrix obtained from \(A'\) by deleting its first column is still an RSB matrix. So \(A'\) is not a nearly \(L\)-matrix and thus is not a CS-matrix by definition.

Combining Cases 1 and 2, we conclude that \(A'\) is not a CS-matrix, so \(A\) is a maximal CS-matrix. \(\square\)

We notice that if in Theorem 5.5 \(A\) is a BCS-matrix, then \((u_1, v_1, A_1)\) is vacuous by Theorem 2.2. So in this case the result of Theorem 5.5 actually asserts that \(A\) is a maximal CS-matrix if and only if \(u_2 = v_2\) and \((u_2, A_2)\) is a maximal CS-matrix. This result is a generalization of the corresponding result for maximal \(S\)-matrices in [3, Theorem 4.4.3].

**References**


