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# Existence and Uniqueness for a Nonlinear Fractional Differential Equation

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We prove existence and uniqueness theorems for some classes of nonlinear fractional differential equations. © 1996 Academic Press, Inc.

## 1. INTRODUCTION

Many papers and books on fractional calculus have appeared recently. Most of them are devoted to the solvability of linear fractional equations in terms of special functions (see for example Miller and Ross [7] and Campos [1]) and to problems of analyticity in the complex domain (see for example Ling and Ding [6]). No contribution exists, as far as we know, concerning nonlinear fractional equations of the form

$$D^s u = f(x, u)$$

where $0 < s < 1$ and $D^s$ is the standard Riemann–Liouville fractional derivative, considered in $\mathbb{R}^+$ or in an interval $(0, h)$ with $h > 0$.

In principle, one may reduce such an equation to an integral equation with weak singularity and apply to it basic techniques of nonlinear analysis (fixed points theorems, Leray–Schauder theory). In practice, to obtain

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explicit results, one has to take into account the peculiarity of the kernel. We proceed in this direction here, presenting a list of theorems of existence and uniqueness.

We shall not give applications in this paper, but we remark that the previous fractional differential equation is easily shown to be equivalent to a nonlinear heat conduction problem by using the Heaviside calculus (cf. Courant and Hilbert [2, Appendix to Chapter V]); this allows an extension to a nonvariational setting of some results of [4]. We finally mention that similar results, for a different fractional differential equation, have been obtained by the first author [3].

The paper is organized as follows. In Section 2 we recall the definitions of fractional integral and derivative and related basic properties used in the text. Section 3 contains results for solutions which are continuous at the origin. Section 4 concerns initial value problems of the type

$$\begin{cases} D^s u = f(x,u) \\ u(a) = b \end{cases}$$

with $a \in \mathbb{R}^+$, $b \in \mathbb{R}$.

## 2. DEFINITIONS AND PRELIMINARY RESULTS

Let us denote by $C^0(\mathbb{R}^+)$ the space of all continuous real functions defined on $\mathbb{R}^+ = \{x \in \mathbb{R},\ x > 0\}$ and by $L^1_{\text{loc}}(\mathbb{R}^+)$ the space of all real functions defined on $\mathbb{R}^+$ which are Lebesgue integrable on every bounded subinterval of $\mathbb{R}^+$. Consider also the space $C^0(\mathbb{R}^+_0)$ of all continuous real functions on $\mathbb{R}^+_0 = \{x \in \mathbb{R},\ x \geq 0\}$, which later on we shall identify, by abuse of notation, with the class of all $f \in C^0(\mathbb{R}^+)$ such that $f(0+) = \lim_{x \to 0+} f(x) \in \mathbb{R}$.

The definitions and the results of the fractional calculus reported below are not exhaustive but rather oriented to the subject of this paper. For the proofs, which are omitted, we refer the reader to Miller and Ross [7] or other texts on basic fractional calculus.

DEFINITION 2.1. The fractional primitive of order $s > 0$ of a function $f\colon \mathbb{R}^+ \to \mathbb{R}$ is given by

$$I^s f(x) = \frac{1}{\Gamma(s)} \int_0^x (x-t)^{s-1} f(t)\, dt$$

provided the right side is pointwise defined on $\mathbb{R}^+$.

For instance, $I^s f$ exists for all $s > 0$, when $f \in C^0(\mathbb{R}^+) \cap L^1_{loc}(\mathbb{R}^+)$; note also that when $f \in C^0(\mathbb{R}_0^+)$ then $I^s f \in C^0(\mathbb{R}_0^+)$ and moreover $I^s f(0) = 0$.

EXAMPLE 2.1.1.

$$I^s x^{\mu} = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + s + 1)} x^{\mu + s}, \qquad s > 0, \mu > -1.$$

Recall that the law of composition

$$I^r I^s = I^{r+s}$$

holds for all $r, s > 0$.

DEFINITION 2.2. The fractional derivative of order $0 < s < 1$ of a continuous function $f: \mathbb{R}^+ \to \mathbb{R}$ is given by

$$D^s f(x) = \frac{1}{\Gamma(1-s)} \frac{d}{dx} \int_0^x (x-t)^{-s} f(t)\, dt$$

provided that the right side is pointwise defined on $\mathbb{R}^+$.

EXAMPLE 2.2.1. As a basic example, we quote for $\lambda > -1$

$$D^s x^{\lambda} = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda - s + 1)} x^{\lambda - s},$$

giving in particular $D^s x^{s-1} = 0$.

We have $D^s I^s f = f$ for all $f \in C^0(\mathbb{R}^+) \cap L^1_{loc}(\mathbb{R})$. From definition 2.2 and Example 2.2.1 we then obtain:

LEMMA 2.3. *Let* $0 < s < 1$. *If we assume* $u \in C^0(\mathbb{R}^+) \cap L^1_{loc}(\mathbb{R}^+)$, *then the fractional differential equation*

$$D^s u = 0$$

*has* $u = cx^{s-1}$, $c \in \mathbb{R}$, *as unique solutions*.

From this lemma we deduce the following law of composition.

PROPOSITION 2.4. *Assume that f is in* $C^0(\mathbb{R}^+) \cap L^1_{loc}(\mathbb{R}^+)$ *with a fractional derivative of order* $0 < s < 1$ *that belongs to* $C^0(\mathbb{R}^+) \cap L^1_{loc}(\mathbb{R}^+)$. *Then*

$$I^s D^s f(x) = f(x) + cx^{s-1}$$

*for some* $c \in \mathbb{R}$.

*When the function f is in* $C^0(\mathbb{R}_0^+)$, *then* $c = 0$.

In all the definitions and results of this section the set $\mathbb{R}^+$ can be substituted by the intervals $(0, a)$ or $(0, a]$, $a > 0$. For simplicity, in the next sections we shall often limit arguments to the choice $a = 1$.

A more precise analysis of the operators $I^s, D^s$ can be given in the frame of the spaces $C_r^0(\mathbb{R}_0^+)$, $r \geq 0$, of all functions $f \in C^0(\mathbb{R}^+)$ such that $x^r f \in C^0(R_0^+)$. We define similarly $C_r^0([0, a])$, which turns out to be a Banach space when endowed with the norm

$$\|f\|_r = \max_{x \in [0, a]} x^r |f(x)|.$$

We have $C_0^0(\mathbb{R}_0^+) = C^0(\mathbb{R}_0^+)$ and $C_0^0([0, a]) = C^0([0, a])$, the Banach space of all continuous functions on $[0, a]$ with norm

$$\|f\| = \max_{x \in [0, a]} |f(x)|.$$

Obviously $C_r^0(\mathbb{R}_0^+) \subset L^1_{\mathrm{loc}}(\mathbb{R}^+)$ if $r < 1$.

Let $0 < s < 1$; if $f \in C_r^0(\mathbb{R}_0^+)$ with $r < s$, then $I^s f \in C^0(\mathbb{R}_0^+)$, with $I^s f(0) = 0$. If $f \in C_s^0(\mathbb{R}_0^+)$ then $I^s f$ is bounded at the origin, whereas if $f \in C_r^0(\mathbb{R}_0^+)$ with $s < r < 1$, then we may expect $I^s f$ to be unbounded at the origin. Concerning Proposition 2.4, the last part can now be stated more precisely:

If $f \in C_r^0(\mathbb{R}_0^+)$ with $r < 1 - s$ and $D^s f \in C^0(\mathbb{R}^+) \cap L^1_{\mathrm{loc}}(\mathbb{R}^+)$, then $I^s D^s f = f$.

## 3. DIFFERENTIAL EQUATIONS OF REAL ORDER: CONTINUOUS SOLUTIONS ON $[0, 1]$

Consider the fractional differential equation

$$D^s u = f(x, u) \tag{3.1}$$

where $0 < s < 1$ and $f: [0, a] \times \mathbb{R} \to \mathbb{R}$, $0 < a \leq +\infty$, is a given function, continuous in $(0, a) \times \mathbb{R}$.

We introduce the following definition of a solution for Eq. (3.1).

DEFINITION 3.1. A real valued function $u \in C^0((0, a)) \cap L^1((0, a))$, or $u \in C^0(\mathbb{R}^+) \cap L^1_{\mathrm{loc}}(\mathbb{R}^+)$ in the case $a = +\infty$, with continuous fractional derivative $D^s u$ on $(0, a)$, is a solution of fractional differential equation (3.1) if

$$D^s u(x) = f(x, u(x))$$

for all $x \in (0, a)$.

*Remark* 3.2. We may apply the results of Section 2, in particular Proposition 2.4 and the subsequent remarks, to reduce (3.1) to an integral equation. In fact, if $u \in C^0([0,a])$, or more generally, $u \in C_r^0([0,a])$ with $r < 1 - s$, and further assumptions guarantee $f(x,u(x)) \in C^0((0,a)) \cap L^1((0,a))$, then the equation (3.1) is equivalent to the integral equation

$$u(x) = I^s f(x, u(x)). \tag{3.2}$$

Such a reduction will be systematically used in this section.

If we allow $u \in C_r^0([0,a])$ with $1 - s \leq r < 1$, then $u$ is a solution of (3.1) if and only if for some $c \in \mathbb{R}$

$$u(x) = I^s f(x, u(x)) + cx^{s-1}. \tag{3.2*}$$

It will be natural in this case to submit $u$ to an initial condition; results in this connection will be given in Section 4.

We first present a local existence theorem.

THEOREM 3.3. *Let* $0 \leq \sigma < s < 1$ *and let* $f: [0,1] \times \mathbb{R} \to \mathbb{R}$ *be a given function continuous in* $(0,1] \times \mathbb{R}$. *Assume that* $t^\sigma f(t,y)$ *is a continuous function on* $[0,1] \times \mathbb{R}$. *Then the fractional differential equation*

$$D^s u = f(x, u) \tag{3.3}$$

*has at least a continuous solution defined on* $[0, \delta]$ *for a suitable* $\delta \leq 1$.

*Proof.* According to Remark 3.2, we are reduced to consider the following nonlinear integral equation

$$u(x) = \frac{1}{\Gamma(s)} \int_0^x (x-t)^{s-1} f(t, u(t))\, dt.$$

Let $T: C^0([0,1]) \to C^0([0,1])$ be the operator defined as

$$(Tu)(x) = \frac{1}{\Gamma(s)} \int_0^x (x-t)^{s-1} f(t, u(t))\, dt.$$

We claim that the operator $T$ is compact. Indeed, the operator is the composition of two simple operators in this way

$$T = A \circ N$$

where

$$(Nu)(x) = x^\sigma f(x, u(x))$$

is a continuous and bounded operator (Nemytskii operator) and

$$(Av)(x) = \frac{1}{\Gamma(s)} \int_0^x (x - t)^{s-1} t^{-\sigma} v(t)\, dt$$

is a compact operator, since $s - \sigma > 0$; since for example [5].

Moreover, from Example 2.1.1 we have for $0 \le x \le \delta \le 1$

$$\begin{aligned} |Av(x)| &\le \Big( \sup_{x \in [0,\delta]} |v(x)| \Big) \frac{1}{\Gamma(s)} \int_0^x (x - t)^{s-1} t^{-\sigma}\, dt \\ &\le \frac{\Gamma(1-\sigma)}{\Gamma(1-\sigma+s)} \delta^{s-\sigma} \sup_{x \in [0,\delta]} |v(x)|; \end{aligned}$$

therefore, taking the norms in $C^0([0, \delta])$,

$$\|Av\| \le \epsilon \|v\|,$$

where we may assume $\epsilon > 0$ as small as we want by shrinking $\delta > 0$.

Now fix $B_r$ as a domain of the operator $T$, where $B_r = \{v \in C^0([0,\delta]) : \|v\| \le r\}$, which is a convex, bounded, and closed subset of the Banach space $C^0([0, \delta])$.

For $\delta$ sufficiently small, we have

$$T(B_r) \subseteq B_r.$$

The Schauder fixed point theorem assures that operator $T$ has at least one fixed point and then (3.3) has at least one continuous solution $u$ defined on $[0, \delta]$, where $\delta \le 1$.

EXAMPLE 3.3.1. From the preceding proof we also obtain that, under the assumptions of Theorem 3.3, all solutions $u \in C^0([0, \delta])$, $\delta > 0$, of (3.3) vanish at the origin.

Observe also that we cannot expect uniqueness for such solutions, in general. Consider for example the equation

$$D^s u = u^r \tag{3.4}$$

with $0 < r, s < 1$, which admits the two solutions $u = 0$ and

$$u(x) = kx^{s/(1-r)},$$

where

$$k = \left( \frac{\Gamma(\mu)}{\Gamma(\mu - s)} \right)^{1/(r-1)} \qquad \textit{with } \mu = \frac{s}{1-r} + 1$$

as we have from Example 2.2.1

EXAMPLE 3.3.2. Theorem 3.3 is a local existence theorem and we cannot expect all the solutions to extend to $\mathbb{R}^+$ or $[0,1)$. Consider for example the equation

$$D^{1/2}u = \frac{Mx^{1/2}}{\pi^{1/2}}(u+M)^2, \qquad M > 0, \tag{3.5}$$

which admits the solution

$$u(x) = (M^{-2} - x)^{-1/2} - M$$

(use formula IIc, page 354, in Miller and Ross [7]). The domain of existence of $u(x)$ is $[0, 1/M^2)$, which shrinks to zero as $M \to +\infty$.

The following theorem shows that uniqueness and global existence can be obtained under an uniform Lipschitz-type assumption. The result applies in particular to the linear case $f(x,u) = g(x) + h(x)u$ with $g, h \in C^0_\sigma([0,1])$, $\sigma < s$.

THEOREM 3.4. *Let* $0 \le \sigma < s < 1$ *and assume* $t^\sigma f(t,y)$ *is continuous on* $[0,1] \times \mathbb{R}$. *Assume further*

$$|f(t,u) - f(t,v)| \le \frac{L}{t^\sigma}|u - v| \tag{3.6}$$

*for some positive constant* $L$ *independent of* $u, v \in \mathbb{R}$, $t \in (0,1]$. *Then the equation*

$$D^s u = f(x,u)$$

*has a unique solution* $u \in C^0([0,1])$.

*Proof.* As in the proof of Theorem 3.3, we are reduced to studying the operator

$$Tu(x) = \frac{1}{\Gamma(s)} \int_0^x (x-t)^{s-1} f(t, u(t))\, dt, \tag{3.7}$$

which is well defined and continuous as a map $T\colon C^0([0,1]) \to C^0([0,1])$, in view of the assumption of continuity on $t^\sigma f(t, y)$. Let us define the iterates of operator $T$ as is standard:

$$T^1 = T \qquad T^n = T \circ T^{n-1}.$$

It will be sufficient to prove that $T^n$ is a contraction operator for $n$ sufficiently large. Actually, we have for $u, v \in C^0([0,1])$

$$|T^n u(x) - T^n v(x)| \le \frac{(KL)^n}{\Gamma(n(s-\sigma)+1)} x^{n(s-\sigma)} \|u - v\| \tag{3.8}$$

where the constant $K$ depends only on $s$ and $\sigma$. In fact,

$$|Tu(x) - Tv(x)| \le \frac{1}{\Gamma(s)} \int_0^x (x-t)^{s-1} |f(t, u(t)) - f(t, v(t))|\, dt$$

which we further estimate using (3.6) and Example 2.1.1 by

$$\frac{\Gamma(1-\sigma)L}{\Gamma(s-\sigma+1)} x^{s-\sigma} \|u - v\|. \tag{3.9}$$

Therefore (3.8) is proved for $n = 1$, if $K \ge \Gamma(1-\sigma)$. Assuming by induction that (3.8) is valid for $n$, we obtain similarly

$$\begin{aligned}
&|T^{n+1}u(x) - T^{n+1}v(x)| \\
&\quad \le \frac{K^n L^{n+1}}{\Gamma(n(s-\sigma)+1)\Gamma(s)} \|u - v\| \int_0^x (x-t)^{s-1} t^{n(s-\sigma)-\sigma}\, dt \\
&\quad = \frac{\Gamma(n(s-\sigma)-\sigma+1)K^n L^{n+1}}{\Gamma(n(s-\sigma)+1)\Gamma((n+1)(s-\sigma)+1)} x^{(n+1)(s-\sigma)} \|u - v\|,
\end{aligned}$$

and then (3.8) follows for $n + 1$ if $K$ is given by

$$K = \max_n K_n, \qquad K_n = \frac{\Gamma(n(s-\sigma)-\sigma+1)}{\Gamma(n(s-\sigma)+1)}. \tag{3.10}$$

Note that (3.10) defines actually a finite $K$, since $K_n \le 1$ for $n \ge (1+\sigma)/(s-\sigma)$. Taking $n$ sufficiently large in (3.8), we have, say, $(KL)^n/\Gamma(n(s-\sigma)+1) \le 1/2$ and therefore

$$\|T^n u - T^n v\| \le \tfrac{1}{2}\|u - v\|,$$

which gives the proof.

Finally, we consider the limit case when in Theorem 3.4 we have $\sigma = s$.

THEOREM 3.5. *Let* $0 < s < 1$. *Assume that* $t^s f(t, y)$ *is continuous on* $[0,1] \times \mathbb{R}$ *and moreover, that*

$$|f(t,u) - f(t,v)| \leq \frac{L}{t^s}|u - v| \tag{3.11}$$

*where*

$$L < \frac{1}{\Gamma(1-s)},$$

*for all* $u, v \in \mathbb{R}$ *and* $t \in (0,1]$. *Then*

$$D^s u(x) = f(x,u)$$

*has a unique solution* $u \in C^0([0,1])$.

*Proof.* We shall prove that under the preceding assumptions $T$: $C^0([0,1]) \to C^0([0,1])$, defined by (3.7) is a contraction operator.

Indeed, setting $\sigma = s$ in (3.9), we have the estimate

$$|Tu(x) - Tv(x)| \leq L\Gamma(1-s)\|u - v\|.$$

Thus we obtain that

$$\|Tu - Tv\| \leq k\|u - v\|$$

where

$$k = L\Gamma(1-s) < 1.$$

Our theorem is proved.

EXAMPLE 3.5.1. The assumption $L < 1/(\Gamma(1-s)$ in Theorem 3.5 is not a technical accident, as the following simple example shows. The linear equation

$$D^s u = \frac{1}{\Gamma(1-s)} x^{-s} u$$

admits the two solutions $u(x) = 0$, $u(x) = 1$.

Therefore, the result of uniqueness fails, in general, if $L = 1/\Gamma(1-s)$ in (3.11).

## 4. INITIAL VALUE PROBLEM: CONTINUOUS SOLUTIONS ON (0, 1]

We open this section with some basic examples, concerning the case when the solutions in $C^0(\mathbb{R}^+)$ are submitted to an initial condition.

PROPOSITION 4.0. *Let* $0 < s < 1$. *For all* $(a, b) \in \mathbb{R}^+ \times \mathbb{R}$ *the initial value problem*

$$\begin{cases} D^s u = 0 \\ u(a) = b \end{cases}$$

*admits* $u = ba^{1-s}x^{s-1}$ *as unique solution in* $C^0(\mathbb{R}^+) \cap L^1_{\mathrm{loc}}(\mathbb{R}^+)$.

This proposition follows directly from Lemma 2.3.

COROLLARY 4.0.1. *Let* $0 < s < 1$. *Assume* $f(x) \in C^0(\mathbb{R}^+) \cap L^1_{\mathrm{loc}}(\mathbb{R}^+)$. *Then for all* $(a, b) \in \mathbb{R}^+ \times \mathbb{R}$ *the initial value problem*

$$\begin{cases} D^s u = f(x) \\ u(a) = b \end{cases}$$

*has a unique solution in* $C^0(\mathbb{R}^+) \cap L^1_{\mathrm{loc}}(\mathbb{R}^+)$ *given by*

$$u(x) = \left( b - \frac{1}{\Gamma(s)} \int_0^a (a-t)^{s-1} f(t)\, dt \right) \frac{x^{s-1}}{a^{s-1}} + \frac{1}{\Gamma(s)} \int_0^x (x-t)^{s-1} f(t)\, dt.$$

Now we introduce the function $e_s \colon \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$e_s(x) = \sum_{k=1}^{+\infty} \frac{x^{ks-1}}{\Gamma(ks)}.$$

The function $e_s(x)$ belongs to $C^0_{1-s}(\mathbb{R}^+_0)$. Indeed, taking the norm in $C^0_{1-s}([0, h])$, $h > 0$, we have

$$\|e_s(x)\|_{1-s} \le \sum_{k=1}^{+\infty} \frac{h^{(k-1)s}}{\Gamma(ks)} < \infty.$$

Remark that the function $e_s(x)$ can be expressed by means of the classical Mittag–Leffler special function (see Miller and Ross [7, Chap. 5]).

THEOREM 4.1. *Let* $0 < s < 1$. *The initial value problem*

$$\begin{cases} D^s u = u \\ u(a) = b, \end{cases} \tag{4.1}$$

*where* $a > 0$ *and* $b \in \mathbb{R}$, *has a unique solution* $u \in C^0_{1-s}([0, h])$, $h > a$, *given by*

$$u(x) = b e_s(a)^{-1} e_s(x).$$

*Proof.* We are reduced to consider the integral equation

$$u(x) = cx^{s-1} + I^s u(x), \qquad c \in \mathbb{R}. \tag{4.1*}$$

Fix $h > a$ and find $u \in C^0_{1-s}([0, h])$.

From (4.1*) we obtain, by iteration,

$$u(x) = c\Gamma(s)\left[\frac{x^{s-1}}{\Gamma(s)} + \frac{x^{2s-1}}{\Gamma(2s)} + \cdots + \frac{x^{sn-1}}{\Gamma(sn)}\right] + I^{ns}u(x).$$

Letting $n \to +\infty$, one has $\|I^{ns}u(x)\|_{1-s} \to 0$ if $u \in C^0_{1-s}([0, h])$, as we deduce easily from Definition 2.1. On the other hand, the sum in the right-hand side tends to $e_s(x)$ in the $C^0_{1-s}([0, h])$-norm. This implies that

$$u(x) = c\Gamma(s)e_s(x).$$

This equality enables us to solve the initial value problem uniquely.

The preceding example suggest we look for solutions $u \in C^0_{1-s}([0, h])$ of the general autonomous initial value problem

$$\begin{cases} D^s u = f(u) \\ u(a) = b, \end{cases} \tag{4.2}$$

where $0 < s < 1$, $a > 0$, and $b \in \mathbb{R}$. As for the function $f\colon \mathbb{R} \to \mathbb{R}$, we shall argue initially under the hypothesis:

$$\begin{cases} f(0) = 0 \\ |f(u) - f(v)| \le L|u - v| \end{cases} \tag{4.3}$$

for some positive constant $L$ independent of $u, v \in \mathbb{R}$.

LEMMA 4.2. *Let $0 < s < 1$. Assume that $f\colon \mathbb{R} \to \mathbb{R}$ satisfies* (4.3). *Then for any fixed $c \in \mathbb{R}$ the nonlinear integral equation*

$$u(x) = cx^{s-1} + \frac{1}{\Gamma(s)}\int_0^x (x-t)^{s-1} f(u(t))\, dt$$

*has a unique solution $u \in C^0_{1-s}([0, h])$ for all $h > 0$.*

*Proof.* Our problem is equivalent to the problem of determination of fixed points of the continuous operator

$$(T_c u)(x) = cx^{s-1} + \frac{1}{\Gamma(s)}\int_0^x (x-t)^{s-1} f(u(t))\, dt.$$

It is immediate to verify that

(i) $T_c\colon C^0_{1-s}([0,h]) \to C^0_{1-s}([0,h])$ is well defined.

Indeed, we have

$$\|(T_c u)(x)\|_{1-s} = \max_{x\in[0,h]} x^{1-s}\left|T_c u(x)\right| \le |c| + R\|u\|_{1-s}$$

since $|f(\xi)| \le L|\xi|$ from assumption, the constant $R$ being given by

$$\max_{x\in[0,h]} L\frac{x^{1-s}}{\Gamma(s)}\int_0^x (x-t)^{s-1}t^{s-1}\,dt = L\frac{\Gamma(s)}{\Gamma(2s)}h^s$$

in view of Example 2.1.1.

(ii) $T_c^n$ *is a contraction operator for n sufficiently large*.

Indeed we have, computing as in (i),

$$x^{1-s}\left|T_c u(x) - T_c v(x)\right| \le \frac{L\Gamma(s)}{\Gamma(2s)}x^s\|u-v\|_{1-s}$$

for any $x \in (0,h)$ and for all $u, v \in C^0_{1-s}([0,h])$ and, by induction, arguing as in the proof of Theorem 3.4 that

$$x^{1-s}\left|T_c^n u(x) - T_c^n v(x)\right| \le \frac{L^n\Gamma(s)}{\Gamma((n+1)s)}x^{ns}\|u-v\|_{1-s}$$

for all $u, v \in C^0_{1-s}([0,h])$.

Thus we get

$$\|T_c^n u - T_c^n v\|_{1-s} \le \frac{L^n h^{ns}\Gamma(s)}{\Gamma((n+1)s)}\|u-v\|_{1-s}.$$

Since

$$\frac{L^n h^{ns}}{\Gamma((n+1)s)} \to 0$$

as $n$ tends to $+\infty$, then for $n$ sufficiently large the operator $T^n$ is a contraction operator. Therefore for any $c$ there exists $u = u(x,c)$ defined on $(0,h)$ for all $h > 0$, satisfying the required equation.

We shall begin by applying Lemma 4.2 to the case of an initial condition at the origin and then pass to considering the problem (4.2).

THEOREM 4.3. *Let* $0 < s < 1$. *Assume that* $f: \mathbb{R} \to \mathbb{R}$ *satisfies* (4.3). *Then for all* $b \in \mathbb{R}$, *the initial value problem*

$$\begin{cases} D^s u = f(u) \\ x^{1-s} u(x)|_{x=0} = b \end{cases} \tag{4.4}$$

*has a unique solution* $u \in C^0_{1-s}([0, h])$ *for all* $h > 0$.

*Proof.* Since we seek a solution $u \in C^0_{1-s}([0, h])$ the initial value problem (4.4) is equivalent to the nonlinear integral equation

$$u(x) = bx^{s-1} + \frac{1}{\Gamma(s)} \int_0^x (x-t)^{s-1} f(u(t))\, dt. \tag{4.5}$$

In fact, in view of Remark 3.2, it suffices to verify that

$$\lim_{x \to 0} x^{s-1} u(x) = b,$$

or, equivalently,

$$\lim_{x \to 0} \frac{x^{1-s}}{\Gamma(s)} \int_0^x (x-t)^{s-1} f(u(t))\, dt = 0$$

which is evident from the first part of the proof of Lemma 4.2. Theorem 4.3 follows then by applying Lemma 4.2.

THEOREM 4.4. *Let* $0 < s < 1$. *Assume that* $f: \mathbb{R} \to \mathbb{R}$ *satisfies* (4.3). *Then the initial value problem* (4.2) *has a unique solution* $u \in C^0_{1-s}([0, h])$ *for all* $h > a$, *provided* $a < a_0$, *where* $a_0$ *is a suitable positive constant depending on* $s$ *and* $L$.

*Proof.* The initial value problem (4.2) will be solved in two steps:

(1) *Local existence*. The integral equation

$$u(x) = \left( b - \frac{1}{\Gamma(s)} \int_0^a (a-t)^{s-1} f(u(t))\, dt \right) a^{1-s} x^{s-1} + \frac{1}{\Gamma(s)} \int_0^x (x-t)^{s-1} f(u(t))\, dt \tag{4.6}$$

is equivalent to the initial value problem (4.2), cf. Corollary 4.0.1.

Note that solving (4.6) is equivalent to finding the fixed points of the operator

$$A: C^0_{1-s}([0, h]) \to C^0_{1-s}([0, h])$$

defined as $Av(x) = u(x)$ where $u(x)$ is the unique solution of

$$u(x) = \left(b - \frac{1}{\Gamma(s)}\int_0^a (a-t)^{s-1} f(v(t))\,dt\right) a^{1-s} x^{s-1} + \frac{1}{\Gamma(s)}\int_0^x (x-t)^{s-1} f(u(t))\,dt.$$

Note also that operator $A$ is well defined from Lemma 4.2.

We shall now prove that $A$ is a contraction.

Indeed, setting $Av_i = u_i$, we have

$$\|u_1 - u_2\|_{1-s} = \|Av_1 - Av_2\|_{1-s} \le \frac{L\Gamma(s)h^s}{\Gamma(2s)}\|u_1 - u_2\|_{1-s} + \frac{L\Gamma(s)a^s}{\Gamma(2s)}\|v_1 - v_2\|_{1-s}$$

for all $v_1, v_2 \in C^0_{1-s}([0,h])$.

Let us assume

$$L\frac{\Gamma(s)}{\Gamma(2s)}a^s < \frac{1}{2},$$

that is,

$$a < a_0 = \left(\frac{\Gamma(2s)}{2L\Gamma(s)}\right)^{1/s}.$$

Taking $h - a > 0$ sufficiently small, we also have

$$\frac{L\Gamma(s)h^s}{\Gamma(2s)} < \frac{1}{2}$$

and then

$$\|Av_1 - Av_2\|_{1-s} \le R\|v_1 - v_2\|_{1-s}$$

with $R = 2L(\Gamma(s)a^s/\Gamma(2s)) < 1$.

Therefore $A$ is a contraction operator.

This shows that initial problem (4.2) has a unique solution.

(2) *Continuation of solution*. Since we know the value of $u(x)$ on $(0, a]$, then we can compute

$$c_* = \left(b - \frac{1}{\Gamma(s)}\int_0^a (a-t)^{s-1} f(u(t))\,dt\right) a^{1-s}.$$

Lemma 4.2 enables us to solve the integral equation

$$w(x) = c_* x^{s-1} + \frac{1}{\Gamma(s)} \int_0^x (x-t)^{s-1} f(w(t))\, dt,$$

obtaining a unique solution $w \in C^0_{1-s}([0,h])$ for all $h > 0$.

Now $u(x)$ and $w(x)$ agree on $(0,a]$. Thus the solution $u(x)$ admits $w(x)$ as its continuation.

The proof is complete.

By using somewhat more sophisticated arguments, we can enlarge the bound $a_0$ given in the preceding proof; however, we do not know whether existence and uniqueness in Theorem 4.4 hold for all $a \in \mathbb{R}^+$, as we had in Theorem 4.1.

Finally we want to study the not-globally-Lipschitzian case. For simplicity, we limit ourselves to analyzing the model $f(t) = t^\alpha$, with $\alpha > 0$, under the initial condition (4.4).

THEOREM 4.5. *Let $0 < s < 1$ and $b \in \mathbb{R}$. If we assume $0 < \alpha < 1/(1-s)$, then the initial value problem*

$$\begin{cases} D^s u = u^\alpha \\ x^{1-s} u(x)|_{x=0} = b \end{cases} \tag{4.7}$$

*has at least a solution in $C^0_{1-s}([0,h])$ for $h > 0$ sufficiently small.*

*Proof.* If $u \in C^0_{1-s}([0,h])$ and $\alpha(s-1) > -1$, then $u^\alpha \in L^1((0,h))$. In view of Remark 3.2, we are therefore reduced again to the nonlinear integral equation

$$u(x) = bx^{s-1} + \frac{1}{\Gamma(s)} \int_0^x (x-t)^{s-1} u^\alpha(t)\, dt. \tag{4.7.i}$$

The existence of a solution to the above problem can be formulated as a fixed point equation $Tu = u$ where

$$(Tu)(x) = bx^{s-1} + \frac{1}{\Gamma(s)} \int_0^x (x-t)^{s-1} u^\alpha(t)\, dt \tag{4.7.ii}$$

in the space $C^0_{1-s}([0,h])$. To prove existence we use Schauder's fixed point theorem.

The proof is divided in two parts:

*A priori estimates*. We seek an a priori estimate of fixed points of operator $T$ defined by (4.7.ii) by means of a closed sphere $\mathbb{S}_r$ defined as

$$\mathbb{S}_r = \left\{ u \in C^0_{1-s}([0,h]) : \|u - bx^{s-1}\|_{1-s} \le r \right\}.$$

To obtain $r > 0$ such that $T\mathbb{S}_r \subseteq \mathbb{S}_r$, we note that

$$\|Tu - bx^{s-1}\|_{1-s} = \max_{x \in [0,h]} \frac{x^{1-s}}{\Gamma(\alpha)} \left| \int_0^x (x-t)^{s-1} t^{\alpha(s-1)} t^{\alpha(1-s)} u^{\alpha}(t)\, dt \right|$$

$$\leq \frac{\Gamma(\alpha(s-1)+1)}{\Gamma(\alpha(s-1)+1+s)} h^{\alpha(s-1)+1} \|u\|_{1-s}^{\alpha}.$$

Since $\|u\|_{1-s} \leq r + |b|$, it will be sufficient to impose

$$\|u - bx^{s-1}\|_{1-s} \leq \text{const.} h^{\alpha(s-1)+1} (r + |b|)^{\alpha} \leq r.$$

In view of the assumption $\alpha(s-1) + 1 > 0$, the second estimate is satisfied if, say, $r = |b|$ and $h$ is chosen sufficiently small.

*Proof of compactness of T*. To prove the compactness of

$$T: C^0_{1-s}([0,h]) \to C^0_{1-s}([0,h])$$

defined by (4.7.ii), it will be sufficient to argue on the operator

$$T_*: C^0([0,h]) \to C^0([0,h])$$

defined in this way:

$$(T_* u)(x) = x^{1-s} T(x^{s-1} u(x)).$$

We have $T_* u = b + T^* u$ where the operator

$$T^* u(x) = \frac{1}{\Gamma(s)} \int_0^x x^{1-s} (x-t)^{s-1} t^{\alpha(s-1)} u^{\alpha}(t)\, dt$$

turns out to be compact from classical sufficient conditions, since $\alpha(s-1) > -1$. The proof is complete.

*Remark* 4.6. If in the preceding theorem one assumes $\alpha \geq 1/(1-s)$, then $u \in C^0_{1-s}([0,h])$ and $b \neq 0$ in (4.7) give $D^s u = u^{\alpha} \notin L^1((0,h))$, so our very reduction to the integral equation fails in this case. Observe also that under the assumptions of the previous theorem we cannot expect uniqueness as shown by Example 3.3.1.

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