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Connection problems for polynomial solutions of nonhomogeneous differential and difference equations[1]

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Abstract

We consider nonhomogeneous hypergeometric-type differential, difference and q-difference equations whose nonhomogeneity is a polynomial $Q_n(x)$. The polynomial solution of these problems is expanded in the $\{Q_n(x)\}$ basis, and also in a basis $\{P_n(x)\}$, related in a natural way with the homogeneous hypergeometric equation. We give an algorithm building a recurrence relation for the expansion coefficients in both bases that we solve explicitly in many cases involving classical orthogonal polynomials. Finally, some concrete applications and extensions are given. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let us consider the hypergeometric equation $\mathscr{L}_{2,\lambda}[z(x)] := (\sigma(x)D^2 + \tau(x)D + \lambda\mathscr{I})[z(x)] = 0$, where $D = d/dx$ is the derivative operator, $\sigma(x)$ and $\tau(x)$ are polynomials of at most second and first degree,

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respectively, and λ is a constant. When the conditions $\lambda=\lambda_n:=-\frac{1}{2}n((n-1)\sigma''(x)+2\tau'(x))$ and $\lambda\neq\lambda_k$ $(k=0,\dots,n-1)$ hold, this equation admits a unique (up to a multiplicative constant) polynomial solution $P_n(x)$ of degree n.

In this paper we shall be concerned with the corresponding nonhomogeneous equation defined as

$$\mathscr{L}_{2,\lambda}[y(x)]=Q_n(x), \tag{1}$$

where the right-hand side $Q_n(x)$ is a polynomial of degree n. Clearly, the conditions

$$\lambda\neq-\tfrac{1}{2}m((m-1)\sigma''(x)+2\tau'(x)), \quad m=0,1,\dots,n, \tag{2}$$

ensure the existence of an unique nth-degree polynomial solution of this nonhomogeneous problem.

On the other hand, the kind of problems defined in Eq. (1) in the context of orthogonal polynomials of a discrete variable will be also considered. In this case Eq. (1) becomes

$$\mathscr{D}_{2,\lambda}[y(x)]\equiv(\sigma(x)\Delta\nabla+\tau(x)\Delta+\lambda\mathscr{I})[y(x)]=Q_n(x) \tag{3}$$

being Δ the forward difference operator $(\Delta f(x)=f(x+1)-f(x))$ and ∇ the backward difference operator $(\nabla f(x)=\Delta f(x-1))$. Here $\sigma(x)$, $\tau(x)$ and λ are defined as in Eq. (1) and the conditions given in Eq. (2) also warranty the existence of an unique nth-degree polynomial solution of Eq. (3).

Moreover, these nonhomogeneous problems will be also studied in the framework of the exponential lattice $x(s)=q^s$ [10], in which case Eq. (1) is

$$\mathscr{Q}_{2,\lambda}[y(s)]\equiv\left(\sigma(s)\frac{\Delta}{\Delta x(s-1/2)}\frac{\nabla}{\nabla x(s)}+\tau(s)\frac{\Delta}{\Delta x(s)}+\lambda\mathscr{I}\right)[y(s)]=Q_n(s), \tag{4}$$

where $\sigma(s):=\sigma_2q^{2s}+\sigma_1q^s+\sigma_0$, $\tau(s):=\tau_1q^s+\tau_0$ and λ is a constant. Here

$$\lambda\neq-[m]_q([m-1]_q\sigma_2+q^{(m-1)/2}\tau_1), \quad m=0,1,\dots,n, \tag{5}$$

are the conditions for the existence of a unique nth-degree polynomial solution, where $[m]_q$ stands for the q-numbers given by $[m]_q=(q^{m/2}-q^{-m/2})/(q^{1/2}-q^{-1/2})$.

Nonhomogeneous equations like the ones defined by Eqs. (1), (3) and (4) appear in a very natural way in many problems of orthogonal polynomials theory. One of the most important example corresponds to the case when the polynomial $Q_n(x)$ is a hypergeometric-type polynomial (continuous, discrete or their q-analogue). An interesting particular case of this situation is related to the first associated polynomials which, as shown in [12], are solutions of nonhomogeneous equations of the form given by Eq. (1) or (3), having as right-hand side the derivative (or difference-derivative) of classical polynomials.

Another interesting example appears in the study of orthogonal polynomials $\{P_n^S(x)\}$ with respect to the *nondiagonal* Sobolev inner product [13]

$$\langle f(x),g(x)\rangle_S=\int_I\mathscr{L}_{2,\lambda}[f(x)]\mathscr{L}_{2,\lambda}[g(x)]\rho(x)\,dx,$$

where $\rho(x)$ is a classical weight function. In this case, the family $\{P_n^S(x)\}$ is solution of a nonhomogeneous problem as the one defined by Eq. (1), i.e.

$$\mathscr{L}_{2,\lambda}[P_n^S(x)]=Q_n(x), \tag{6}$$

where now $\{Q_n(x)\}$ is a classical orthogonal sequence with respect to $\rho(x)$.

Main aim of this paper is to show how the problem of finding polynomial solutions of these kinds of nonhomogeneous equations is equivalent to the one of solving certain connection problems between some polynomial families closely related with the corresponding homogeneous equation. Interest of this equivalence comes from the existence of an algorithm developed in [1, 4, 14, 15, 17] and also in [6, 7] which allows us to solve recurrently these connection problems and so, to find the searched polynomial solution of the nonhomogeneous equation.

The outline of the paper is as follows: In Section 2 the aforementioned equivalence is shown to be an almost direct consequence of the structure of the nonhomogeneous equations we are dealing with. Moreover, the algorithm for solving the corresponding connection problems [1, 4, 14, 15, 17] is briefly described. In Section 3, we consider several examples of the nonhomogeneous equations of type (1), (3) and (4), which are explicitly solved. Section 4 is devoted to the study of some particular cases involving first associated polynomials and a family of *nondiagonal* Sobolev orthogonal polynomials. Finally, an extension is given in Section 5 where some representation problems appearing in *diagonal* Sobolev orthogonal polynomial context are considered.

2. Connection problem approaches for solving nonhomogeneous equations

As pointed out in the introduction, we consider the nonhomogeneous problem

$$\mathscr{R}_{2,\lambda}[y(x)] \equiv (\sigma(x)D^2 + \tau(x)D + \lambda\mathscr{I})[y(x)] = Q_n(x), \tag{7}$$

where the hypergeometric-type operator $\mathscr{R}_{2,\lambda}$ can be either $\mathscr{L}_{2,\lambda}$, $\mathscr{R}_{2,\lambda}$ or $\mathscr{Q}_{2,\lambda}$ defined in Eqs. (1), (3) or (4), respectively, where now D denotes

$$D \equiv \frac{\mathrm{d}}{\mathrm{d}x},\ \Delta,\ \text{or}\ \frac{\Delta}{\Delta x(s)} \quad \text{and} \quad D^2 \equiv \frac{\mathrm{d}^2}{\mathrm{d}x^2},\ \Delta\nabla,\ \text{or}\ \frac{\Delta}{\Delta x(s-1/2)}\frac{\nabla}{\nabla x(s)},$$

if we are dealing with the continuous, discrete or q-analogue case, respectively. In what follows we assume that conditions given by Eq. (2) or Eq. (5) hold, so that Eq. (7) always has an unique nth-degree polynomial solution.

Moreover, the right-hand side ($Q_n(x)$ in Eq. (7)) is a hypergeometric-type polynomial satisfying the equation $\mathscr{S}[Q_n(x)] = 0$, where the coefficients defining this $\mathscr{S}$ operator are, in general, different from the ones characterizing the above $\mathscr{R}_{2,\lambda}$.

In these conditions two different approaches can be devised to solve Eq. (7) by means of a specific connection problem. An extension of these techniques to the case in which neither the operator $\mathscr{R}_{2,\lambda}$ nor the polynomial $Q_n(x)$ are of hypergeometric-type will be given in Section 5.

2.1. First method

Let us consider the sequence of hypergeometric-type operators $\{\mathscr{R}_{2,\lambda_m}\}_{m=0}^{n}$ obtained from Eq. (7) by choosing $\lambda = \lambda_m := -(m/2)((m-1)\sigma'' + 2\tau')$ in the continuous and discrete cases or $\lambda = \lambda_m := -[m]_q([m-1]_q\sigma_2 + q^{(m-1)/2}\tau_1)$ in the q-situation. Assuming that, for each $m = 0, 1, \ldots, n$, the conditions $\lambda_m \neq \lambda_k$ $(k = 0, 1, \ldots, m-1)$ are satisfied, this sequence of operators defines a unique family of monic hypergeometric-type polynomials, to be denoted by $\{P_m(x)\}$, each of them satisfying the homogeneous equation

$$\mathscr{R}_{2,\lambda_m}[P_m(x)]=0, \quad m=0,1,\ldots,n. \tag{8}$$

The nth-degree polynomial solution $y(x)$ of the non-homogeneous equation (7) can be expanded now in the $\{P_m(x)\}$ family as

$$y(x)=\sum_{m=0}^{n} C_m(n)P_m(x), \tag{9}$$

so the obtention of coefficients $C_m(n)$ gives the solution of Eq. (7). To compute these expansion coefficients, we insert Eq. (9) into Eq. (7) to obtain

$$Q_n(x)=\mathscr{R}_{2,\lambda}\left[\sum_{m=0}^{n} C_m(n)P_m(x)\right]=\sum_{m=0}^{n} C_m(n)\left[(\lambda-\lambda_m)P_m(x)\right], \tag{10}$$

where the last equality is a consequence of Eq. (8). Notice that, in this latter equation, $\lambda\neq\lambda_m$ ($m=0,1,\ldots,n$), which is the condition for the existence of a unique nth-degree polynomial solution of the nonhomogeneous equation (7). In this way the relation (10) shows that the solution of Eq. (7) can be computed by solving the connection problem using our algorithm briefly summarized as follows (see [1, 4, 14, 15, 17] for details). Since $Q_n(x)$ is solution of an equation of hypergeometric-type, i.e. $\mathscr{S}[Q_n(x)]=0$, action of this $\mathscr{S}$ operator on Eq. (10) gives rise to the identity:

$$0=\sum_{m=0}^{n} C_m(n)(\lambda-\lambda_m)\mathscr{S}[P_m(x)],$$

where the expression $\mathscr{S}[P_m(x)]$ is a sum of terms of the form $x^jD^i[P_m]$ $(0\leqslant i,j\leqslant 2)$. Then, using the properties satisfied by the $\{P_m(x)\}$ family (basically, the three term recurrence relation and the structure relation [4, 15]) these terms can be written as linear combination with constants coefficients of the polynomials $P_m(x)$ themselves (or its first or second derivatives, see [1, 4, 14, 15, 17] for a complete description). In doing so, we arrive to a triangular linear system which is solved recurrently giving all $C_m(n)$ coefficients $(m=0,1,\ldots,n-1)$ in terms of the $C_n(n)$ one, which is computed by direct matching in Eq. (10). It is clear that if $Q_n(x)$ is monic then $C_n(n)=1/(\lambda-\lambda_n)$.

We should mention here that these kind of connection problems could be also solved by using another existing methods, e.g. [5, 6, 8].

2.2. Second method

An alternative approach to the method we have just described can be derived when the right-hand side $Q_n(x)$ in Eq. (7) belongs to a hypergeometric-type polynomial family $\{Q_m(x)\}_{m=0}^{n}$ satisfying for each $m=0,1,\ldots,n$

$$\bar{\mathscr{S}}_{2,\bar{\lambda}_m}[Q_m(x)]\equiv(\bar{\sigma}(x)D^2+\bar{\tau}(x)D+\bar{\lambda}_m\mathscr{I})[Q_m(x)]=0,$$

where it is assumed that $\bar{\lambda}_m=-(m/2)((m-1)\bar{\sigma}''(x)+2\bar{\tau}'(x))$ and $\bar{\lambda}_m\neq\bar{\lambda}_k$ $(k=0,1,\ldots,m-1)$. In this situation, expansion of the polynomial solution $y(x)$ of the nonhomogeneous equation (7) as

$$y(x)=\sum_{m=0}^{n} \bar{C}_m(n)Q_m(x), \tag{11}$$

gives rise to

$$Q_n(x) = \mathscr{R}_{2,\lambda}[y(x)] = \sum_{m=0}^{n} \bar{C}_m(n)[\sigma(x)D^2[Q_m(x)] + \tau(x)D[Q_m(x)] + \lambda Q_m(x)]. \tag{12}$$

Then, the properties satisfied by the $\{Q_m\}$ family allow to express both sides of Eq. (12) as linear combination of the polynomials $Q_m(x)$ themselves or its first or second derivatives. As in the first method, this procedure gives rise to a linear system which can be also solved recurrently being the initial conditions computed by direct matching in the final expression.

Remark 1. It is interesting to notice that if, in Eq. (7), the operator $\mathscr{R}_{2,\lambda}$ satisfies the conditions required by the first method and the polynomial $Q_n(x)$ also satisfies the requirements of the second method, then we can compute two different representations for the same polynomial solution of (7).

3. Some examples

We are going to show three examples where $Q_n(x)$ in Eq. (7) is solution of a second order differential equation of hypergeometric type, first order difference equation or a second order difference equation of hypergeometric type on the exponential lattice, respectively. From now on, monic polynomials will be considered.

3.1. Classical continuous case: Laguerre operator–Laguerre polynomials

Let us consider the monic Laguerre polynomial, $L_n^{(\beta)}(x)$, of degree n with $\beta > -1$. The polynomial solution of the nonhomogeneous problem

$$[xD^2 + (-x + \alpha + 1)D + \lambda\mathscr{I}](y(x)) = L_n^{(\beta)}(x), \quad (\alpha \neq \beta), \tag{13}$$

can be computed by using the second method we have just described. Assuming that $\lambda \neq k$ ($k = 0, \ldots, n$), Eq. (13) has a unique polynomial solution which can be written as $y(x) = \sum_{m=0}^{n} \bar{C}_m(n) L_m^{(\beta)}(x)$. Then, our method gives for the $\bar{C}_m(n)$-coefficients the following two-term recurrence relation

$$m(\alpha - \beta + \lambda - m)\bar{C}_m(n) + (\lambda - m + 1)\bar{C}_{m-1}(n) = 0,$$

valid for $1 \leqslant m \leqslant n - 1$, with the initial conditions

$$\bar{C}_n(n) = \frac{1}{\lambda - n}, \qquad \bar{C}_{n-1}(n) = \frac{n(\beta - \alpha)}{(\lambda - n)(1 + \lambda - n)}.$$

The explicit solution of this recurrence relation is

$$\bar{C}_m(n) = \frac{(\alpha - \beta)(-n)_{n-m}(\alpha - \beta + \lambda - n + 1)_{n-m-1}}{(\lambda - n)_{n-m+1}},$$

valid for $0 \leqslant m \leqslant n-1$, where $(A)_s$ denotes the Pochhammer symbol. So, the polynomial solution of (13) is given by

$$y(x) = \sum_{m=0}^{n-1} \frac{(\alpha-\beta)(-n)_{n-m}(\alpha-\beta+\lambda-n+1)_{n-m-1}}{(\lambda-n)_{n-m+1}} L_m^{(\beta)}(x) + \frac{L_n^{(\beta)}(x)}{\lambda-n}.$$

3.2. Classical discrete case: Hahn operator – falling factorial basis

Let us choose in Eq. (7), $Q_n(x) = x^{[n]} = x(x-1)\ldots(x-n+1)$ (the falling factorial basis). Although it is not an orthogonal family, the falling factorial sequence of polynomials satisfies a first order difference equation and also a two term recurrence relation. So, the second method described above can be used to compute the polynomial solution of the nonhomogeneous problem,

$$[\sigma(x)\Delta\nabla + \tau(x)\Delta + \lambda\mathscr{I}](y(x)) = x^{[n]}.$$

In doing so, we arrive at an expression of type (10), which is now an inversion problem [17] for classical discrete orthogonal polynomials.

For instance, for the monic Hahn polynomials $\{h_n^{(\alpha,\beta)}(x)\}$ [10], our first method gives the expression

$$y(x) = \sum_{m=0}^{n} \binom{n}{m} \frac{(N-n)_{n-m}(n+\beta)^{[n-m]}}{(n+m+\alpha+\beta+1)^{[n-m]}(\lambda - m(\alpha+\beta+m+1))} h_m^{(\alpha,\beta)}(x) \tag{14}$$

for the polynomial solution of the nonhomogeneous problem

$$[x(N+\alpha-x)\Delta\nabla + ((\beta+1)(N-1)-(\alpha+\beta+2)x)\Delta + \lambda\mathscr{I}](y(x)) = x^{[n]},$$

where it is assumed that $\lambda \neq k(\alpha+\beta+k+1)$ $(k=0,\ldots,n)$, which ensures the uniqueness of the solution. Representation (14) is also a simple consequence of the explicit solution to the inversion problem for the Hahn polynomials given in [3, p. 188, Eq. (4.2)].

3.3. q-analogue of classical discrete case: Charlier q-polynomials

In the nonuniform lattice $x(s) = q^s$, monic Charlier q-polynomials $\{c_n^{(b)}(x;q)\}$ are solution of the second order difference equation of hypergeometric type

$$\left(\bar{\sigma}(s)\frac{\Delta}{\Delta x(s-1/2)}\frac{\nabla}{\nabla x(s)} + \bar{\tau}(s)\frac{\Delta}{\Delta x(s)} + \bar{\lambda}_n\mathscr{I}\right)[c_n^{(b)}(s,q)] = 0,$$

where the following data $\bar{\sigma}(s)$, $\bar{\tau}(s)$ and $\bar{\lambda}_n$ are chosen like in [10]

$$\bar{\sigma}(s) = \frac{q^{s-1/2}(q^s-1)}{q-1}, \qquad \bar{\tau}(s) = \frac{1+b(q-1)\sqrt{q}-q^s}{(q-1)^2}, \qquad \bar{\lambda}_n = \frac{q-q^{1-n}}{(q-1)^3}. \tag{15}$$

For these polynomials we have computed the expression of the difference-derivative representation, which for monic polynomials is

$$c_n^{(b)}(s,q) = e_n\frac{\Delta c_{n+1}^{(b)}(s,q)}{\Delta x(s)} + f_n\frac{\Delta c_n^{(b)}(s,q)}{\Delta x(s)}, \qquad e_n = \frac{q-1}{q^{n+1}-1}, \qquad f_n = b(q-1)^2 q^{n-1/2}.$$

Let us consider the following nonhomogeneous problem:

$$\left(\sigma(s)\frac{\Delta}{\Delta x(s-1/2)}\frac{\nabla}{\nabla x(s)}+\tau(s)\frac{\Delta}{\Delta x(s)}+\lambda\mathscr{I}\right)[y(s)]=c_n^{(b)}(s,q), \tag{16}$$

where $\sigma(s)=\bar{\sigma}(s)$ is given in (15),

$$\tau(s)=\frac{1+a(q-1)\sqrt{q}-q^s}{(q-1)^2}, \quad (a\neq b),$$

and we assume that $\lambda\neq\bar{\lambda}_k$ $(k=0,\dots,n)$ in order to have a unique solution.

Once more, the second method allows us to compute recurrently the $\bar{C}_m(n)$-coefficients in the expansion

$$y(s)=\sum_{m=0}^{n}\bar{C}_m(n)c_m^{(b)}(s,q)$$

giving the polynomial solution of Eq. (16). They satisfy now the following two-term recurrence relation:

$$[(a-b)q^{1/2}+(q-1)f_m(\lambda-\bar{\lambda}_m)]\bar{C}_m(n)+(q-1)e_{m-1}(\lambda-\bar{\lambda}_{m-1})\bar{C}_{m-1}(n)=0$$

valid for $1\leqslant m\leqslant n-1$ with the initial conditions

$$\bar{C}_n(n)=\frac{1}{\lambda-\bar{\lambda}_n}, \qquad \bar{C}_{n-1}(n)=\frac{(b-a)q^{1/2}}{(q-1)e_{n-1}(\lambda-\bar{\lambda}_{n-1})(\lambda-\bar{\lambda}_n)}.$$

The solution of this recurrence relation is

$$\bar{C}_m(n)=\frac{(a-b)q^{1/2}(q^{m+1};q)_{n-m}\prod_{j=1}^{n-m-1}[(a-b)q^{1/2}+(q-1)f_{n-j}(\lambda-\bar{\lambda}_{n-j})]}{(1-q)^{2(n-m)}\prod_{j=0}^{n-m}(\lambda-\bar{\lambda}_{n-j})}$$

valid for $0\leqslant m\leqslant n-2$, where $(a;q)_n=(1-a)(1-aq)\dots(1-aq^{n-1})$.

4. Two particular cases: first associated polynomials and nondiagonal Sobolev orthogonal polynomials

4.1. First associated polynomials

If $Q_n(x)=KP'_{n+1}(x)$, where K is a constant and $P_n(x)$ is any classical monic (continuous) family of orthogonal polynomials, problem (1) appears in relation with the first associated family of $\{P_n(x)\}$ denoted by $\{P_{n-1}^{(1)}(x)\}$ [12] since

$$\mathscr{L}^*_{2,\lambda_{n+1}}[P_n^{(1)}(x)]=KP'_{n+1}(x) \quad (K=\sigma''-2\tau'), \tag{17}$$

where $\mathscr{L}^*_{2,\lambda}$ is the formal adjoint of $\mathscr{L}_{2,\lambda}$, i.e.: $\mathscr{L}^*_{2,\lambda}=\sigma D^2+(2\sigma'-\tau)D+(\lambda+\sigma''-\tau')\mathscr{I}$, which is an operator of hypergeometric-type.

Since the right-hand side of Eq. (17) is again a classical orthogonal polynomial, our algorithm gives a representation of the first associated polynomials $P_{n-1}^{(1)}(x)$ of a given classical family $\{P_n(x)\}$ in terms of their derivatives $\{P_n'(x)\}$, i.e.,

$$P_{n-1}^{(1)}(x) \equiv y(x) = \sum_{m=1}^{n} \bar{C}_m(n-1) P_m'(x). \tag{18}$$

To obtain a recurrence relation for the coefficients $\bar{C}_m(n-1)$, we follow the steps of the method, so first we apply the $\mathscr{L}_{2,\lambda_n}^*$ operator to the above expression and from Eq. (17) we get

$$KP_n'(x) = \mathscr{L}_{2,\lambda_n}^*[y(x)] \sum_{m=0}^{n} \bar{C}_m(n-1) \mathscr{L}_{2,\lambda_n}^*[P_m'(x)].$$

Then, the appropriate use of properties of the family $\{P_n(x)\}$ allows us to express both sides of the latter identity as a linear combination (with constant coefficients) of the polynomials $\{P_n''(x)\}$ (see [1, 4] for details). In this way, a linear system of equations satisfied by $\bar{C}_m(n-1)$ is obtained, and in some cases it is possible to give an explicit expression for them, as in the following example. Another recurrent approach was discussed in [6] where the expansion of the numerator polynomial $P_{n-1}^{(1)}(x)$ in terms of the derivatives of the classical orthogonal polynomials might be obtained from the more standard expansion by using the derivative representation.

4.2. Gegenbauer polynomials ($P_n^{(\alpha,\alpha)}(x), \alpha > -1$)

The explicit expression for the $\bar{C}_m(n-1)$ coefficients in the expansion

$$[P_{n-1}^{(\alpha,\alpha)}(x)]^{(1)} = \sum_{m=1}^{n} \bar{C}_m(n-1)(P_m^{(\alpha,\alpha)}(x))'$$

has the form

$$\bar{C}_m(n-1) = \begin{cases} -(1+2\alpha)^2 n^{[n-m]} f(n,m) & \text{if } n-m=2p, \\ 0 & \text{if } n-m=2p+1, \end{cases} \tag{19}$$

valid for $1 \leqslant m \leqslant n-2$, where $f(n,m)$ is defined by

$$f(n,m) = \frac{\prod_{j=0}^{p-2}((1+2j-2\alpha)(j-n-2\alpha))}{\prod_{j=1}^{2p-1}(1-2j+2n+2\alpha)\prod_{j=0}^{p}((j-n)(1+2j+2\alpha))}$$

being $\bar{C}_n(n-1) = 1/n$ and $\bar{C}_{n-1}(n-1) = 0$. In [5], the connection of the associated Jacobi (with any real positive association parameter) with the Jacobi polynomials has been examined, and a nice explicit result for the associated Gegenbauer polynomials was given [6].

From (19) when $\alpha = -\frac{1}{2}$ (Chebyshev polynomials of first kind $T_n(x)$) we recover the well known relation $[T_{n-1}(x)]^{(1)} = (T_n(x))'/n = U_{n-1}(x)$ where $U_n(x)$ denotes monic Chebyshev polynomial of second kind of degree n ($\alpha = \frac{1}{2}$).

Moreover, if $\alpha=0$ (Legendre polynomials) the coefficients (19) reduce to

$$\bar{C}_m(n-1)=\frac{2^{3+m-n}(-n)_{n-m}}{(-1+m-n)(1+m-n)(m+n)(2+m+n)(\frac{1}{2}-n)_{n-m-1}}$$

if $n-m$ is even and $0\leqslant m\leqslant n-2$.

4.3. Nondiagonal Sobolev orthogonal polynomials

Let $\mathscr{L}_{2,\lambda}[y(x)]:=xy''(x)+(\alpha+1-x)y'(x)+\lambda y(x)$ be the Laguerre operator with $\lambda\neq k$ $(k=0,1,\ldots,n)$. Then, a representation of the polynomials $P_n^{\mathrm{S}}(x)$ orthogonal with respect to the inner product

$$\langle f,g\rangle=\int_0^{\infty}\mathscr{L}_{2,\lambda}[f(x)]\mathscr{L}_{2,\lambda}[g(x)]\mathrm{e}^{-x}x^{\beta}\,\mathrm{d}x \tag{20}$$

can be obtained by solving the nonhomogeneous problem

$$xz_n''(x)+(\alpha+1-x)z_n'(x)+\lambda z_n(x)=L_n^{(\beta)}(x).$$

The following expression for these $P_n^{\mathrm{S}}(x)$ polynomials is obtained using our first method

$$z_n(x)\equiv P_n^{\mathrm{S}}(x)=\sum_{m=0}^{n}(-1)^{n-m}\binom{n}{m}\frac{(\beta-\alpha)_{n-m}}{\lambda-m}L_m^{(\alpha)}(x). \tag{21}$$

This formula can be also derived from (10) and the classical formula [11, p. 119, Eq. (2)].

5. An extension: classical Sobolev orthogonal polynomials

Given a Sobolev inner product

$$(f,g)_{\mathrm{S}}=\sum_{i=0}^{p}\int_{\mathbb{R}}f^{(i)}(x)g^{(i)}(x)\,\mathrm{d}\mu_i(x), \tag{22}$$

where p is a nonnegative integer and $\mathrm{d}\mu_i(x)$ are positive Borel measures, the corresponding orthogonal polynomials with respect to (22) appear as solutions of the nonhomogeneous problem

$$\mathscr{F}(y(x))=\sum_{i=n-k}^{n+1}\xi_i q_i(x), \tag{23}$$

where $\mathscr{F}$ is now the self-adjoint differential operator of order r $(r\geqslant 2)$ for the Sobolev scalar product used, and $q_i(x)$ are classical orthogonal polynomials.

It turns out that, in general, the operator $\mathscr{F}$ is not of hypergeometric-type. Moreover, the right-hand side of Eq. (23) is a linear combination of classical orthogonal polynomials. So, the nonhomogeneous problem (23) does not fulfill the requirements of the methods described in Section 2. Nevertheless, the ideas of them can be extended in such a way that a representation of these polynomials in terms of the classical family $\{q_i(x)\}$ can be devised.

As illustration, let us consider the *diagonal* Sobolev inner product in the Laguerre case ($\alpha > -1$, $\lambda \geqslant 0$), i.e.,

$$\langle f, g \rangle_S = \int_0^\infty f(x)g(x)x^\alpha e^{-x}\,dx + \lambda \int_0^\infty f'(x)g'(x)x^\alpha e^{-x}\,dx. \tag{24}$$

For this inner product, the differential operator $\mathscr{F}$ is given by

$$\mathscr{F} \equiv x\mathscr{I} - \lambda(\alpha - x)\frac{d}{dx} - \lambda x \frac{d^2}{dx^2} \tag{25}$$

which is self-adjoint with respect to the inner product (24), i.e., it satisfies the relation $\langle \mathscr{F}p, q \rangle_S = \langle p, \mathscr{F}q \rangle_S$, $\forall p, q \in \mathscr{P}$. Then, the nth-degree polynomial $y_n(x)$ orthogonal with respect to (24) satisfies the nonhomogeneous equation

$$\mathscr{F}[y_n(x)] = L_{n+1}^{(\alpha)}(x) + b_n^{(n)} L_n^{(\alpha)}(x) + b_{n-1}^{(n)} L_{n-1}^{(\alpha)}(x), \tag{26}$$

where the coefficients $b_n^{(n)}$ and $b_{n-1}^{(n)}$ are given in [9, p. 33]. One can see from (26) and the definition of $\mathscr{F}$ in (25), this operator maps polynomials of degree n to polynomials of degree $n+1$.

To compute the solution $y_n(x)$ we consider its expansion in the Laguerre family $y_n(x) = \sum_{m=0}^n C_m(n) L_m^{(\alpha)}(x)$. Now, the action of the $\mathscr{F}$ operator on it together with Eq. (26) generates to the expression

$$\sum_{m=0}^{n} C_m(n)\mathscr{F}[L_m^{(\alpha)}(x)] = L_{n+1}^{(\alpha)}(x) + b_n^{(n)} L_n^{(\alpha)}(x) + b_{n-1}^{(n)} L_{n-1}^{(\alpha)}(x).$$

The properties satisfied by Laguerre polynomials (basically, the three term recurrence relation and the structure relation) allow us to express both sides of the latter equation as a linear combination with constant coefficients of the family $\{[L_n^{(\alpha)}(x)]''\}$, giving rise to a linear system of equation for the $C_m(n)$-coefficients which can be solved recurrently. In this case, the following five-term recurrence is obtained

$$\begin{aligned} &C_{m-3}(n) + (-5 + \alpha + \lambda(-2 + m) + 4m)C_{m-2}(n) + (-1 + m)(3\alpha + (3 + \lambda)(-1 + 2m))C_{m-1}(n) \\ &\quad + (-1 + m)m(1 + 3\alpha + \lambda + (4 + \lambda)m)C_m(n) + m(1 + \alpha + m)(-1 + m^2)C_{m+1}(n) = 0. \end{aligned}$$

Solution of this recurrence with the appropriate initial conditions leads us to the same representation already given in [9], where a completely different approach were used. After this paper was submitted to this Journal, Prof. S. Lewanowicz kindly informed us that the above recurrence relation can be replaced by a four-term recurrence relation [8].

The same technique works in more general Sobolev situations: for any classical continuous or discrete orthogonal polynomials even if the number of derivatives is larger than one.

Finally, we should mention here that many of the computations involved in this paper have been performed with the help of Mathematica [16] symbolic language.

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