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On the ultimate normalized chromatic difference sequence of a graph

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Abstract

For graphs G and H, the Cartesian product $G \times H$ is defined as follows: the vertex set is $V(G) \times V(H)$, and two vertices (g,h) and (g',h') are adjacent in $G \times H$ if either $g = g'$ and $hh' \in E(H)$ or $h = h'$ and $gg' \in E(G)$. Let G^k denote the Cartesian product of k copies of G. The chromatic difference sequence $cds(G)$ is defined by $cds(G) = (a_1, a_2 - a_1, \ldots, a_t - a_{t-1}, \ldots)$ where a_t denotes the maximum number of vertices of t-colorable subgraph of G. The normalized chromatic difference sequence $ncds(G)$ is defined by $ncds(G) = cds(G)/|V(G)|$. This paper studies the ultimate normalized chromatic difference sequence of a graph $NCDS(G)$ which is equal to the limit of $ncds(G^k)$ as k goes to infinity. We study $NCDS(G)$ under the context of other graph theoretical properties: star chromatic number, hom-regularity, and graph homomorphism. We have provided new upper and lower bounds for $NCDS(G)$. We have also proved, among others, that if there is a homomorphism from a graph G to a graph H, then $NCDS(G)$ dominates $NCDS(H)$.

1. Introduction

For a graph G, $\alpha_t(G)$ denotes the maximum number of vertices of t-colorable subgraph of G, $i_t(G)$ the *t-coloring ratio* of G (i.e., $i_t(G) = \alpha_t(G)/|V(G)|$), and $\chi = \chi(G)$ the *chromatic number* of G. The *chromatic difference sequence* $cds(G)$ [1] is defined by

$$cds(G) = (\alpha_1(G), \alpha_2(G) - \alpha_1(G), \ldots, \alpha_t(G) - \alpha_{t-1}(G), \ldots, \alpha_\chi(G) - \alpha_{\chi-1}(G)).$$

The *normalized chromatic difference sequence* $ncds(G)$ is defined by

$$ncds(G) = cds(G)/|V(G)|$$
$$= (i_1(G), i_2(G) - i_1(G), \ldots, i_t(G) - i_{t-1}(G), \ldots, i_\chi(G) - i_{\chi-1}(G)).$$

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The n-term sequence (x_k) is said to *dominate* the n-term sequence (y_k), written $(x_k) \succcurlyeq (y_k)$ or $(y_k) \preccurlyeq (x_k)$, if:

$$(1) \qquad \sum_{k=1}^{n} x_k = \sum_{k=1}^{n} y_k,$$

and

$$(2) \qquad \sum_{k=1}^{p} x_k \geqslant \sum_{k=1}^{p} y_k \quad \text{for } p = 1, 2, \ldots, n-1.$$

The n-term sequence (y_k) is said to be between the n-term sequence (x_k) and (z_k) if either $(x_k) \succcurlyeq (y_k) \succcurlyeq (z_k)$ or $(x_k) \preccurlyeq (y_k) \preccurlyeq (z_k)$. For graphs G and H, the *Cartesian product* $G \times H$ is defined as follows: the vertex set is $V(G) \times V(H)$, and two vertices (g,h) and (g',h') are adjacent in $G \times H$ just if either $g = g'$ and $hh' \in E(H)$ or $h = h'$ and $gg' \in E(G)$. We use G^k to denote the *Cartesian product* of k copies of G. We are interested in the *ultimate normalized chromatic difference sequence* $NCDS(G)$ of a graph G, defined by

$$NCDS(G) = \lim_{k \to \infty} ncds(G^k).$$

If we denote $I_t(G) = \lim_{k\to\infty} i_t(G^k)$, then $NCDS(G) = (I_1(G), I_2(G) - I_1(G), \ldots, I_t(G) - I_{t-1}(G), \ldots, 1 - I_{\chi-1}(G))$. We note that $ncds(G \times H) \preccurlyeq ncds(G)(ncds(H))$, $ncds(G^k)$ is nonincreasing with respect to k in the sense of dominance, and so the limit $NCDS(G)$ always exists and lies between $ncds(G)$ and the flat sequence $(1/\chi(G))(1,1,\ldots,1)$, by Theorem 4.1, Corollary 4.2, and Corollary 4.3 of [11].

A *homomorphism* of G to H is a mapping $f : V(G) \to V(H)$ such that $gg' \in E(G)$ implies $f(g)f(g') \in E(H)$. We write $G \to H$ to denote that there is a homomorphism of G to H. A homomorphism is a useful tool in studying the *NCDS* as well as the *ncds*, see also [12].

The study of the ultimate normalized chromatic difference sequence can be viewed in the spirit of investigating the limiting behaviour of graph parameters under graph products. The work in [4, 5, 7–9] deal with other graph theoretical parameters of other types of graph products.

We have some partial results in [11], and will contribute more results in this paper, in which the limit *NCDS* can be evaluated. In all our results, both in [11] and in this paper, the limit is actually equal to either the upper or the lower bound. In [11], we work on the classes of graphs whose *cds* can be calculated. In this paper, we work mainly on the sufficient conditions of the graphs whose *ncds* is stable, i.e., *NCDS* = *ncds*, see Theorems 7 and 9. We also obtain a sufficient condition under which *NCDS* reaches the lower bound mentioned above, see Corollary 14. We obtain new lower and upper bounds for *NCDS* in the sense of dominance: see Theorem 1 which gives the lower bound in terms of star chromatic number and chromatic number; see Corollary 13 which gives the upper bound in terms of maximum clique number. Both Corollaries 13 and 14 are derived from the main theorem of this paper: Theorem 10, i.e., if there is a homomorphism from a graph G to a graph H, then $NCDS(G)$ dominates $NCDS(H)$. Our main ideas originate from [4,6,13] which concentrated on the first term of $NCDS$.

2. *NCDS* and star chromatic numbers

We start with the definition of the star chromatic number of a graph [3,10]. Let k and d be positive integers such that $k \geqslant 2d$. Set $[k] = \{0, 1, \ldots, k-1\}$. A (k,d)-coloring of a graph $G = (V,E)$ is a mapping $c : V \to [k]$ such that, for each edge $(u,v) \in E, d \leqslant |c(u) - c(v)| \leqslant k - d$. The *star chromatic number* $\chi^*(G)$ of G is defined by $\chi^*(G) = \inf\{k/d : G \text{ has a } (k,d)\text{-coloring}\}$, and can be calculated by

$$\chi^*(G) = \min\{k/d : G \text{ has a } (k,d)\text{-coloring for } 2d \leqslant k \leqslant |V(G)|\}.$$

It has been proved that $\chi(G) - 1 < \chi^*(G) \leqslant \chi(G)$. It has been further proved that a graph G is (k,d)-colorable if and only if there is a homomorphism from G to G_k^d, where G_k^d has vertex set $\{0, 1, \ldots, k-1\}$ and edge set $\{(i,j) : d \leqslant |i-j| \leqslant k-d \text{ for } i,j \in [k]\}$. See [3,10] for details. Since $NCDS(G) = ncds(G)$ for any circulant graph G [11] it follows that

$$NCDS(G_k^d) = ncds(G_k^d) = \left(\frac{d}{k}, \frac{d}{k}, \ldots, \frac{d}{k}, \frac{k - \lfloor \frac{k}{d} \rfloor d}{k}\right).$$

Therefore, we can apply a result of Albertson and Collins [2], i.e., if H is vertex transitive and $G \to H$, then $ncds(G) \geqslant ncds(H)$, to obtain a new lower bound for the $NCDS$ in the sense of dominance.

Theorem 1. *For any graph G,*

$$NCDS(G) \geqslant \left(\frac{1}{\chi^*}, \frac{1}{\chi^*}, \ldots, \frac{1}{\chi^*}, 1 - \frac{\chi - 1}{\chi^*}\right),$$

where $\chi = \chi(G)$ and $\chi^* = \chi^*(G)$.

As corollaries, we get Theorem 1 of [13], i.e., $I_1(G) \geqslant 1/\chi^*(G)$ for any graph G, and that $\chi(G) = \chi^*(G)$ provided $I_1(G) = 1/\chi(G)$.

3. *NCDS* and hom-regular graphs

For graphs G and H, a *t-colorable subgraph cover* of G with respect to H is a family $\{S_h : h \in V(H)\}$ such that

(i) each S_h is a maximum t-colorable subgraph in G,

(ii) $\bigcap_{h \in V(H')} S_h = \emptyset$ for any $(t+1)$-chromatic subgraph H' of H, and

(iii) for each S_h, $h \in V(H)$, there exists a t-coloring c_{S_h} of S_h such that for any subgraph H' of H with $\chi(H') \leqslant t$, any $v \in \bigcap_{h \in V(H')} S_h$, there exists a proper coloring $c_{H'}$ of H' such that $c_{S_h}(v) = c_{H'}(h)$ for any $h \in H'$.

It is not hard to check that the conditions (i) and (ii) are equivalent to the condition (iv):

(iv) For each S_h, $h \in V(H)$, there exists a t-coloring c_{S_h} of S_h such that for any subgraph H' of H, if $\bigcap_{h\in V(H')} S_h \neq \emptyset$, then for any $v \in \bigcap_{h\in V(H')} S_h$, the coloring defined by $c_{H'}(h) = c_{S_h}(v)$ is a proper coloring of H'.

The condition (iv) is also equivalent to the following condition (v):

(v) There is a family of t-coloring c_h: $S_h \to \{1,\dots,t\}$ such that if $hh' \in E(H)$ and $v \in S_h \cap S_{h'}$, then $c_h(v) \neq c_{h'}(v)$.

A t-colorable subgraph cover of G is just a t-colorable subgraph cover of G with respect to itself.

For graphs G and H, if, for any t : $1 \leqslant t \leqslant \chi(G) - 1$, there exists a t-colorable subgraph cover of G with respect to H, then we say that G has a *chromatic-complete subgraph cover* with respect to H. A chromatic-complete subgraph cover of G is just a chromatic-complete subgraph cover of G with respect to itself. We have already proved that $i_t(G \times H) \leqslant i_t(G)$ (see 11, Theorem 4.1] or the argument contained in the proof of the following proposition). Furthermore, we have the following proposition.

Proposition 2. *For* $1 \leqslant t \leqslant \chi(G) - 1$, $i_t(G \times H) = i_t(G)$ *if and only if* G *has a* t-*colorable subgraph cover with respect to* H.

Proof. Since the restriction of a maximum t-colorable subgraph of $G \times H$ on $V(G) \times \{h\}$ is a t-colorable subgraph for $h \in V(H)$, it follows that $\alpha_t(G \times H) \leqslant |V(H)|\alpha_t(G)$. If G has a t-colorable subgraph cover $\{S_h : h \in V(H)\}$ with respect to H, then it is easy to check that the union of the sets $S_h \times \{h\}$ is a t-colorable subgraph of $G \times H$ of cardinality $|V(H)|\alpha_t(G)$. Hence $\alpha_t(G \times H) = |V(H)|\alpha_t(G)$. Conversely, if $\alpha_t(G \times H) = |V(H)|\alpha_t(G)$, then take a maximum t-colorable subgraph S of $G \times H$. Let $S_h = \{g : (g,h) \in S\}$. By the pigeon hole principle, it is easy to see that each S_h is a maximum t-colorable subgraph of G. Let H' be a $(t+1)$-chromatic subgraph of H. Then $\bigcap_{h\in V(H')} S_h = \emptyset$. Otherwise, let $v \in \bigcap_{h\in V(H')} S_h$, then $\{v\} \times V(H')$ induces a $(t+1)$-chromatic subgraph of S. This is a contradiction since S is t-colorable. The restriction of a t-coloring c_S of S on S_h is a t-coloring c_{S_h} on S_h. For any subgraph H' of H with $\chi(H') \leqslant t$, any $v \in \bigcap_{h\in V(H')} S_h$, there is a proper coloring of H' defined by $c_{H'}(h) = c_S((v,h)) = c_{S_h}(v)$ to satisfy (iii). Therefore, $\{S_h : h \in V(H)\}$ is a t-colorable subgraph cover of G with respect to H. □

We now focus on a particular class of graphs. We say that G is hom-regular if $G^2 \to G$. The importance of these graphs can be seen by the following facts:

Proposition 3. *If* G *is hom-regular and* $1 \leqslant t \leqslant \chi(G) - 1$, *then* $I_t(G) = i_t(G)$ *if and only if* $i_t(G^2) = i_t(G)$.

Proof. If $I_t(G) = i_t(G)$, then clearly $i_t(G^2) = i_t(G)$. Assume that $i_t(G^2) = i_t(G)$. Since G is hom-regular, we have $G^k \to G$ by induction. Let f be a homomorphism of G^k to G, and let $\{S_g : g \in V(G)\}$ be a t-colorable subgraph cover of G. Such a cover exists by Proposition 2 and the fact that $i_t(G^2) = i_t(G)$. If we can prove that $\{S_{f(u)} : u \in V(G^k)\}$ is a t-colorable subgraph cover of G with respect to G^k, then $i_t(G^k) = i_t(G)$ for every k.

It is easy to see that for any $u \in V(G^k), f(u) \in V(G), S_{f(u)}$ is a maximum t-colorable subgraph. For any subgraph H' of G^k with $\chi(H') = t+1$, the graph $f(H')$ induced by $f(V(H'))$ in G has chromatic number at least $t+1$. Therefore, $\bigcap_{u \in V(H')} S_{f(u)} = \bigcap_{g \in f(V(H'))} S_g = \emptyset$. For any subgraph H' of G^k with $\chi(H') \leqslant t$, if $\chi(f(H')) > t$, then $\bigcap_{u \in V(H')} S_{f(u)} = \emptyset$. So we may assume that $\chi(f(H')) \leqslant t$. For any $v \in \bigcap_{u \in V(H')} S_{f(u)} = \bigcap_{g \in f(V(H'))} S_g$, we need to prove that there exists a proper coloring $c_{H'}$ of H' such that $c_{S_{f(u)}}(v) = c_{H'}(u)$. Since $\{S_g : g \in V(G)\}$ is a t-colorable subgraph cover of G, there exists a proper coloring $c_{f(H')}$ of $f(H')$ such that $c_{S_g}(v) = c_{f(H')}(g)$. Now the composition of $c_{f(H')}$ and f, $c_{H'} = c_{f(H')} \cdot f$, is a proper coloring of H' such that $c_{S_{f(u)}}(v) = c_{H'}(u)$. □

Corollary 4. *If G is hom-regular, then $NCDS(G) = ncds(G)$ if and only if $ncds(G^2) = ncds(G)$.*

A graph G is a core if each homomorphism $G \to G$ is an automorphism of G, i.e., is a bijection. For hom-regular cores, we show that $I_t(G) = i_t(G)$ for $t = 1, 2, \ldots, \chi(G)-1$, i.e., $NCDS(G) = ncds(G)$. We need to introduce the concept of $Aut(G)$, the automorphism graph of G: The vertices of $Aut(G)$ are automorphisms of G, and ff' is an edge of $Aut(G)$ just if $f(g)f'(g) \in E(G)$ for each vertex g of G.

Proposition 5. *A core G is hom-regular if and only if $G \to Aut(G)$.*

Proof. See [6] for the proof. It is also mentioned in [6] that hom-regular cores have a more standard kind of regularity: Any hom-regular core is vertex transitive. □

Proposition 6. *If $G \to Aut(G)$, then G has a chromatic-complete subgraph cover.*

Proof. We prove that for every $t, 1 \leqslant t \leqslant \chi(G) - 1$, G has a t-colorable subgraph cover. Let $f : G \to Aut(G)$ be a homomorphism, and S a maximum t-colorable subgraph of G. We prove that the family $\{f(g)(S) : g \in V(G)\}$ is a t-colorable subgraph cover of G.

Each $f(g)(S)$ is a maximum t-colorable subgraph of G since $f(g)$ is an automorphism and S is a maximum t-colorable subgraph of G. In order to prove that the family $\{f(g)(S) : g \in V(G)\}$ satisfies (ii) and (iii) required by the definition of t-colorable subgraph cover, we prove the following fact first. Let K be a subgraph of G and $v \in \bigcap_{g \in V(K)} (f(g))(S)$. Assume further that

$$f(g_1)(s_1) = f(g_2)(s_2) = \cdots = f(g_m)(s_m) = v,$$

where $V(K) = \{g_1, g_2, \ldots, g_m\}$ and $s_i \in V(S)$ for $i = 1, 2, \ldots, m$. Define $\psi : \psi(g_i) = s_i$ $(i = 1, 2, \ldots, m)$. We claim that ψ is a homomorphism from K to S. Let $g_i g_j$ be an edge of the subgraph K, where $i, j \in \{1, 2, \ldots, m\}$ and $i \neq j$. Then $f(g_i)f(g_j)$ is an edge of $Aut(G)$. First, we prove that $s_i \neq s_j$. Otherwise, $vv = f(g_i)(s_i)f(g_j)(s_j) = f(g_i)(s_i)$ $f(g_j)(s_i) \in E(G)$. This is a contradiction. Second, we prove that $s_i s_j \in E(G)$. Let $f(g_i)(s_j) = w$. Then $f(g_i)(s_j)f(g_j)(s_j) = wv \in E(G)$. Since $f(g_i)$ is an automorphism, we have $s_i s_j = (f(g_i))^{-1}(v)(f(g_i))^{-1}(w) \in E(G)$. Therefore, ψ is a homomorphism.

Now we can conclude that $\bigcap_{g \in V(K)} f(g)(S) = \emptyset$ for any $(t+1)$-chromatic subgraph K of G. For otherwise, there exists a homomorphism from K to S, which implies $t + 1 = \chi(K) \leqslant \chi(S)$, a contradiction. For checking the condition (iii), we note that if c is a t-coloring of S, then there is a natural t-coloring $c_{f(g)(S)}$ of $f(g)(S)$ defined by $c_{f(g)(S)}[f(g)(s)] = c(s)(s \in S)$ for every $g \in V(G)$. For any j-chromatic subgraph K of G $(1 \leqslant j \leqslant t)$, any $v \in \bigcap_{g \in V(K)} f(g)(S)$, let $V(K) = \{g_1, g_2, \ldots g_m\}$, and $s_i \in S$ $(i = 1, 2, \ldots, m)$ such that $f(g_i)(s_i) = v$ $(i = 1, 2, \ldots, m)$. As we proved above, the mapping ψ defined by $\psi(g_i) = s_i$ $(i = 1, 2, \ldots, m)$ is a homomorphism from K to S. Hence we can define a coloring c_K on K by $c_K(g_i) = c \cdot \psi(g_i) = c(s_i)$ for $i = 1, 2, \ldots, m$. Now it is obvious that $c_K(g_i) = c(s_i) = c_{f(g_i)(S)}[f(g_i)(s_i)] = c_{f(g_i)(S)}(v)$ for any $g_i \in V(K)$. □

Theorem 7. *A hom-regular core G has NCDS(G) = ncds(G).*

Proof. A hom-regular core G has $G \to Aut(G)$ by Proposition 5 and a chromatic-complete subgraph cover by Proposition 6. Now Proposition 2 implies that $i_t(G^2) = i_t(G)$, and Proposition 3 that $I_t(G) = i_t(G)$ for $t = 1, 2, \ldots, \chi(G) - 1$. □

It is easy to see that a Cayley graph G of a commutative group has $G \to Aut(G)$ (using left multiplications). Thus if G is also a core, $I_t(G) = i_t(G)$. We will see below that the condition of being a core is not necessary.

Let $V(H)$ be a commutative group, with the operation written as $+$. A *strong t-colorable subgraph cover* of G with respect to H (or just "of G" if $G = H$) is a t-colorable subgraph cover $\{S_h : h \in V(H)\}$ of G with respect to H, such that

(a) for any $(t+1)$-chromatic subgraph K of H, $\bigcap_{g \in V(K)} S_{g+x} = \emptyset$ for any $x \in V(H)$; and

(b) for any $u \in V(H)$, there exists a t-coloring c_{S_u} of S_u such that for any $x \in V(H)$, any j-chromatic subgraph K of H $(1 \leqslant j \leqslant t)$, and any $v \in \bigcap_{g \in V(K)} S_{g+x}$, there exists a j-coloring c_K of K, which induces a natural j-coloring on $K + x$, $c_{K+x}(g + x) = c_K(g)$ for $g \in V(K)$, such that $c_{K+x}(g + x) = c_{S_{g+x}}(v)$ for any $g \in V(K)$.

For graphs G and H, if for any $t, 1 \leqslant t \leqslant \chi(G) - 1$, there exists a strong t-colorable subgraph cover of G with respect to H, then we say that G has a *strong chromatic-complete subgraph cover* with respect to H.

Proposition 8. *If G has a strong chromatic-complete subgraph cover, then each G^k $(k = 1, 2, \ldots.)$ has a strong chromatic-complete subgraph cover with respect to G.*

Proof. By induction on k. If $\{S_g : g \in V(G)\}$, is a strong t-colorable ($t \in \{1,2,\ldots,\chi(G)-1\}$) subgraph cover of G^k with respect to G, we define $\{S'_g : g \in V(G)\}$ as follows:

$$S'_g = \bigcup_{x \in V(G)} (S_{g+x} \times \{x\}).$$

It is easy to see that each S'_g ($g \in V(G)$) is a maximum t-colorable subgraph of G^{k+1}. Furthermore, we claim that $\{S'_g : g \in V(G)\}$ is a strong t-colorable subgraph cover of G^{k+1} with respect to G. Let K be a $(t+1)$-chromatic subgraph of G. Then

$$\bigcap_{g \in V(K)} \left(\bigcup_{y \in V(G)} (S_{g+x+y} \times \{y\}) \right) = \bigcup_{y \in V(G)} \left(\bigcap_{g \in V(K)} (S_{g+x+y} \times \{y\}) \right)$$
$$= \bigcup_{y \in V(G)} \left(\left(\bigcap_{g \in V(K)} S_{g+x+y} \right) \times \{y\} \right) = \emptyset$$

by induction hypothesis.

For any $v \in S'_g$, let $v = (u, y_v)$ where $u \in S_{g+y_v}, y_v \in V(G)$. We color v by the color of u in the t-coloring of S_{g+y_v}. Now assume that $x \in V(G)$, K is a j-chromatic subgraph of G ($j \leqslant t$), and

$$v \in \bigcap_{g \in V(K)} S'_{g+x} = \bigcap_{g \in V(K)} \left(\bigcup_{y \in V(G)} (S_{g+x+y} \times \{y\}) \right)$$
$$= \bigcup_{y \in V(G)} \left(\bigcap_{g \in V(K)} (S_{g+x+y} \times \{y\}) \right)$$
$$= \bigcup_{y \in V(G)} \left(\left(\bigcap_{g \in V(K)} S_{g+x+y} \right) \times \{y\} \right).$$

Then there exists $y_v \in V(G)$ such that $v \in (\bigcap_{g \in V(K)} S_{g+x+y_v}) \times \{y_v\}$, i.e., for any $g \in V(K)$, there exists $u \in S_{g+x+y_v}$ such that $v = (u, y_v)$. By applying the induction hypothesis and the definition of coloring $c_{S'_{g+x}}$ on S'_{g+x}, we have

$$c_{K+x}(g+x) = c_{S_{g+x+y_v}}(u) = c_{S'_{g+x}}(v)$$

for any $v \in V(K)$ and any $x \in V(G)$. Therefore, $\{S'_g : g \in V(G)\}$ is a strong t-colorable subgraph cover of G^{k+1} with respect to G. □

Now the following theorem follows from Propositions 2 and 8.

Theorem 9. *If G has a strong chromatic-complete subgraph cover, then $NCDS(G) = ncds(G)$. In particular, $NCDS(G) = ncds(G)$ for Cayley graphs of commutative groups, since it has a strong chromatic-complete subgraph cover.* □

4. *NCDS* and homomorphisms

If we get rid of the condition of vertex transitivity of the graph H in the so-called "no-homomorphism lemma" of [2] (see the statement of this lemma and the notation just before Theorem 1 of this paper), then the dominance will not hold. Let G be a triangle. Let H have vertices a, b, c, d, e, f; and edges $ab, bc, ca, db, dc, ea, ec, fa$ and fb. Then $G \to H, ncds(G) = \frac{1}{3}(1,1,1,)$, and $ncds(H) = (\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$. $ncds(G)$ does not dominate $ncds(H)$. It is very interesting that the dominance relationship between the *NCDS* of the two graphs will still hold.

Theorem 10. *If $G \to H$, then $NCDS(G) \geqslant NCDS(H)$.*

We shall begin by proving two propositions of independent interest.

Proposition 11. *Let G be a subgraph of H. Then $I_t(G) \geqslant I_t(H)$ for $t = 1, 2, \ldots, \chi(G)$.*

Proof. Without loss of generality we assume, in this proof and the proof of next proposition, that $V(G) = \{1, 2, \ldots, n\}$ and $V(H) = \{1, 2, \ldots, m\}$ are the vertex sets of G and H, respectively. For each $k \geqslant 1$, consider the subset S_k of $V(H^k)$ defined by

$$S_k = \{x : x_r \leqslant n \text{ for some } r = 1, \ldots, k\},$$

that is, the set of those vertices $x = (x_1, \ldots, x_k)$ of H^k for which at least one coordinate x_r belongs to $V(G)$. We claim that $i_t(G) \geqslant i_t(S_k)$.

In order to prove the claim, we partition S_k into $S_{k,1} \cup S_{k,2} \cup \cdots \cup S_{k,k}$, and show that $i_t(G) \geqslant i_t(S_{k,r})$ for each $r = 1, \ldots, k$. We define $S_{k,1} = \{x : x_1 \leqslant n\}$, and $S_{k,r} = \{x : x_r \leqslant n \text{ and } x_j > n \text{ for } j = 1, \ldots, r-1\}, r = 2, \ldots, k$. In other words, x belongs to $S_{k,r}$ just if r is its first coordinate with $x_r \leqslant n$.

Now observe that each $S_{k,r}$ is the disjoint union of sets of the form

$$\{(x_1, \ldots, x_{r-1}, y, x_{r+1}, \ldots, x_k) : y = 1, \ldots, n\},$$

where $x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_k$ are fixed and $x_j > m$ for $j < r$. Since each of these disjoint sets induces, in H^k, a graph isomorphic to G, $i_t(G) \geqslant i_t(S_{k,r})$ for each r, and hence also

$$\begin{aligned}
i_t(S_k) &= \frac{\alpha_t(S_k)}{|S_k|} \leqslant \frac{\alpha_t(S_{k,1}) + \alpha_t(S_{k,2}) + \cdots + \alpha_t(S_{k,k})}{|S_k|} \\
&= \frac{\alpha_t(S_{k,1})}{|S_{k,1}|}\frac{|S_{k,1}|}{|S_k|} + \cdots + \frac{\alpha_t(S_{k,k})}{|S_{k,k}|}\frac{|S_{k,k}|}{|S_k|} \\
&= i_t(S_{k,1})\frac{|S_{k,1}|}{|S_k|} + \cdots + i_t(S_{k,k})\frac{|S_{k,k}|}{|S_k|} \\
&\leqslant i_t(G)\left(\frac{|S_{k,1}|}{|S_k|} + \cdots + \frac{|S_{k,k}|}{|S_k|}\right) = i_t(G). \qquad (1)
\end{aligned}$$

This proves the claim.

To finish the proof of this lemma, observe that the complement of S_k in $V(H^k)$ contains $(m-n)^k$ vertices. Now clearly

$$\begin{aligned} i_t(H^k) &= \frac{\alpha_t(H^k)}{|V(H^k)|} \leqslant \frac{\alpha_t(S_k)+\alpha_t(H^k\backslash S_k)}{|V(H^k)|} \\ &= \frac{\alpha_t(S_k)}{|S_k|}\frac{|S_k|}{|V(H^k)|} + \frac{\alpha_t(H^k\backslash S_k)}{|V(H^k)|} \leqslant \frac{\alpha_t(S_k)}{|S_k|} + \frac{(H^k\backslash S_k)}{|V(H^k)|} \\ &= i_t(S_k) + \frac{|V(H^k)\backslash S_k|}{|V(H^k)|} = i_t(S_k) + \left(1-\frac{n}{m}\right)^k \\ &= i_t(G) + \left(1-\frac{n}{m}\right)^k. \end{aligned} \tag{2}$$

Taking the limit of both sides, as k goes to infinity, we obtain that $I_t(H) \leqslant i_t(G)$. For any integer $k \geqslant 1$, G^k is a subgraph of H^k. By the similar argument as above, we obtain that $I_t(H) \leqslant i_t(G^k)$. Let k goes to infinity, we obtain the desired conclusion. □

Let G be a graph on n vertices and let $p_1,\ldots,p_n$ be positive integers. We say that a graph H is a $(p_1,\ldots,p_n)$*-multiple* of G if it is obtained by replacing each vertex x_i of G by a set $x_{i1},\ldots,x_{ip_i}$ of new vertices with an edge betweeen x_{ij} and $x_{i'j'}$ if and only if there is an edge between x_i and $x_{i'}$ in G. A multiple is said to be *p-regular* if $p_1 = \cdots p_n = p$.

Proposition 12. *Let H be a multiple of a graph G. Then $I_t(G) = I_t(H)$ for $t = 1,2,\ldots,\chi(G)-1$.*

Proof. We prove the lemma in two steps.

(i) Assume first that H is a p-regular multiple of G and let S be a maximum t-colorable subgraph of G^k. Then

$$S' = \{(x_{1j_1},x_{2j_2},\ldots,x_{kj_k}) : (x_1,\ldots,x_k) \in S \text{ and } 1 \leqslant j_1,\ldots,j_k \leqslant p\}$$

is a t-colorable subgraph of G^k of size $p^k|S|$. Hence

$$i_t(H^k) \geqslant \frac{|S'|}{(np)^k} = \frac{|S|}{n^k} = i_t(G^k),$$

and $I_t(H) \geqslant I_t(G)$. Combining this with $I_t(H) \leqslant I_t(G)$ obtained from Lemma 11 (since G is a subgraph of H), we get $I_t(G) = I_t(H)$ in this case.

(ii) Let H be an arbitrary $(p_1,\ldots,p_n)$-multiple of G, and let $p = \max\{p_1,\ldots,p_n\}$. Let F be the regular p-multiple of G. By the preceding lemma, we have $I_t(F) \leqslant I_t(H) \leqslant I_t(G)$ since G is a subgraph of H, and H is a subgraph of F. We have already proved $I_t(F) = I_t(G)$ in part (i) and so we conclude that $I_t(G) = I_t(H) = I_t(F)$. □

We can now prove Theorem 10.

Proof of Theorem 10. Let f be a homomorphism from G to H. Let $F = f(G)$ be the image of G. Let W be a p-regular multiple of F with p sufficiently large ($p = |V(G)|$ is enough). Now F is a subgraph of H and G is a subgraph of W and so, by Lemma 11, $I_t(H) \leqslant I_t(F)$ and $I_t(W) \leqslant I_t(G)$. Using Lemma 12 we get $I_t(F) = I_t(W)$, and hence $I_t(H) \leqslant I_t(G)$. This argument is true for any $t = 1, 2, \ldots, \chi(G) - 1$. □

Corollary 13. *For any graph G,*

$$NCDS(G) \leqslant \left(\frac{1}{\omega(G)}, \frac{1}{\omega(G)}, \ldots, \frac{1}{\omega(G)}, 0, \ldots, 0\right),$$

where there are $\chi(G) - \omega(G)$ zero's and $\omega(G)$ denotes the size of a maximum complete subgraph of G.

Proof. Let K be a maximum complete subgraph of G. Then $K \to G$ and $NCDS(K) = (\frac{1}{\omega(G)})(1, 1, \ldots, 1)$ for a complete graph K by [11]. □

It follows that we can exactly evaluate $NCDS(G)$ for perfect graphs G. In fact, we have a more general result:

Corollary 14. *If $\omega(G) = \chi(G)$, then $NCDS(G) = (\frac{1}{\chi(G)})(1, 1, \ldots, 1)$.*

Proof. This follows from Corollary 13 and the lower bound $NCDS(G) \geqslant (\frac{1}{\chi(G)})(1, 1, \ldots, 1)$. □

In particular, $NCDS(G) = (\frac{1}{2}, \frac{1}{2})$ if G is bipartite.

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