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Journal of Algebra 319 (2008) 2259–2270

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

Normal form of m-by-n-by-2 matrices for equivalence

Genrich Belitskii [a,*,1], Maxim Bershadsky [a], Vladimir V. Sergeichuk [b,2]

[a] *Department of Mathematics, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel*
[b] *Institute of Mathematics, Tereshchenkivska 3, Kiev, Ukraine*

Received 17 April 2006

Available online 21 December 2007

Communicated by Efim Zelmanov

Abstract

We study $m \times n \times 2$ matrices up to equivalence and give a canonical form of $m \times 2 \times 2$ matrices over any field.

Keywords: Spatial matrices; Tensors; Classification

1. Introduction and the main results

Complex $2 \times 2 \times 2$ matrices up to equivalence were classified by Schwartz [9] and Duschek [3]. Canonical forms of complex and real $2 \times 2 \times 2$ matrices for equivalence were given by Oldenburger [6–8]; they are presented in [10, Section IV, Theorem 1.1]. Ehrenborg [4] also got a canonical form of complex $2 \times 2 \times 2$ matrices for equivalence basing on a collection of covariants that separates the canonical matrices.

In this paper we give a canonical form of $m \times 2 \times 2$ matrices for equivalence over any field $\mathbb{F}$, but first we establish when $m \times n \times 2$ matrices, whose two $m \times n \times 1$ submatrices are in the Kronecker canonical form for matrix pencils, are equivalent over $\mathbb{F}$. Using an alternative method,

* Corresponding author.
E-mail addresses: genrich@cs.bgu.ac.il (G. Belitskii), maximb@math.bgu.ac.il (M. Bershadsky), sergeich@imath.kiev.ua (V.V. Sergeichuk).

1 Partially supported by Israel Science Foundation, Grant 216/98.
2 Partially supported by FAPESP (São Paulo, Brazil), processo 05/59407-6.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2007.10.040

the authors recently obtained in [1] a canonical form of $m \times 2 \times 2$ matrices for equivalence over a field of characteristic different from 2.

Note that the canonical form problem for $m \times n \times 3$ matrices for equivalence is wild; this means that it contains the problem of classifying pairs of linear operators and therefore it contains the problem of classifying an arbitrary system of linear operators (see, for example, [2, Theorems 4.5 and 2.1]).

All matrices and spatial matrices in this article are considered over an arbitrary field $\mathbb{F}$. By an $m \times n \times q$ *spatial matrix* over $\mathbb{F}$ we mean an array

$$\mathcal{A} = [a_{ijk}]_{i=1}^{m}{}_{j=1}^{n}{}_{k=1}^{q}, \quad a_{ijk} \in \mathbb{F}. \tag{1}$$

Two $m \times n \times q$ matrices $\mathcal{A} = [a_{ijk}]$ and $\mathcal{B} = [b_{ijk}]$ are *equivalent* if there exist nonsingular $m \times m$, $n \times n$, and $q \times q$ matrices

$$R = [r_{ii'}], \qquad S = [s_{jj'}], \qquad T = [t_{kk'}] \tag{2}$$

such that

$$b_{i'j'k'} := \sum_{ijk} a_{ijk} r_{ii'} s_{jj'} t_{kk'}. \tag{3}$$

This notion arises in the theory of forms: each trilinear form $f : U \times V \times W \to \mathbb{F}$ on vector spaces with bases $\{u_i\}_{i=1}^m$, $\{v_j\}_{j=1}^n$, and $\{w_k\}_{k=1}^q$ is given by the spatial matrix (1) with $a_{ijk} := f(u_i, v_j, w_k)$. Its entries change by (3) if we go to other bases with the transition matrices (2).

We will give the spatial matrix (1) by the q-tuple of $m \times n$ matrices

$$\mathcal{A} = \| A_1 \mid \ldots \mid A_q \|, \quad A_k = [a_{ijk}]_{ij}$$

(that is, by the list of its *horizontal slices*).

The transfer from $\mathcal{A}$ to $\mathcal{B}$ given by (3) can be realized in two steps: by the *simultaneous equivalence transformation* with the horizontal slices

$$\| C_1 \mid \ldots \mid Cq \| := \| R^T A_1 S \mid \ldots \mid R^T A_q S \|, \tag{4}$$

and then by the nonsingular linear substitution

$$B_1 = C_1 t_{11} + \cdots + C_q t_{q1}, \qquad \ldots, \qquad B_q = C_1 t_{1q} + \cdots + C_q t_{qq}, \tag{5}$$

where R, S, and T are the matrices (2). The last transformation can be made by *elementary operations* on the set $\{C_1, \ldots, Cq\}$ of horizontal slices: interchange any two slices, multiply one slice by a nonzero scalar, and add a scalar multiple of one slice to another one. This implies the following lemma.

Lemma 1. *Two spatial matrices are equivalent if and only if one can be transformed to the other by a sequence of*

(i) *simultaneous equivalence transformations with all horizontal slices, and*
(ii) *elementary operations on the set of horizontal slices.*

We denote the m-by-n zero matrix by 0_{mn}. The numbers m and n may be zero: the matrices 0_{m0} and 0_{0n} represent the linear mappings $0 \to \mathbb{F}^m$ and $\mathbb{F}^n \to 0$. For every $p \times q$ matrix M_{pq} we have

$$M_{pq} \oplus 0_{m0} = \begin{bmatrix} M_{pq} \\ 0_{mq} \end{bmatrix}, \qquad M_{pq} \oplus 0_{0n} = [\, M_{pq} \quad 0_{pn} \,].$$

For each natural number r, we define the $(r-1) \times r$ matrices

$$F_r := \begin{bmatrix} 1 & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}, \qquad G_r := \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{bmatrix}. \tag{6}$$

For each polynomial

$$\chi(x) = x^l - u_1 x^{l-1} - \cdots - u_l \in \mathbb{F}[x], \quad l \geqslant 1,$$

we define the $l \times l$ matrix

$$\Phi_\chi := \begin{bmatrix} 0 & & 0 & u_l \\ 1 & \ddots & & \vdots \\ & \ddots & 0 & u_2 \\ 0 & & 1 & u_1 \end{bmatrix}, \tag{7}$$

whose characteristic polynomial is $\chi(x)$.

We also define the direct sum of matrix pairs:

$$(A, B) \oplus (A', B') := (A \oplus A', B \oplus B').$$

The next theorem will be proved in Section 2, it extends Theorem 4.4 of [2] dealing with spatial matrices over an algebraically closed field.

Theorem 1. *Over any field $\mathbb{F}$, every $m \times n \times 2$ matrix $\mathcal{A} = \|A_1 \mid A_2\|$, in which $\min(m, n)$ is less than or equal to the number of elements of $\mathbb{F}$, is equivalent to some $\mathcal{B} = \|B_1 \mid B_2\|$, in which*

$$(B_1, B_2) = \bigoplus_{i=1}^{p_1} (F_{r_i}, G_{r_i}) \oplus \bigoplus_{j=1}^{p_2} \left(F_{s_j}^T, G_{s_j}^T\right) \oplus \bigoplus_{k=1}^{q} (I_{l_k}, \Phi_{\chi_k}), \tag{8}$$

p_1, p_2, q are nonnegative integers, all r_i, s_j, l_k are natural numbers, and each polynomial χ_k has degree l_k and is a power of an irreducible polynomial. This sum is determined by $\mathcal{A}$ uniquely, up to permutation of summands and up to simultaneous replacement of all Φ_{χ_k} by Φ_{η_k} with

$$\eta_k(x) := \varepsilon_k (d - xb)^{l_k} \chi_k\left(\frac{xa - c}{d - xb}\right), \tag{9}$$

where

- *a, b, c, d are arbitrary elements of $\mathbb{F}$ satisfying $ad - bc \neq 0$ and*

$$a + b\lambda_k \neq 0 \quad \text{if } \chi_k(x) = (x - \lambda_k)^{l_k}, \tag{10}$$

- *each ε_k is a nonzero element of $\mathbb{F}$ that makes the coefficient of the highest order term of $\eta_k(x)$ equaling* 1 (*the characteristic polynomial $\eta_k(x)$ must be* monic).

Let $\mathcal{A} = [a_{ijk}]_{i=1}^{m}{}_{j=1}^{n}{}_{k=1}^{q}$ be a spatial matrix. Consider the sets

$$\mathcal{S} = \{A_1, \ldots, A_q\}, \qquad \tilde{\mathcal{S}} = \{\tilde{A}_1, \ldots, \tilde{A}_n\}, \qquad \tilde{\tilde{\mathcal{S}}} = \{\tilde{\tilde{A}}_1, \ldots, \tilde{\tilde{A}}_m\} \tag{11}$$

of its $m \times n$, $m \times q$, and $n \times q$ submatrices

$$A_k := [a_{ijk}]_{ij}, \qquad \tilde{A}_j := [a_{ijk}]_{ik}, \qquad \tilde{\tilde{A}}_i := [a_{ijk}]_{jk}.$$

We say that $\mathcal{A}$ is *regular* if each of the sets (11) is linearly independent.

Suppose $\mathcal{A}$ is nonregular and let q', n', m' be the ranks of the sets (11). Make the first q' matrices in $\mathcal{S}$ linearly independent and the others zero by elementary operations on the set $\mathcal{S}$. Reduce the "new" $\tilde{\mathcal{S}}$ and then the "new" $\tilde{\tilde{\mathcal{S}}}$ in the same way. We obtain a spatial matrix $\mathcal{B} = [b_{ijk}]$, whose $m' \times n' \times q'$ submatrix

$$\mathcal{B}' = [b_{ijk}]_{i=1}^{m'}{}_{j=1}^{n'}{}_{k=1}^{q'}$$

is regular, and whose entries outside of $\mathcal{B}'$ are zero; $\mathcal{B}'$ is called a *regular part* of $\mathcal{A}$. Two spatial matrices of the same size are equivalent if and only if their regular parts are equivalent [2, Lemma 4.7]. Hence, it suffices to give canonical forms of regular spatial matrices. The following theorem will be proved in Section 3.

Theorem 2. *Over any field $\mathbb{F}$, each regular $m \times n \times q$ matrix $\mathcal{A}$ with $n \leqslant 2$ and $q \leqslant 2$ is equivalent to one of the spatial matrices*:

$$\| 1 \| \quad (1 \times 1 \times 1), \tag{12}$$

$$\left\| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right\| \quad (2 \times 2 \times 1), \tag{13}$$

$$\left\| \begin{matrix} 1 \\ 0 \end{matrix} \middle| \begin{matrix} 0 \\ 1 \end{matrix} \right\| \quad (2 \times 1 \times 2), \tag{14}$$

$$\| 1\ 0 \mid 0\ 1 \| \quad (1 \times 2 \times 2), \tag{15}$$

$$\left\| \begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix} \middle| \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{matrix} \right\| \quad (3 \times 2 \times 2), \tag{16}$$

$$\left\| \begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix} \middle| \begin{matrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{matrix} \right\| \quad (3 \times 2 \times 2), \tag{17}$$

$$\left\| \begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{matrix} \middle| \begin{matrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{matrix} \right\| \quad (4 \times 2 \times 2), \tag{18}$$

$$\mathcal{A}(v) := \left\| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \middle| \begin{matrix} 0 & v \\ 1 & 0 \end{matrix} \right\| \quad (v \in \mathbb{F}, 2 \times 2 \times 2), \tag{19}$$

$$\mathcal{B}(v) := \left\| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \middle| \begin{matrix} 0 & v \\ 1 & 1 \end{matrix} \right\| \quad (\operatorname{char} \mathbb{F} = 2;\ v \in \mathbb{F}, 2 \times 2 \times 2). \tag{20}$$

These spatial matrices are pairwise inequivalent except for the following cases:

- *If* $\operatorname{char} \mathbb{F} \neq 2$, *then* $\mathcal{A}(v)$ *is equivalent to each* $\mathcal{A}(v')$ *with*

$$v' = vz, \quad 0 \neq z \in \mathbb{F}^2 := \{a^2 \mid a \in \mathbb{F}\}. \tag{21}$$

- *If* $\operatorname{char} \mathbb{F} = 2$, *then* $\mathcal{A}(v)$ *is equivalent to each* $\mathcal{A}(v')$ *with*

$$v' = \frac{\alpha v + \beta}{\gamma v + \delta}, \quad \alpha, \beta, \gamma, \delta \in \mathbb{F}^2,\ \alpha\delta + \beta\gamma \neq 0,\ \gamma v + \delta \neq 0, \tag{22}$$

  and $\mathcal{B}(v)$ *is equivalent to each* $\mathcal{B}(v')$ *with*

$$v' = v + \beta + \beta^2, \quad \beta \in \mathbb{F}. \tag{23}$$

In particular, if $\mathbb{F}$ *is algebraically closed, then each regular* $m \times n \times q$ *matrix* $\mathcal{A}$ *with* $n \leqslant 2$ *and* $q \leqslant 2$ *is equivalent to exactly one of the following spatial matrices*: (12)–(18), $\mathcal{A}(0)$, *and*

$$\left\| \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} \middle| \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} \right\| \quad (2 \times 2 \times 2). \tag{24}$$

2. Proof of Theorem 1

We say that two pairs of matrices of the same size are *equivalent* if the matrices of the first pair are simultaneously equivalent to the matrices of the second pair.

Lemma 2. *Let* (I_l, Φ_χ) *and* (I_l, Φ_η) *be two matrix pairs given by arbitrary monic polynomials* χ *and* η *of degree* l. *Let*

$$T := \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \quad ad - bc \neq 0,$$

be a nonsingular matrix.

(a) *If the pair*

$$(aI_l + b\Phi_\chi, cI_l + d\Phi_\chi) \tag{25}$$

is equivalent to (I_l, Φ_η)*, then*

$$\eta(x) = \varepsilon(d - xb)^l \chi_k\left(\frac{xa - c}{d - xb}\right) \tag{26}$$

for some $\varepsilon \in \mathbb{F}$.

(b) *If* (26) *holds then the characteristic polynomials of*

$$(cI_l + d\Phi_\chi) \cdot (aI_l + b\Phi_\chi)^{-1} \tag{27}$$

and Φ_η *are equal.*

Proof. (a) Since the pair (25) is equivalent to (I_l, Φ_η), $aI_l + b\Phi_\chi$ is nonsingular, and so the pair (25) is equivalent to

$$\big(I_l, (cI_l + d\Phi_\chi) \cdot (aI_l + b\Phi_\chi)^{-1}\big). \tag{28}$$

Hence (27) is similar to Φ_η and their characteristic polynomials are equal:

$$\begin{aligned}
\eta(x) &= \det\big[xI_l - (cI_l + d\Phi_\chi) \cdot (aI_l + b\Phi_\chi)^{-1}\big] \\
&= \det\big[\big[x(aI_l + b\Phi_\chi) - (cI_l + d\Phi_\chi)\big] \cdot (aI_l + b\Phi_\chi)^{-1}\big] \\
&= \det\big[(xa - c)I_l - (d - xb)\Phi_\chi\big] \cdot \det(aI_l + b\Phi_\chi)^{-1} \\
&= (d - xb)^l \det\left(\frac{xa - c}{d - xb} I_l - \Phi_\chi\right) \cdot \det(aI_l + b\Phi_\chi)^{-1} \\
&= (d - xb)^l \chi\left(\frac{xa - c}{d - xb}\right) \cdot \det(aI_l + b\Phi_\chi)^{-1}.
\end{aligned} \tag{29}$$

This proves (26).

(b) This statement follows from (29). □

Recall [11] that each square matrix A over an arbitrary field $\mathbb{F}$ is similar to a matrix of the form $\Phi = \Phi_{\chi_1} \oplus \cdots \oplus \Phi_{\chi_q}$, where $\chi_1, \ldots, \chi_q$ are powers of an irreducible polynomials and Φ_{χ_k} are defined in (7). The matrix Φ is called the *Frobenius canonical form* of A and is determined by A uniquely up to permutations of summands.

Each pair (A_1, A_2) of matrices of the same size is equivalent to a pair of the form

$$(B_1, B_2) = \bigoplus_{i=1}^{p_1} (F_{r_i}, G_{r_i}) \oplus \bigoplus_{j=1}^{p_2} \big(F_{s_j}^T, G_{s_j}^T\big) \oplus \bigoplus_{k=1}^{q_1} (I_{l_k}, \Phi_{\chi_k}) \oplus \bigoplus_{k=q_1+1}^{q} \big(J_{l_k}(0), I_{l_k}\big), \tag{30}$$

where p_1, p_2, q_1, q_2 are nonnegative integers, F_r and G_r are defined in (6), each polynomial χ_k has degree l_k and is a power of an irreducible polynomial, and

$$J_l(\lambda) = \begin{bmatrix} \lambda & & & 0 \\ 1 & \lambda & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda \end{bmatrix} \quad (l\text{-by-}l).$$

The pair (30) is determined by (A_1, A_2) uniquely up to permutation of summands and is called the *Kronecker canonical form* of (A_1, A_2) (see, for example, [5, Section 1.8]).

Proof of Theorem 1. *Step* 1. Let (A_1, A_2) be a pair of matrices of the same size and let (30) be its Kronecker canonical form. In this step, we prove that for each nonsingular matrix

$$T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \mathbb{F}^{2\times 2}, \quad ad - bc \neq 0,$$

the Kronecker canonical form of the pair

$$(C_1, C_2) = (aB_1 + bB_2, cB_1 + dB_2) \tag{31}$$

has the same number $p_1 + p_2 + q$ of direct summands as (30) and, after a suitable permutation of its summands, it has the same first $p_1 + p_2$ summands as (30) and the same sizes $l_1 \times l_1, \dots, l_q \times l_q$ of the remaining q summands as (30).

A matrix pair is *decomposable* if it is equivalent to a direct sum of pairs of smaller sizes. All direct summands in (30) are indecomposable. The transformation (31) takes them into indecomposable matrix pairs. Indeed, if it takes a summand $\mathcal{P}$ into a decomposable $\mathcal{R}$, then the inverse transformation (given by the matrix T^{-1}) takes $\mathcal{R}$ into a decomposable one, which is equivalent to $\mathcal{P}$, contrary to the indecomposability of all direct summands of (30).

All indecomposable pairs of $(r-1)\times r$ or $r \times (r-1)$ matrices are equivalent to (F_r, G_r) or, respectively, (F_r^T, G_r^T). Hence, though transformations (31) may spoil the direct summands (F_{r_i}, G_{r_i}) and $(F_{s_j}^T, G_{s_j}^T)$ in (30), but they are restored by equivalence transformations.

Step 2. Suppose $\mathcal{A} = \|A_1 \mid A_2\|$ satisfies the hypotheses of Theorem 1. In this step, we reduce $\mathcal{A}$ by equivalence transformations to some $\mathcal{B} = \|B_1 \mid B_2\|$ with (B_1, B_2) of the form (8).

From the start, we reduce (A_1, A_2) to the form (30).

Thereupon in the case $q_1 < q$ we reduce the pair (30) to a pair of the form (8) (with other $\chi_1, \dots, \chi_{q_1}$) as follows. The transformation (31) with (30) given by

$$T = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}, \quad b \neq 0,$$

takes the direct sum of the last q summands into

$$\bigoplus_{k=1}^{q_1} (I_{l_k} + b\Phi_{\chi_k}, \Phi_{\chi_k}) \oplus \bigoplus_{k=q_1+1}^{q} \big(J_{l_k}(b), I_{l_k}\big). \tag{32}$$

If some $I_{l_k} + b\Phi_{\chi_k}$ is singular, then $\chi_k(x) = (x - b^{-1})^{l_k}$. Indeed, 0 is an eigenvalue of $I_{l_k} + b\Phi_{\chi_k}$, hence $I_{l_k} + b\Phi_{\chi_k}$ has an eigenvalue in $\mathbb{F}$, and so Φ_{χ_k} is similar to a Jordan block. Further, this Jordan block must be $J_{l_k}(-b^{-1})$.

In view of the hypotheses of Theorem 1, $\min(m, n)$ is less than or equal to the number of elements of $\mathbb{F}$. Since $q_1 < q \leqslant \min(m, n)$, the number q_1 of the summands (I_{l_k}, Φ_{χ_k}) in (30) is less than or equal to the number of nonzero elements of $\mathbb{F}$.

First suppose that one of these summands is $(I_{l_k}, J_{l_k}(0))$. Then there exists a nonzero $b \in \mathbb{F}$ such that $\chi_k(x) \neq (x - b^{-1})^{l_k}$ for all $k \leqslant q_1$, this means that all $I_{l_k} + b\Phi_{\chi_k}$ are nonsingular. We take such b and reduce (32) to the form

$$(I_{l_1}, \Phi_{\eta_1}) \oplus \cdots \oplus (I_{l_q}, \Phi_{\eta_q}) \tag{33}$$

by equivalence transformations.

Now suppose that there are no summands $(I_{l_k}, J_{l_k}(0))$. Then the second matrix in each of the last q summands of (30) is nonsingular. We interchange the matrices B_1 and B_2 in the pair (30) and reduce its last q summands to the form (33).

Step 3. Suppose $\mathcal{A} = \|A_1 \mid A_2\|$ is equivalent both to $\mathcal{B} = \|B_1 \mid B_2\|$ with (B_1, B_2) of the form (8) and to another $\mathcal{B}' = \|B'_1 \mid B'_2\|$ with

$$\left(B'_1, B'_2\right) = \bigoplus_{i=1}^{p'_1} (F_{r'_i}, G_{r'_i}) \oplus \bigoplus_{j=1}^{p'_2} \left(F^T_{s'_j}, G^T_{s'_j}\right) \oplus \bigoplus_{k=1}^{q'} (I_{l'_k}, \Phi_{\eta_k}). \tag{34}$$

Let us prove that (34) coincides, after a suitable permutation of its summands, with (8) except for χ_k and η_k, and that (9) is fulfilled.

Since $\mathcal{B}$ and $\mathcal{B}'$ are equivalent, by Lemma 1 (B'_1, B'_2) is the Kronecker canonical form of some pair (C_1, C_2) of the form (31) with $ad - bc \neq 0$. In view of Step 1, p'_1, p'_2, q' and all r'_i, s'_j, l'_k coincide with p_1, p_2, q and all r_i, s_j, l_k after a suitable permutation of the summands of (34).

The transformation (31) converts each summand (I_{l_k}, Φ_{χ_k}) of (8) to the matrix pair

$$(aI_{l_k} + b\Phi_{\chi_k}, cI_{l_k} + d\Phi_{\chi_k}), \tag{35}$$

which is equivalent to (I_{l_k}, Φ_{η_k}). The matrix $aI_{l_k} + b\Phi_{\chi_k}$ is nonsingular; this means that if Φ_{χ_k} is similar to some Jordan block $J_{l_k}(\lambda_k)$, then $a + b\lambda_k \neq 0$; we have the condition (10). Due to Lemma 2(a), $\eta_k(x)$ is represented in the form (9).

Conversely, let (B_1, B_2) of the form (8) and (34) coincide with except for χ_k and η_k that satisfy (9). By Lemma 2(b), the characteristic polynomials of the matrices

$$(cI_{l_k} + d\Phi_{\chi_k}) \cdot (aI_{l_k} + b\Phi_{\chi_k})^{-1} \tag{36}$$

and Φ_{η_k} are equal for each k. Since (I_{l_k}, Φ_{χ_k}) is indecomposable, in view of Step 1 the matrix pair (35) is indecomposable too, hence the matrix (36) is indecomposable with respect to similarity and its Frobenius canonical form is Φ_{η_k}. Therefore, each $\|I_{l_k} \mid \Phi_{\chi_k}\|$ is equivalent to $\|I_{l_k} \mid \Phi_{\eta_k}\|$, and so $\mathcal{A}$ is equivalent to $\mathcal{B}$. □

3. Proof of Theorem 2

Lemma 3. *A spatial matrix*

$$\mathcal{D}(u, v) := \left\| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \middle| \begin{matrix} 0 & v \\ 1 & u \end{matrix} \right\|, \quad u, v \in \mathbb{F}, \tag{37}$$

is equivalent to $\mathcal{D}(u', v')$ *if and only if there exist* $a, b, c, d \in \mathbb{F}$ *such that*

$$ad - bc \neq 0, \qquad a^2 + uab - vb^2 \neq 0 \tag{38}$$

and

$$u' = \frac{2ac + uad + ucb - 2vbd}{a^2 + uab - vb^2}, \qquad v' = \frac{-c^2 - ucd + vd^2}{a^2 + uab - vb^2}. \tag{39}$$

Proof. Notice that

$$\mathcal{D}(u, v) = \|I_2 \mid \Phi_\chi\|, \qquad \mathcal{D}(u', v') = \|I_2 \mid \Phi_\eta\|,$$

where

$$\chi(x) := x^2 - ux - v, \qquad \eta(x) := x^2 - u'x - v'.$$

"$\Longrightarrow$." Let $\|I_2 \mid \Phi_\chi\|$ and $\|I_2 \mid \Phi_\eta\|$ be equivalent. By Lemma 1, there exists a nonsingular matrix

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}, \quad ad - bc \neq 0,$$

such that the pairs

$$(aI_2 + b\Phi_\chi, cI_2 + d\Phi_\chi), \quad (I_2, \Phi_\eta) \tag{40}$$

are equivalent. Then $aI_2 + b\Phi_\chi$ is nonsingular; i.e.,

$$\det(aI_2 + b\Phi_\chi) = a^2 + uab - vb^2 \neq 0.$$

By Lemma 2(a), $\eta(x)$ satisfies (26), this means that for some nonzero ε

$$\begin{aligned} \eta(x) &= \varepsilon\big[(xa - c)^2 - u(xa - c)(d - xb) - v(d - xb)^2\big] \\ &= \varepsilon\big[x^2\big(a^2 + uab - vb^2\big) + x(-2ac - uad - ucb + 2vbd) + \big(c^2 + ucd - vd^2\big)\big] \\ &= x^2 - u'x - v'. \end{aligned} \tag{41}$$

Therefore, $\varepsilon = (a^2 + uab - vb^2)^{-1}$ and the conditions (38) and (39) hold true.

"$\Longleftarrow$." Conversely, let (38) and (39) hold. Then (41) is fulfilled and we have (26). By Lemma 2(b), the characteristic polynomials of

$$(cI_2 + d\Phi_\chi) \cdot (aI_2 + b\Phi_\chi)^{-1} \tag{42}$$

and Φ_η are equal. Since (42) is 2-by-2, this implies that its Frobenius canonical form is either Φ_η, or a direct sum of two 1-by-1 Frobenius blocks $\lambda I_1 \oplus \mu I_1$ for some $\lambda, \mu \in \mathbb{F}$.

In the last case, $\eta(x) = (x - \lambda)(x - \mu)$. But $\eta(x)$ is a power of an irreducible polynomial. Hence, $\lambda = \mu$ and (42) is λI_2. We get consecutively

$$cI_2 + d\Phi_\chi = \lambda(aI_2 + b\Phi_\chi), \qquad (c - \lambda a)I_2 = (\lambda b - d)\Phi_\chi,$$
$$c - \lambda a = \lambda b - d = 0, \qquad (c, d) = \lambda(a, b),$$

contrary to $ad - bc = 0$.

Therefore, (42) is similar to Φ_η, the pairs (40) are equivalent, and so $\|I_2 \mid \Phi_\chi\|$ is equivalent to $\|I_2 \mid \Phi_\eta\|$. □

Proof of Theorem 2. Let $\mathcal{A}$ be a regular $m \times n \times q$ matrix with $n \leqslant 2$ and $q \leqslant 2$.

Step 1. Let us prove that $\mathcal{A}$ is equivalent to at least one of the spatial matrices (12)–(20). This is clear if $\mathcal{A}$ is $m \times n \times 1$ with $n \leqslant 2$: indeed, since $\mathcal{A} = \|A\|$ is regular, it reduces by elementary transformations (4) to (12) or (13).

So we suppose that $\mathcal{A}$ is $m \times n \times 2$ with $n \leqslant 2$. By Theorem 1, $\mathcal{A}$ is equivalent to some $\mathcal{B} = \|B_1 \mid B_2\|$ with (B_1, B_2) of the form (8). Since $\mathcal{A}$ is regular, (8) does not have the summands (F_1, G_1) and (F_1^T, G_1^T). If $m = 1$ or $n = 1$, then (B_1, B_2) is (F_2, G_2) or (F_2^T, G_2^T), we have (15) or (14).

It remains to consider $\mathcal{A}$ of size $m \times 2 \times 2$ with $m \geqslant 2$. Then (B_1, B_2) is one of the pairs:

$$\left(F_3^T, G_3^T\right), \quad \left(F_2^T, G_2^T\right) \oplus \left(F_2^T, G_2^T\right), \quad \left(F_2^T, G_2^T\right) \oplus \left(I_1, J_1(\lambda)\right), \tag{43}$$
$$\left(I_1, J_1(\lambda)\right) \oplus \left(I_1, J_1(\mu)\right), \quad (I_2, \Phi_\chi). \tag{44}$$

The first and the second pairs give (17) and (18). In the third pair we take $\lambda = 0$ (because $\|1 \mid \lambda\|$ and $\|1 \mid 0\|$ are equivalent) and obtain (16). In the fourth pair, $\lambda \neq \mu$ since $\mathcal{A}$ is regular, and so it is equivalent to (I_2, Φ_χ) with $\chi(x) = (x - \lambda)(x - \mu)$.

Hence, the spatial matrices that are given by (44) are equivalent to $\mathcal{D}(u, v)$ defined in (37).

If $\operatorname{char} \mathbb{F} \neq 2$, then each $\mathcal{D}(u, v)$ is equivalent to $\mathcal{D}(0, v')$ for some v' due to Lemma 3: substituting

$$(a, b, c, d) := (1, 0, -u/2, 1)$$

in (38) and (39), we obtain $u' = 0$. This gives (19).

If $\operatorname{char} \mathbb{F} = 2$, then each $\mathcal{D}(u, v)$ is equivalent to $\mathcal{D}(0, v')$ or $\mathcal{D}(1, v')$: for each $u \neq 0$ we get $u' = 1$ putting

$$(a, b, c, d) := \left(1, 0, 0, u^{-1}\right)$$

in (38) and (39). This gives (19) and (20).

Step 2. Let us prove that $\mathcal{A}$ is equivalent to exactly one of the spatial matrices (12)–(20) up to replacements (21)–(23).

Let two distinct spatial matrices among (12)–(20) be equivalent. Then they have the same size, and so they are $3 \times 2 \times 2$ or $2 \times 2 \times 2$. The spatial matrices (16) and (17) are inequivalent in view of Theorem 1 since the corresponding decompositions (8) are $(F_2^T, G_2^T) \oplus (I_1, J_1(0))$ and (F_3^T, G_3^T). Hence, they are (19) or (20).

Let $\operatorname{char}\mathbb{F} \neq 2$, and let $\mathcal{A}(v)$ be equivalent to $\mathcal{A}(v')$. By Lemma 3 there exist a, b, c, d satisfying (38) and (39) with $u = u' = 0$. Then the equalities (39) ensure

$$ac - vbd = 0, \qquad v' = \frac{-c^2 + vd^2}{a^2 - vb^2},$$

and so

$$\begin{aligned} v'\big(a^2 - vb^2\big)^2 &= \big(a^2 - vb^2\big)\big(-c^2 + vd^2\big) = -a^2c^2 + a^2vd^2 + vb^2c^2 - v^2b^2d^2 \\ &= -(ac - vbd)^2 + v(ad - bc)^2 = v(ad - bc)^2. \end{aligned}$$

We have (21) with

$$z = \left(\frac{a^2 - vb^2}{ad - bc}\right)^2.$$

Conversely, if (21) holds, then $\mathcal{A}(v)$ is equivalent to $\mathcal{A}(v')$ due to Lemma 3 since the conditions (38) and (39) are fulfilled with

$$u = u' = 0, \qquad (a, b, c, d) := \big(1, 0, 0, z^{-1/2}\big).$$

Let $\operatorname{char}\mathbb{F} = 2$. If $\mathcal{D}(0, v)$ is equivalent to $\mathcal{D}(u', v')$, then by (39) $u' = 0$. Hence $\mathcal{A}(v)$ and $\mathcal{B}(v')$ (defined in (19) and (20)) are inequivalent for all v and v'.

Due to Lemma 3, $\mathcal{A}(v)$ and $\mathcal{A}(v')$ are equivalent if and only if the conditions (38) and (39) with $u = u' = 0$ hold for some $a, b, c, d \in \mathbb{F}$. The first condition in (39) is the identity, putting

$$(\alpha, \beta, \gamma, \delta) := \big(d^2, c^2, b^2, a^2\big)$$

in the other conditions gives the conditions (22).

Let $\mathcal{B}(v)$ be equivalent to $\mathcal{B}(v')$. Then there exist a, b, c, d such that the conditions (38) and (39) hold for $u = u' = 1$.

We first suppose that $b = 0$. The conditions (39) take the form $1 = ad/a^2$ (and so $a = d \neq 0$) and

$$v' = \frac{c^2 + ca + va^2}{a^2} = v + \frac{c}{a} + \frac{c^2}{a^2};$$

this gives (23) with $\beta = c/a$.

Let now $b \neq 0$. Denote

$$\alpha := a/b, \qquad \gamma := c/b, \qquad \delta := d/b.$$

Remembering that $\operatorname{char}\mathbb{F} = 0$ and $u = u' = 1$, rewrite (39) in the form

$$1 = \frac{\alpha\delta + \gamma}{\alpha^2 + \alpha + v}, \qquad v' = \frac{\gamma^2 + \gamma\delta + v\delta^2}{\alpha^2 + \alpha + v}.$$

From the first equality determine

$$\gamma = v + \alpha + \alpha^2 + \alpha\delta,$$

substitute it to the second:

$$\begin{aligned} v' &= \frac{(v+\alpha+\alpha^2)^2 + (\alpha\delta)^2 + (v+\alpha+\alpha^2)\delta + \alpha\delta^2 + v\delta^2}{\alpha^2+\alpha+v} \\ &= v + \alpha + \alpha^2 + \delta + \delta^2 = v + (\alpha+\delta) + (\alpha+\delta)^2, \end{aligned}$$

and obtain (23).

Conversely, let $v' = v + \beta + \beta^2$ for some nonzero $\beta \in \mathbb{F}$. The conditions (38) and (39) hold for $u = u' = 1$ and

$$(a, b, c, d) := \begin{cases} (\beta, 1, v', 0) & \text{if } v' \neq 0, \\ (1, 0, \beta, 1) & \text{If } v' = 0; \end{cases}$$

hence $\mathcal{B}(v)$ and $\mathcal{B}(v')$ are equivalent by Lemma 3.

Step 3. Let $\mathbb{F}$ be algebraically closed. Then $\mathbb{F}^2 = \mathbb{F}$. If $\operatorname{char}\mathbb{F} \neq 2$, each $\mathcal{A}(v)$ is equivalent to $\mathcal{A}(0)$ or $\mathcal{A}(1)$ (if $v \neq 0$, we put $z = 1/v$ in (21)). The spatial matrix $\mathcal{A}(1)$ is equivalent to (24) since it reduces to (24) by the following transformations: add the first slice I_2 to the second, reduce the second to the form $J_1(0) \oplus J_1(2)$ by simultaneous similarity transformations with the slices, divide the second by 2, and subtract the second slice from the first:

$$\mathcal{A}(1) \to \left\| \begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array} \right\| \to \left\| \begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{array} \right\| \to \left\| \begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{array} \right\| \to \left\| \begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right\|.$$

Suppose $\operatorname{char}\mathbb{F} = 2$. Then all $\mathcal{A}(v)$ are equivalent to $\mathcal{A}(0)$ since the conditions (22) hold for $v' = 0$ and $(\alpha, \beta, \gamma, \delta) := (1, v, 0, 1)$. All $\mathcal{B}(v)$ are equivalent to $\mathcal{B}(0)$ since Eq. (23) with $v' = 0$ is solvable for β. We reduce the second slide of $\mathcal{B}(0)$ to the form $J_1(0) \oplus J_1(1)$ by simultaneous similarity transformations with the slices, and then subtract the second slice from the first. □

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