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# Common Transversals

T. C. BROWN*

*Department of Mathematics, Simon Fraser University, Burnaby, British Columbia, V5A 1S6*

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Given $t$ families, each family consisting of $s$ finite sets, we show that if the families "separate points" in a natural way, and if the union of all the sets in all the families contains more than $(s+1)^t - s^{t-1} - 1$ elements, then a common transversal of the $t$ families exists. In case each family is a covering family, the bound is $s^t - s^{t-1}$. Both of these bounds are best possible. This work extends recent work of Longyear [2].

## 1. INTRODUCTION AND STATEMENT OF RESULTS

Throughout this paper, the symbol $\mathscr{F}$ will always denote a family of $t$ families of sets, each of the $t$ families consisting of $s$ finite, but not necessarily distinct or nonempty, sets. The symbol $\Omega$ will always denote the union of all of the sets contained in all of the $t$ families. Thus $\mathscr{F} = (F_1, F_2, ..., F_t)$, where for each $j$, $1 \leqslant j \leqslant t$, $F_j = (F_j(1), F_j(2), ..., F_j(s))$ is a family of $s$ (finite, but not necessarily distinct or nonempty) sets, and $\Omega = \bigcup \{F_j(i) \mid 1 \leqslant j \leqslant t, 1 \leqslant i \leqslant s\}$ (or more briefly, $\Omega = \bigcup \bigcup \mathscr{F}$). We always assume that $\mathscr{F}$ *separates points* of $\Omega$ in the following sense. Letting $F_j(0) = \Omega \setminus \bigcup F_j$, $1 \leqslant j \leqslant t$, we require

$$\left| \bigcap \{F_j(a_j) \mid 1 \leqslant j \leqslant t\} \right| \leqslant 1$$

for every $t$-tuple $(a_1, a_2, ..., a_t)$, where $0 \leqslant a_j \leqslant s$, $1 \leqslant j \leqslant t$. Note that this immediately implies $|\Omega| \leqslant (s+1)^t - 1$ (since $|\bigcap \{F_j(0) \mid 1 \leqslant j \leqslant t\}| = 0$), and that in the case where each $F_j$ covers $\Omega$ (so that $F_j(0) = \varnothing$) we have $|\Omega| \leqslant s^t$.

Recall that the set $T$ is a *transversal* (sometimes called a *system of distinct representatives* or SDR) of the family $F_j$ if there is a bijection $\varphi: T \to$

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$\{1, 2, \ldots, s\}$ such that $x \in F_j(\varphi(x))$ for all $x \in T$. The set $T$ is a *common transversal* of $F_1, F_2, \ldots, F_t$ if $T$ is simultaneously a transversal of each $F_j$, $1 \leqslant j \leqslant t$.)

In this paper the following results are proved, which extend recent results of Judith Q. Longyear [2]. Longyear proved, among other things, Theorem 1(b) below in the case where each family $F_j$ is assumed to consist of mutually disjoint sets. Theorem 1(b) can be obtained as a corollary to her result. We give an alternative proof.

THEOREM 1. *Let $\mathscr{F}$ and $\Omega$ be as in the first paragraph of this paper, and assume further that each family $F_j$ covers $\Omega$, that is, $\bigcup F_j = \Omega$, $1 \leqslant j \leqslant t$.*

(a) *If $|\Omega| > s^t - s^{t-1}$ then each family $F_j$ has a transversal, and $s^t - s^{t-1}$ is best possible.*

(b) *If $|\Omega| > s^t - s^{t-1}$ then a common transversal of $F_1, F_2, \ldots, F_t$ exists, and $s^t - s^{t-1}$ is best possible.*

THEOREM 2. *Let $\mathscr{F}$ and $\Omega$ be as in the first paragraph of this paper.*

(a) *If $|\Omega| > (s+1)^t - (s+1)^{t-1} - 1$ then each family $F_j$ has a transversal, and $(s+1)^t - (s+1)^{t-1} - 1$ is best possible.*

(b) *If $|\Omega| > (s+1)^t - s^{t-1} - 1$ then a common transversal of $F_1, F_2, \ldots, F_t$ exists, and $(s+1)^t - s^{t-1} - 1$ is best possible.*

## 2. PROOFS

Let us show first of all that the bounds given in Theorems 1 and 2 are best possible.

For Theorem 1(a) and Theorem 1(b) let

$$\Omega = \{(a_1, a_2, \ldots, a_t) \mid 1 \leqslant a_j \leqslant s,\ 1 \leqslant j \leqslant t-1,\ 1 \leqslant a_t \leqslant s-1\}.$$

For all $j, i$, $1 \leqslant j \leqslant t$, $1 \leqslant i \leqslant s$, let $F_j(i) = \{\omega \in \Omega \mid$ the $j$th coordinate of $\omega$ equals $i\}$. Note that $F_t(s) = \varnothing$, so that $F_t$ has no transversal. It is easy to see that $|\Omega| = s^t - s^{t-1}$, each $F_j$ covers $\Omega$, $1 \leqslant j \leqslant t$, and $\mathscr{F} = (F_1, F_2, \ldots, F_t)$ separates points.

For Theorem 2(a), we let

$$\Omega = \{(a_1, a_2, \ldots, a_t) \mid 0 \leqslant a_j \leqslant s,\ 1 \leqslant j \leqslant t-1,\ 0 \leqslant a_t \leqslant s-1\} \backslash \{(0, 0, \ldots, 0)\}.$$

For all $j, i$, $1 \leqslant j \leqslant t$, $1 \leqslant i \leqslant s$, let

$$F_j(i) = \{\omega \in \Omega \mid \text{the } j\text{th coordinate of } \omega \text{ equals } i\}.$$

Again $F_t(s) = \varnothing$ so $F_t$ has no transversal, and it is easy to see that $|\Omega| = (s+1)^t - (s+1)^{t-1} - 1$, $\Omega = \bigcup \bigcup \mathscr{F}$, and $\mathscr{F} = (F_1, F_2, \ldots, F_t)$ separates points.

For Theorem 2(b), let $\Omega$ be the set of all $t$-tuples $(a_1, a_2, \ldots, a_t)$, $0 \leqslant a_j \leqslant s$, $1 \leqslant j \leqslant t$, *excluding* the set $(\{(a_1, a_2, \ldots, a_t) \mid 1 \leqslant a_j \leqslant s, 1 \leqslant j \leqslant t-1, a_t = s\} \cup \{(0, 0, \ldots, 0)\})$. For all $j, i$, $1 \leqslant j \leqslant t$, $1 \leqslant i \leqslant s$, let

$$F_j(i) = \{\omega \in \Omega \mid \text{the } j\text{th coordinate of } \omega \text{ equals } i\}.$$

Then any element $\omega$ of $F_t(s)$ must have its $j_0$th coordinate equal to 0 for some $j_0 \neq t$, and hence $\omega$ cannot represent any set in the family $F_{j_0}$, therefore $\omega$ cannot belong to any common transversal of $F_1, F_2, \ldots, F_t$. Therefore no common transversal exists. Again it is easy to see that $|\Omega| = (s+1)^t - s^{t-1} - 1$, $\Omega = \bigcup \bigcup \mathscr{F}$, and $\mathscr{F} = (F_1, F_2, \ldots, F_t)$ separates points.

Throughout the remaining proofs, the following notation will be used. It is therefore fixed once and for all. For $t \geqslant 2$, let $X$ be the set of all $(t-1)$-tuples $(a_1, a_2, \ldots, a_{t-1})$, where each $a_j$, $1 \leqslant j \leqslant t-1$, satisfies $1 \leqslant a_j \leqslant s$. Note that $|X| = s^{t-1}$. For each $x = (a_1, a_2, \ldots, a_{t-1}) \in X$, we denote by $f(x)$ the set $\bigcap \{F_j(a_j) \mid 1 \leqslant j \leqslant t-1\}$. Then since $\mathscr{F}$ distinguishes points and each $F_j$ covers $\Omega$ we have $|f(x) \cap F_t(i)| \leqslant 1$ for all $x \in X$ and all $i$, $1 \leqslant i \leqslant s$, and $\Omega = \bigcup \{f(x) \mid x \in X\}$.

*Proof of Theorem* 1(a). The case $t = 1$ follows from the various definitions, so we assume $t \geqslant 2$, and without loss of generality we restrict our attention to $F_t$. We shall make use of the classical result of P. Hall [1] according to which $F_t$ has a transversal if and only if $|\bigcup \{F_t(i) \mid i \in I\}| \geqslant |I|$ for all $I \subset \{1, 2, \ldots, s\}$. Suppose that $F_t$ does not have a transversal, and that $|F_t(i_1) \cup F_t(i_2) \cup \cdots \cup F_t(i_k)| < k$, where $k$ is as small as possible. Then $F_t(i_k) \subset F_t(i_1) \cup F_t(i_2) \cup \cdots \cup F_t(i_{k-1})$ $(F_t(i_k) = \varnothing$ if $k = 1)$, hence $\Omega = \bigcup \{F_t(i) \mid 1 \leqslant i \leqslant s, i \neq i_k\}$. Then

$$\begin{aligned}
|\Omega| &= \left|\left(\bigcup \{f(x) \mid x \in X\}\right) \cap \left(\bigcup \{F_t(i) \mid 1 \leqslant i \leqslant s, i \neq i_k\}\right)\right| \\
&= \left|\bigcup \{f(x) \cap F_t(i) \mid x \in X, 1 \leqslant i \leqslant s, i \neq i_k\}\right| \\
&\leqslant |\{(x, i) \mid x \in X, 1 \leqslant i \leqslant s, i \neq i_k\}| \\
&= s^{t-1}(s-1) = s^t - s^{t-1},
\end{aligned}$$

contrary to the hypothesis of the Theorem. Hence $F_t$, and similarly each $F_j$, has a transversal.

*Proof of Theorem* 1(b). Since the family $F_t = (F_t(1), F_t(2),..., F_t(s))$ has a transversal (by Theorem 1(a)) and covers $\Omega$, we can replace $F_t$ by a *partition* $G = (G(1), G(2),..., G(s))$ such that $G(i) \subset F_t(i)$ for all $i$, $1 \leqslant i \leqslant s$. (The partition $G$ can be constructed as follows. Let $\{\omega_1, \omega_2,..., \omega_s\}$ be a transversal of $F_t$, where $\omega_i \in F_t(i)$, $1 \leqslant i \leqslant s$. Let

$$\begin{aligned} G(1) &= F_t(1)\backslash\{\omega_2,..., \omega_s\}, \\ G(2) &= F_t(2)\backslash(G(1) \cup \{\omega_3,..., \omega_s\}), \\ G(3) &= F_t(3)\backslash(G(1) \cup G(2) \cup \{\omega_4,..., \omega_s\}), \\ &\vdots \\ G(s) &= F_t(s)\backslash(G(1) \cup G(2) \cup \cdots \cup G(s-1)). \end{aligned}$$

Then $\mathscr{F}' = (F_1, F_2,..., F_{t-1}, G)$ distinguishes points, hence $|f(x) \cap G(i)| \leqslant 1$ for all $x \in X$ and all $i$, $1 \leqslant i \leqslant s$, and any common transversal of $F_1, F_2,..., F_{t-1}, G$ is a common transversal of $F_1, F_2,..., F_t$.

At this point we could in fact replace *every* $F_j$ by a partition (since we know by Theorem 1(a) that every $F_j$ has a transversal); however, it is not necessary, and so we do not.

We now demonstrate the existence of a common transversal of $F_1, F_2,..., F_{t-1}, G$.

To this end we define a *diagonal* of $X$ to be a subset $D$ of $X$ such that $|D| = s$ and for each $j$, $1 \leqslant j \leqslant t-1$, the $j$th coordinates of the elements of $D$ run through $\{1, 2,..., s\}$ in some order. Note that whenever $D = \{x_1, x_2,..., x_s\}$ is a diagonal, $\omega_k \in f(x_k)$, $1 \leqslant k \leqslant s$, and $\omega_1, \omega_2,..., \omega_s$ are all distinct, then $\{\omega_1, \omega_2,..., \omega_s\}$ is a common transversal of $F_1, F_2,..., F_{t-1}$.

Now we let $\mathscr{D}$ be some fixed collection of *mutually disjoint* diagonals of $X$ whose union is $X$, $X = \bigcup \mathscr{D}$. (The existence of $\mathscr{D}$ can be shown by induction on $t$.)

Since $|X| = s^{t-1}$ and every diagonal has $s$ elements, we have $|\mathscr{D}| = s^{t-2}$. For any diagonal $D$, let $f(D) = \bigcup\{f(x) \mid x \in D\}$. Then

$$\Omega = \bigcup\{f(x) \mid x \in X\} = \bigcup\{f(D) \mid D \in \mathscr{D}\}.$$

Now

$$s^t - s^{t-1} < |\Omega| \leqslant \sum_{D \in \mathscr{D}} |f(D)| \leqslant |\mathscr{D}| \max\{|f(D)| \mid D \in \mathscr{D}\},$$

hence $s^2 - s < \max\{|f(D)| \mid D \in \mathscr{D}\}$.

We now fix a diagonal $D$ with $s^2 - s < |f(D)|$. Let $D = \{x_1, x_2,..., x_s\}$, and define, for $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant s$,

$$e_{ij} = \begin{cases} 1 & \text{if } f(x_i) \cap G(j) \neq \varnothing \\ 0 & \text{otherwise.} \end{cases}$$

Then since $|f(x_i) \cap G(j)| \leqslant 1$ for all $i$, $j$, and $G$ is a partition of $\Omega$, the $s \times s$ 0-1 matrix $(e_{ij})$ contains exactly $|f(D)|$, and hence more than $s^2 - s$, 1's. Therefore there exist (as can be shown by induction on $s$) indices $i_1 j_1$, $i_2 j_2$,..., $i_s j_s$ such that $e_{i_1 j_1} = e_{i_2 j_2} = \cdots = e_{i_s j_s} = 1$ and $\{i_1, i_2,..., i_s\} = \{j_1, j_2,..., j_s\} = \{1, 2,..., s\}$.

Now let $\{\omega_k\} = f(x_{i_k}) \cap G(j_k)$, $1 \leqslant k \leqslant s$. Then since $G$ is a partition, $\omega_1, \omega_2,..., \omega_s$ are all distinct, and hence $\{\omega_1, \omega_2,..., \omega_s\}$ is not only a transversal of $G$ but is also (since $\{x_1, x_2,..., x_s\}$ is a diagonal) a common transversal of $F_1, F_2,..., F_{t-1}$. Therefore $\{\omega_1, \omega_2,..., \omega_s\}$ is a common transversal of $F_1, F_2,..., F_{t-1}$, $G$, and hence of $F_1, F_2,..., F_t$.

This completes the proof of Theorem 1(b).

*Proof of Theorem* 2(a). Recall that for each $j$, $1 \leqslant j \leqslant t$, $F_j(0)$ denotes the complement in $\Omega$ of the union of the family $F_j$, that is, $F_j(0) = \Omega \backslash \bigcup \{F_j(i) \mid 1 \leqslant i \leqslant s\}$. If now for each $j$, $1 \leqslant j \leqslant t$, we let

$$G_j = (F_j(0), F_j(1),..., F_j(s)),$$

and let $\mathscr{G} = (G_1, G_2,..., G_t)$, then $\mathscr{G}$ separates points and each family $G_j$ covers $\Omega$, therefore we may proceed exactly as in the proof of Theorem 1(a), where now we have $t$ families with $s + 1$ sets in each family. Furthermore, since we know that $\bigcap \{F_j(0) \mid 1 \leqslant j \leqslant t\} = \varnothing$ (this is so because $\bigcup \{F_j(i) \mid 1 \leqslant j \leqslant t, 1 \leqslant i \leqslant s\} = \Omega$), the last inequality in the proof of Theorem 1(a) can be replaced by

$$\begin{aligned} |\,\Omega\,| &\leqslant |\{(x, i) \mid x \in X, 0 \leqslant i \leqslant s, i \neq i_k, (i, x) \neq (0, (0, 0,..., 0))\}| \\ &= (s + 1)^t - (s + 1)^{t-1} - 1\,. \end{aligned}$$

This proves Theorem 2(a).

*Proof of Theorem* 2(b). For each $j$, $1 \leqslant j \leqslant t$, let $\Omega_j = \bigcup F_j$, and let

$$\Omega_0 = \bigcap \{\Omega_j \mid 1 \leqslant j \leqslant t\}.$$

For each $j$, $i$, $1 \leqslant j \leqslant t$, $1 \leqslant i \leqslant s$, let $G_j(i) = F_j(i) \cap \Omega_0$, $G_j = (G_j(1), G_j(2),..., G_j(s))$, and

$$\mathscr{G} = (G_1, G_2,..., G_t).$$

Then for each $j$, $1 \leqslant j \leqslant t$, $\Omega_0 = \bigcup G_j$. Also, since $G_j(i) \subset F_j(i)$ for all $j$, $i$, the family $\mathscr{G}$ separates points. Thus, by Theorem 1(b), it suffices now to show that $|\,\Omega_0\,| > s^t - s^{t-1}$, since any common transversal of $G_1, G_2,..., G_t$ is also a common transversal of $F_1, F_2,..., F_t$. Since $\mathscr{F}$ separates points, the cardinal of $\Omega \backslash \Omega_0$ cannot exceed the cardinal of the set of all those

$t$-tuples $(a_1, a_2, ..., a_t)$, $0 \leqslant a_j \leqslant s$, having at least one coordinate equal to 0 (excluding $(0, 0, ..., 0)$). That is, $| \Omega \backslash \Omega_0 | \leqslant (s+1)^t - s^t - 1$. Hence

$$\begin{aligned}(s+1)^t - s^{t-1} - 1 < | \Omega | &= | \Omega_0 | + | \Omega \backslash \Omega_0 | \\ &\leqslant | \Omega_0 | + (s+1)^t - s^t - 1,\end{aligned}$$

and therefore

$$| \Omega_0 | > s^t - s^{t-1}.$$

This completes the proof of Theorem 2(b).

## References

1. P. Hall, On representatives on subsets, *J. London Math. Soc.* **10** (1935), 26–30.
2. J. Q. Longyear, Common transversal in partitioning families, to appear.