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Normality and closed projections of products with a cardinal factor

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Abstract

For a space X and a regular uncountable cardinal κ, we discuss when $X \times \kappa$ is normal if and only if the projection $\pi : X \times \kappa \to X$ is closed.

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1. Introduction

Throughout this paper, let κ be a regular uncountable cardinal with the usual order topology, and we denote by π the projection from a product onto one factor.

It had been shown in [7] that

$$\pi : X \times Y \to X \text{ is closed } \Rightarrow X \times Y \text{ is normal}$$

is true for a paracompact space X and a normal space Y. Using this, it was shown in [6] that

$$X \times \kappa \text{ is orthocompact } \Rightarrow \pi : X \times \kappa \to X \text{ is closed} \Rightarrow X \times \kappa \text{ is normal}$$

is true for a paracompact space X. Moreover, for a metacompact space X, the orthocompactness of $X \times \kappa$ is equivalent that X has orthocaliber κ (see [6]). On the other hand, it was essentially shown in [7,8] that

$$X \times \kappa^+ \text{is normal} \Leftrightarrow t(X) \leqslant \kappa \Leftrightarrow \pi : X \times \kappa^+ \to X \text{ is closed}$$

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is true for a compact space X. This was generalized in [6] by replacing κ^+ and $t(X)$ with κ and $t^+(X)$, respectively. It means that the normality of the product of a compact space X and a cardinal factor κ gives an internal characterization of the space X in terms of tightness.

From the facts mentioned above, it is natural to raise the question of what internal characterization of a general space X is given by the normality of $X \times \kappa$. As is seen later, it is not difficult to give an internal characterization of X when $\pi : X \times \kappa \to X$ is closed. So we may regard this question as when the normality of $X \times \kappa$ makes the projection $\pi : X \times \kappa \to X$ closed for a generalized compact (or ordered) space X. Conversely, it should be noted that by Kunen's theorem in [10, Corollary 3.7] if $X \times \kappa$ is normal, then X is normal and $<\kappa$-paracompact (i.e., τ-paracompact for each $\tau < \kappa$). So we can assume in our discussion that X is normal and $<\kappa$-paracompact.

All spaces discussed here are assumed to be Hausdorff.

2. A preliminary result

Let us recall some definitions. Let X be a space. For each $p \in X$, we define

$$t^+(p, X) = \min\{\lambda\colon \text{ for each } A \subset X, \text{ there is a } B \subset A \text{ with } |B| < \lambda \text{ and } p \in \mathrm{Cl}\, B\},$$

and define $t^+(X) = \sup\{t^+(p, X)\colon p \in X\}$. The definition of the usual *tightness* $t(p, X)$ and $t(X)$ are obtained by replacing $|B| < \lambda$ in the definition of the above $t^+(p, X)$ by $|B| \leqslant \lambda$.

A sequence $\{x_\alpha\colon \alpha \in \kappa\}$ in a space X is a *free sequence of length* κ if for each $\alpha \in \kappa$, $\mathrm{Cl}\{x_\beta\colon \beta < \alpha\} \cap \mathrm{Cl}\{x_\beta\colon \beta \geqslant \alpha\} = \emptyset$.

First of all, we give a slight generalization of known results in [1,8,7].

Lemma 2.1. *Let X be a compact space with $p \in X$. If, for every collection $\mathcal{H}$ of nonempty closed G_δ-sets in X with $|\mathcal{H}| < \kappa$, there is an open neighborhood U of p in X such that $H - U \neq \emptyset$ for each $H \in \mathcal{H}$, then there is a free sequence of length κ in X.*

The proof which is due to Arhangel'skii (see [2, p. 67]) is found in that of [4, Theorem 7.10].

Proposition 2.2. *For a compact space X, the following are equivalent.*

(a) *$X \times \kappa$ is normal.*

(b) *The projection $\pi : X \times \kappa \to X$ is closed.*

(c) *There is no free sequence of length κ in X.*

Proof.

(a) $\Rightarrow$ (b): Assume $X \times \kappa$ is normal, or equivalently $t^+(X) \leqslant \kappa$ (see [6, Theorem 3.5]). Then it follows from [6, Lemma 3.4] (essentially due to [7]) that π is closed.

(b) $\Rightarrow$ (c): Assume there is a free sequence $\{x_\alpha\colon \alpha \in \kappa\}$ in X. Since X is compact, we can take some $p \in \bigcap_{\alpha\in\kappa} \mathrm{Cl}\{x_\beta\colon \beta \geqslant \alpha\}$. Let $K = \mathrm{Cl}\{\langle x_\alpha, \alpha\rangle\colon \alpha \in \kappa\}$. Then it is straightforward to show that $p \in \mathrm{Cl}\,\pi(K) - \pi(K)$.

(c) $\Rightarrow$ (a): Assume $X \times \kappa$ is not normal. By [6, Theorem 3.5], we have $t^+(X) > \kappa$. Note that $t(X) \geqslant \kappa$. If $t(X) > \kappa$, there is a free sequence of length κ^+ in X by Arhangel'skii's theorem in [1]. So we may assume that $t(X) = \kappa$ and $t^+(X) = \kappa^+$. There are a subset A and a point p in X such that $p \in \mathrm{Cl}\,A$ and $p \notin \mathrm{Cl}\,B$ for each $B \subset A$ with $|B| < \kappa$. Moreover, we may assume that A is dense in X and $|A| = \kappa$. By our assumption (c), there is no free sequence of length κ in X, so by Lemma 2.1, there is a collection $\mathcal{H}$ of nonempty closed G_δ-sets in X with $|\mathcal{H}| < \kappa$ such that each open neighborhood U of p in X contains some member of $\mathcal{H}$. For each $H \in \mathcal{H}$, choose a sequence $\{G_n(H)\colon n \in \omega\}$ of open sets in X such that $H = \bigcap_{n\in\omega} G_n(H)$ and $\mathrm{Cl}\,G_{n+1}(H) \subset G_n(H)$ for each $n \in \omega$. Find an $a_{H,n} \in G_n(H) \cap A$ for each $H \in \mathcal{H}$ and $n \in \omega$. Let $B_0 = \{a_{H,n}\colon H \in \mathcal{H}, n \in \omega\}$. Then $B_0 \subset A$ with $|B_0| < \kappa$. Let U be an open neighborhood of p and take an $H \in \mathcal{H}$ with $H \subset U$. Since X is compact, there is an $n \in \omega$ such that $G_n(H) \subset U$. Then U contains $a_{H,n} \in B_0$. This means $p \in \mathrm{Cl}\,B_0$. This contradicts the choice of A. □

3. Normality of $X \times \kappa$ versus closed projections

In this section, we discuss the relation of the normality of $X \times \kappa$ and the closed projection $\pi : X \times \kappa \to X$, and give a generalization of Proposition 2.2.

First, we consider when the closed projection $\pi : X \times \kappa \to X$ implies the normality of $X \times \kappa$. Recall that a space X is *κ-compact* if there is not a closed discrete subspace of size κ and *κ-paracompact* if every open cover of X with cardinality κ has a locally finite open refinement. Observe that both Lindelöf spaces and countably compact spaces are ω_1-compact.

For each $C \subset \kappa$, let $\mathrm{Lim}(C) = \{\alpha \in \kappa\colon \alpha = \sup(\alpha \cap C)\}$.

Lemma 3.1. *Assume that X is κ-compact and the projection $\pi : X \times \kappa \to X$ is closed. Then for each pair of disjoint closed sets K_0 and K_1 in $X \times \kappa$, there is an $\alpha \in \kappa$ such that $\pi(K_0 \cap X \times (\alpha, \kappa)) \cap \pi(K_1 \cap X \times (\alpha, \kappa)) = \emptyset$.*

Proof. Assume the contrary. Let $F_\alpha = \pi(K_0 \cap X \times (\alpha, \kappa)) \cap \pi(K_1 \cap X \times (\alpha, \kappa))$ for each $\alpha \in \kappa$. Then each F_α is a nonempty closed set in X.

First, assume we can pick some $p \in \bigcap_{\alpha\in\kappa} F_\alpha$. For each $\alpha \in \kappa$ and $i \in 2 = \{0, 1\}$, take a $\beta_i(\alpha)$ with $\alpha < \beta_i(\alpha) < \kappa$ such that $\langle p, \beta_i(\alpha)\rangle \in K_i$. We can inductively choose a sequence $\{\alpha_j\colon j \in \omega\}$ in κ such that $\beta_i(\alpha_j) \leqslant \alpha_{j+1}$ for each $j \in \omega$ and $i \in 2$. Let $\alpha_\omega = \sup_{j\in\omega} \alpha_j$ ($= \sup\{\beta_i(\alpha_j)\colon j \in \omega,\ i \in 2\}$). So we have $\langle p, \alpha_\omega\rangle \in K_0 \cap K_1$, which is a contradiction. This establishes $\bigcap_{\alpha\in\kappa} F_\alpha = \emptyset$.

For each $\alpha \in \kappa$, take an $x_\alpha \in F_\alpha$ and a $\theta(\alpha) > \alpha$ such that $x_\alpha \notin F_{\theta(\alpha)}$. Moreover, take a $\beta_i(\alpha) > \alpha$ such that $\langle x_\alpha, \beta_i(\alpha)\rangle \in K_i$ for each $\alpha \in \kappa$ and $i \in 2$. Then

$$C = \{\alpha \in \kappa\colon \forall \delta \in \alpha\ \forall i \in 2\ (\theta(\delta) < \alpha \text{ and } \beta_i(\delta) < \alpha)\}$$

is a closed unbounded set in κ. Since X is κ-compact, $\{x_\alpha\colon \alpha \in C\}$ has an accumulation point. Let

$$\gamma = \min\{\gamma' \leqslant \kappa\colon \{x_\alpha\colon \alpha \in C \cap \gamma'\} \text{ has an accumulation point}\},$$

and let y be an accumulation point of $\{x_\alpha\colon \alpha \in C \cap \gamma\}$. By the minimality of γ, we have $\gamma \in \mathrm{Lim}(C)$ or $\gamma = \kappa$, and observe that $\{x_\alpha\colon \alpha \in C \cap \delta\}$ is closed discrete in X for each $\delta < \gamma$. Therefore y is an accumulation point of $\{x_\alpha\colon \alpha \in C \cap (\delta, \gamma)\}$ for each $\delta < \gamma$. Now, assume $\gamma = \kappa$. Then there is a $\xi \in C$ such that $y \notin F_\xi$. Since F_ξ is a closed set containing $\{x_\alpha\colon \alpha \in C - \xi\}$, it follows that $y \notin \mathrm{Cl}\{x_\alpha\colon \alpha \in C - \xi\}$. So, y is an accumulation point of $\{x_\alpha\colon \alpha \in C \cap \xi\}$. This contradicts $\gamma = \kappa$. Hence we have $\gamma \in \mathrm{Lim}(C)$.

Take any open neighborhood W of y and any $\delta < \gamma$. Since y is an accumulation point of $\{x_\alpha\colon \alpha \in C \cap (\delta, \gamma)\}$, there is an $\eta \in C \cap (\delta, \gamma)$ such that $x_\eta \in W$. Then we have $\eta < \beta_i(\eta) < \gamma$. Therefore it follows that $\langle x_\eta, \beta_i(\eta)\rangle \in K_i \cap W \times (\delta, \gamma]$. This shows $\langle y, \gamma\rangle \in \mathrm{Cl}\, K_i = K_i$ for each $i \in 2$. This contradicts that K_0 and K_1 are disjoint. □

Proposition 3.2. *Let X be a normal, $<\kappa$-paracompact and κ-compact space. If the projection $\pi : X \times \kappa \to X$ is closed, then $X \times \kappa$ is normal.*

Proof. Let K_0 and K_1 be any disjoint closed sets in $X \times \kappa$. By Lemma 3.1, take an $\alpha \in \kappa$ such that $\{\pi(K_i \cap X \times (\alpha, \kappa))\colon i \in 2\}$ are disjoint closed sets in X. By the normality of X, $\{K_i \cap X \times (\alpha, \kappa)\colon i \in 2\}$ can be separated by disjoint open sets in the closed-open subspace $X \times (\alpha, \kappa)$ of $X \times \kappa$. Since X is $|\alpha + 1|$-paracompact, it follows from Kunen's theorem (see [10, Corollary 3.7]) that $X \times (\alpha + 1)$ is normal. So $\{K_i \cap X \times (\alpha + 1)\colon i \in 2\}$ can be separated by disjoint open sets in $X \times (\alpha + 1)$. Hence K_0 and K_1 can be separated by disjoint open sets in $X \times \kappa$. □

As stated in the Introduction, it is rather easy to get an internal characterization of a space X such that the projection $\pi : X \times \kappa \to X$ is closed.

A well-ordered collection $\{A_\alpha\colon \alpha \in \kappa\}$ of subsets in a set X is *monotone decreasing* if $A_\alpha \subset A_\beta$ whenever $\beta < \alpha$.

Proposition 3.3. *For a space X, the projection $\pi : X \times \kappa \to X$ is closed if and only if $\bigcap_{\alpha \in \kappa} U_\alpha$ is open in X for every monotone decreasing collection $\{U_\alpha\colon \alpha \in \kappa\}$ of open sets.*

Proof. The "only if" part: Let $\{U_\alpha\colon \alpha \in \kappa\}$ be a monotone decreasing collection of open sets in X. We may assume $\bigcap_{\alpha \in \kappa} U_\alpha \neq \emptyset$. Pick a $p \in \bigcap_{\alpha \in \kappa} U_\alpha$. Let $U = \bigcup\{U_\alpha \times (\alpha + 1)\colon \alpha \in \kappa\}$. Since U is an open set in $X \times \kappa$ with $\{p\} \times \kappa \subset U$, there is an open neighborhood V of p in X with $V \times \kappa \subset U$ by the closedness of π. Since $V \times (\alpha + 1) \subset U_\alpha \times (\alpha + 1)$ for each $\alpha \in \kappa$, we have $V \subset \bigcap\{U_\alpha \colon \alpha \in \kappa\}$.

The "if" part: Take a point $p \in X$ and an open set U in $X \times \kappa$ with $\{p\} \times \kappa \subset U$. For each $\alpha \in \kappa$, let $U_\alpha = \bigcup\{W\colon W$ is open set in X with $W \times (\alpha + 1) \subset U\}$. Then $\{U_\alpha\colon \alpha \in \kappa\}$ is a monotone decreasing collection of open neighborhoods of p in X. By the assumption, $V = \bigcap_{\alpha\in\kappa} U_\alpha$ is an open neighborhood of p in X such that $V \times \kappa \subset U$. □

We can also show the following lemma similar to Proposition 3.3.

Proposition 3.4. *Let X be a space and F a closed set of X. Then the following are equivalent.*

(a) *For each closed set K in $X \times \kappa$ which is disjoint from $F \times \kappa$, K and $F \times \kappa$ are separated by disjoint open sets in $X \times \kappa$.*

(b) *For each monotone decreasing collection $\{U_\alpha\colon \alpha \in \kappa\}$ of open sets containing F, there is a monotone decreasing collection $\{V_\alpha\colon \alpha \in \kappa\}$ of open sets containing F such that $\bigcap_{\beta<\alpha} \operatorname{Cl} V_\beta \subset U_\alpha$ for each $\alpha \in \kappa$.*

(c) *For each monotone decreasing collection $\{U_\alpha\colon \alpha \in \kappa\}$ of open sets containing F, there is a monotone decreasing collection $\{V_\alpha\colon \alpha \in \kappa\}$ of open sets containing F such that $\{\alpha \in \kappa\colon \bigcap_{\beta<\alpha} \operatorname{Cl} V_\beta \not\subset U_\alpha\}$ is not stationary in κ.*

Proof.

(a) $\Rightarrow$ (b): Let $\{U_\alpha\colon \alpha \in \kappa\}$ be a monotone decreasing collection of open sets containing F in X. Let $U = \bigcup\{U_\alpha \times (\alpha+1)\colon \alpha \in \kappa\}$. Since U is an open set containing $F \times \kappa$, there is an open set V in $X \times \kappa$ such that $F \times \kappa \subset V \subset \operatorname{Cl} V \subset U$. For each $\alpha \in \kappa$, let $V_\alpha = \bigcup\{W\colon W$ is open in X with $W \times (\alpha+2) \subset V\}$. Then $\{V_\alpha\colon \alpha \in \kappa\}$ is a monotone decreasing collection of open sets containing F in X. Pick an $\alpha \in \kappa$, and let $x \notin U_\alpha$. Then we have $\langle x, \alpha\rangle \notin U$, so $\langle x, \alpha\rangle \notin \operatorname{Cl} V$. Take a $\beta < \alpha$ and a neighborhood W_x of x such that $W_x \times (\beta, \alpha] \cap V = \emptyset$. Now, assume $x \in \operatorname{Cl} V_\beta$. Then taking a $y \in W_x \cap V_\beta$, we have $\langle y, \beta+1\rangle \in V_\beta \times (\beta+2) \cap W_x \times (\beta, \alpha] \subset V \cap W_x \times (\beta, \alpha] = \emptyset$. This is a contradiction. So we obtain $x \notin \operatorname{Cl} V_\beta$. This implies $\bigcap_{\beta<\alpha} \operatorname{Cl} V_\beta \subset U_\alpha$ for each $\alpha \in \kappa$.

(b) $\Rightarrow$ (c): Evident.

(c) $\Rightarrow$ (a): Let U be an open set in $X \times \kappa$ with $F \times \kappa \subset U$. Let $U_\alpha = \bigcup\{W\colon W$ is open in X with $W \times (\alpha + 1) \subset U\}$ for each $\alpha \in \kappa$. Then $\{U_\alpha\colon \alpha \in \kappa\}$ is a monotone decreasing collection of open sets containing F. There are a monotone decreasing collection $\{V_\alpha\colon \alpha \in \kappa\}$ of open sets containing F in X and a closed unbounded set C in κ such that $\bigcap_{\beta<\alpha} \operatorname{Cl} V_\beta \subset U_\alpha$ for each $\alpha \in C$. Note that $\operatorname{Lim}(C)$ is also closed unbounded in κ and $\operatorname{Lim}(C) \subset C$. Define $H_\alpha = \bigcap_{\beta<\alpha} \operatorname{Cl} V_\beta$ for each $\alpha \in \operatorname{Lim}(C)$, $V = \bigcup\{V_\alpha \times (\alpha + 1)\colon \alpha \in \operatorname{Lim}(C)\}$ and $H = \bigcup\{H_\alpha \times (\alpha + 1)\colon \alpha \in \operatorname{Lim}(C)\}$. Then we have $F \times \kappa \subset V \subset H \subset U$. So it suffices to show that H is closed in $X \times \kappa$. Let $\langle x, \alpha\rangle \notin H$. Consider two cases.

Case 1: $\alpha \in \operatorname{Lim}(C)$. Since $\langle x, \alpha\rangle \notin H \supset H_\alpha \times (\alpha+1)$, we have $x \notin H_\alpha$. So there is a $\beta < \alpha$ such that $x \notin \operatorname{Cl} V_\beta$. Then $(X - \operatorname{Cl} V_\beta) \times (\beta, \alpha]$ is a neighborhood of $\langle x, \alpha\rangle$ disjoint from H.

Case 2: $\alpha \notin \operatorname{Lim}(C)$. Define $\gamma = \min\{\gamma' \in \operatorname{Lim}(C)\colon \alpha \leqslant \gamma'\}$ and $\delta = \sup(C \cap \alpha)$. Since $\alpha \notin \operatorname{Lim}(C)$, we have $\delta < \alpha < \gamma$. Assume $(X - H_\gamma) \times (\delta, \alpha]$ meets H. Then there is a $\beta \in \operatorname{Lim}(C)$ such that $(X - H_\gamma) \times (\delta, \alpha] \cap H_\beta \times (\beta + 1) \neq \emptyset$. Since $(X - H_\gamma) \cap H_\beta \neq \emptyset$ and $(\delta, \alpha] \cap (\beta + 1) \neq \emptyset$, we have $\beta < \gamma$ and $\delta < \beta$ respectively. Therefore $\beta \in \operatorname{Lim}(C) \cap (\delta, \gamma) \subset C \cap (\delta, \gamma)$. But this contradicts the definition of δ. Moreover, since $\alpha \in (\delta, \gamma)$, $C \cap (\delta, \gamma) = \emptyset$ and $\langle x, \alpha\rangle \notin H$, by the definition of H, we know $X - H_\gamma$ is a neighborhood of x. So $(X - H_\gamma) \times (\delta, \alpha]$ is a neighborhood of $\langle x, \alpha\rangle$ disjoint from H. □

Note that if X is normal and the projection $\pi : X \times \kappa \to X$ is closed, then one (hence all) of the clauses of Proposition 3.4 holds.

Recall that a point x in X said to be a (*complete*) *accumulation point* of a subset A of X if $U \cap A$ is infinite (of size $|A|$) for each neighborhood U of x.

Lemma 3.5. *If $X \times \kappa$ is normal, then every free sequence of length κ in X has a subsequence of length κ with no complete accumulation point.*

Proof. Assume the contrary. There is a free sequence $\{x_\alpha\colon \alpha \in \kappa\}$ in X such that $\{x_\alpha\colon \alpha \in S\}$ has a complete accumulation point for each $S \subset \kappa$ of size κ. For each $S \subset \kappa$ of size κ, let $F(S)$ be the set of all complete accumulation points of $\{x_\alpha\colon \alpha \in S\}$. Then each $F(S)$ is a nonempty closed set in X. Let $U_\alpha = X - \operatorname{Cl}\{x_\beta\colon \beta \leqslant \alpha\}$ for each $\alpha \in \kappa$. Since $\{x_\alpha\colon \alpha \in \kappa\}$ is a free sequence, $\{U_\alpha\colon \alpha \in \kappa\}$ is a monotone decreasing collection of open sets containing $F(\kappa)$. Since $X \times \kappa$ is normal, it follows from Proposition 3.4 that there is a monotone decreasing collection $\{V_\alpha\colon \alpha \in \kappa\}$ of open sets containing $F(\kappa)$ such that $\bigcap_{\beta<\alpha} \operatorname{Cl} V_\beta \subset U_\alpha$ for each $\alpha \in \kappa$. Since $x_\alpha \notin U_\alpha$, there is a $\beta(\alpha) < \alpha$ with $x_\alpha \notin \operatorname{Cl} V_{\beta(\alpha)}$. Then it follows from the pressing down lemma that there is a $\gamma < \kappa$ and a stationary set S in κ such that $\beta(\alpha) = \gamma$ for each $\alpha \in S$. This means $\operatorname{Cl} V_\gamma \cap \{x_\alpha\colon \alpha \in S\} = \emptyset$. Hence we have $V_\gamma \cap \operatorname{Cl}\{x_\alpha\colon \alpha \in S\} = \emptyset$. On the other hand, since $F(S) \subset \operatorname{Cl}\{x_\alpha\colon \alpha \in S\}$ and $F(S) \subset F(\kappa) \subset V_\gamma$, it follows that $V_\gamma \cap \operatorname{Cl}\{x_\alpha\colon \alpha \in S\} \supset F(S) \neq \emptyset$. This is a contradiction. □

Now, we are ready to prove the main result, which is the converse of Proposition 3.2.

Theorem 3.6. *Let X be a normal, $<\kappa$-paracompact and κ-compact space. Then $X \times \kappa$ is normal if and only if the projection $\pi : X \times \kappa \to X$ is closed.*

Proof. Since the "if" part is just Proposition 3.2, we show the "only if" part. Assume π is not closed. It follows from Proposition 3.3 that there is a monotone decreasing collection $\{U_\alpha\colon \alpha \in \kappa\}$ of open sets in X such that $\bigcap_{\alpha\in\kappa} U_\alpha$ is not open. Pick some $p \in \bigcap_{\alpha\in\kappa} U_\alpha - \operatorname{int}(\bigcap_{\alpha\in\kappa} U_\alpha)$. Since $X \times \kappa$ is normal, it follows from Proposition 3.4 that there is a monotone decreasing collection $\{V_\alpha\colon \alpha \in \kappa\}$ of open neighborhoods of p such that $\bigcap_{\beta<\alpha} \operatorname{Cl} V_\beta \subset U_\alpha$ for each $\alpha \in \kappa$. In particular, observe that $\operatorname{Cl} V_\alpha \subset U_{\alpha+1} \subset U_\alpha$ for each $\alpha \in \kappa$. Since V_α is a neighborhood of p but $\bigcap_{\alpha\in\kappa} U_\alpha$ is not, we take an $f(\alpha) > \alpha$ and an $x_\alpha \in V_\alpha - U_{f(\alpha)}$ for each $\alpha \in \kappa$. Put $C = \{\alpha \in \kappa\colon \forall\beta < \alpha(f(\beta) < \alpha)\}$. Since C is unbounded in κ and X is κ-compact, there is an accumulation point of $\{x_\alpha\colon \alpha \in C\}$. Let

$$\gamma = \min\{\gamma' \leqslant \kappa\colon \text{ there is an accumulation point of } \{x_\alpha\colon \alpha \in C \cap \gamma'\}\},$$

and let y be an accumulation point of $\{x_\alpha\colon \alpha \in C \cap \gamma\}$. By the minimality of γ, we have $\gamma \in \operatorname{Lim}(C)$ or $\gamma = \kappa$.

Case 1: $\gamma \in \operatorname{Lim}(C)$. Since $x_\alpha \notin U_{f(\alpha)}$ and $f(\alpha) < \gamma$ for each $\alpha \in C \cap \gamma$, it follows that $y \in \operatorname{Cl}\{x_\alpha\colon \alpha \in C \cap \gamma\} \subset X - U_\gamma$. So there is a $\delta < \gamma$ with $y \notin \operatorname{Cl} V_\delta$. By the minimality of γ, $\{x_\alpha\colon \alpha \in C \cap \delta\}$ is closed discrete in X. Hence $y \in \operatorname{Cl}\{x_\alpha\colon \alpha \in C \cap [\delta, \gamma)\}$. Since $X - \operatorname{Cl} V_\delta$ is a neighborhood of y, there is a $\xi \in C \cap [\delta, \gamma)$ such that $x_\xi \in X - \operatorname{Cl} V_\delta$. This contradicts $x_\xi \in V_\xi \subset V_\delta$.

Case 2: $\gamma = \kappa$. First, notice that $\{x_\alpha\colon \alpha \in C\}$ is a free sequence by enumerating C with the increasing order. In fact, pick an $\alpha \in C$. Since $x_\beta \notin U_{f(\beta)} \supset U_\alpha$ for each $\beta \in C \cap \alpha$ and $x_\beta \in V_\beta \subset V_\alpha$ for each $\beta \in C - \alpha$, it follows that $\operatorname{Cl}\{x_\beta\colon \beta \in C \cap \alpha\} \subset X - U_\alpha$ and $\operatorname{Cl}\{x_\beta\colon \beta \in C - \alpha\} \subset \operatorname{Cl} V_\alpha$. By $\operatorname{Cl} V_\alpha \subset U_\alpha$, $\{x_\alpha\colon \alpha \in C\}$ is a free sequence in X. So it follows from Lemma 3.5 that there is some $S \subset C$ of size κ such that $\{x_\alpha\colon \alpha \in S\}$ has no complete accumulation point. On the other hand, the κ-compactness of X assures the existence of an accumulation point z of $\{x_\alpha\colon \alpha \in S\}$. By the minimality of $\gamma = \kappa$, this z must be a complete accumulation point of $\{x_\alpha\colon \alpha \in S\}$. This is a contradiction. □

Our Theorem 3.6 immediately yields the following corollaries.

Corollary 3.7. *Let X be a regular Lindelöf space. Then $X \times \kappa$ is normal if and only if the projection $\pi : X \times \kappa \to X$ is closed.*

Corollary 3.8. *Let X be normal, countably paracompact and ω_1-compact space. Then $X \times \omega_1$ is normal if and only if the projection $\pi : X \times \omega_1 \to X$ is closed.*

These corollaries are not true for a paracompact space X, because of

Example 3.9. There is a paracompact space Y such that $Y \times \kappa$ is normal, but the projection $\pi : Y \times \kappa \to Y$ is not closed.

Let Y_0 be the set of maps f on κ to $\{0, 1\}$ such that $f(\alpha) = 0$ for all but finitely many $\alpha \in \kappa$. Let $f_0 : \kappa \to \{0, 1\}$ be the function which assumes the constant value 1. For each $q \in [\kappa]^{<\omega} = \{q \subset \kappa\colon |q| < \omega\}$, put $U(q) = \{f_0\} \cup \{f \in Y_0\colon f(\alpha) = 1 \text{ for all } \alpha \in q\}$. Set $Y = \{f_0\} \cup Y_0$. Topologize Y as follows. Let $\{U(q)\colon q \in [\kappa]^{<\omega}\}$ be a neighborhood base of f_0, and other points isolated. This example described in [9, p. 342]. Then Y is paracompact and $|Y| = \kappa$. Since κ is a collectionwise normal space with weight $\leqslant \kappa$, $Y \times \kappa$ is (collectionwise) normal. By $|Y| = \kappa$, let $Y \setminus \{f_0\} = \{g_\alpha\colon \alpha \in \kappa\}$. Let $U_\alpha = Y \setminus \operatorname{Cl}\{g_\beta\colon \beta < \alpha\}$ for $\alpha \in \kappa$. Then $\{U_\alpha\colon \alpha \in \kappa\}$ is a monotone decreasing collection of open neighborhoods of f_0 in Y, and $\bigcap_{\alpha\in\kappa} U_\alpha$ is not open in Y. It follows from Proposition 3.3 that π is not closed.

4. GO-spaces and related results

In this section, we consider the case X is a GO-space. It is well known that every GO-space is normal and countably paracompact. First, we recall the notations in [5]. Let X be a GO-space. A linearly ordered compactification of X is a compact linearly ordered space which contains X as a dense subspace and whose linear order is an extension of the original order on X. It is known that there always exists a minimal linearly ordered compactification lX of a GO-space X in the sense that, for each linearly ordered compactification cX of X, there is a continuous map $f : cX \to lX$ such that all points of X are pointwise fixed. It is also shown in [5, Lemma 2.1] that, for a linearly ordered compactification cX of a GO-space X, $cX = lX$ if and only if $(a,b)_{cX} \neq \emptyset$ for each $a, b \in cX - X$ with $a < b$, where $(a,b)_{cX}$ denotes the open interval in cX.

Next, for each $x \in lX$, let

$$0\text{-}\operatorname{cf} x = \min\bigl\{|A|:\ A \subset (\leftarrow, x) \text{ and } \forall y < x\ \exists a \in A\ (y \leqslant a)\bigr\},$$

and let

$$1\text{-}\operatorname{cf} x = \min\bigl\{|A|:\ A \subset (x, \rightarrow) \text{ and } \forall y > x\ \exists a \in A\ (y \geqslant a)\bigr\}.$$

Then we can fix a strictly increasing sequence $\langle x(\alpha) : \alpha \in 0\text{-}\operatorname{cf} x\rangle$ in $(\leftarrow, x)_{lX}$ such that for each $y < x$ there is an $\alpha < 0\text{-}\operatorname{cf} x$ such that $y \leqslant x(\alpha)$, and $x(\alpha) = \sup\{x(\beta):\ \beta < \alpha\}$ for each limit ordinal $\alpha < 0\text{-}\operatorname{cf} x$. We call this sequence a *0-normal sequence for x*. Analogously, we can define a 1-normal sequence for x.

Lemma 4.1 [5, Theorem 4.3]. *Let X be a $<\kappa$-paracompact GO-space. Then $X \times \kappa$ is normal if and only if $i\text{-}\operatorname{cf} x \neq \kappa$ for each $x \in X$ and $i \in 2$.*

Theorem 4.2. *Let X be a GO-space. Then $i\text{-}\operatorname{cf} x \neq \kappa$ for each $x \in X$ and $i \in 2$ if and only if the projection $\pi : X \times \kappa \to X$ is closed.*

Proof. The "only if" part: Let $\{U_\alpha:\ \alpha \in \kappa\}$ be a monotone decreasing collection of open sets. It suffices from Proposition 3.3 to show that $\bigcap_{\alpha\in\kappa} U_\alpha$ is open in X, so pick an $x \in \bigcap_{\alpha\in\kappa} U_\alpha$.

Claim 1. There is a $y_0 \in lX$ with $y_0 < x$ such that $(y_0, x]_{lX} \cap X \subset \bigcap_{\alpha\in\kappa} U_\alpha$. In fact, put $\lambda = 0\text{-}\operatorname{cf} x \neq \kappa$, and fix a 0-normal sequence $\langle x(\beta) : \beta \in \lambda\rangle$ for x. Since U_α is open in X and $x \in U_\alpha$, we can take a $\beta(\alpha) < \lambda$ such that $(x(\beta(\alpha)), x]_{lX} \cap X \subset U_\alpha$ for each $\alpha \in \kappa$.

Case 1: $\lambda < \kappa$. Applying the pressing down lemma, find a stationary set $S \subset \kappa$ and a $\gamma < \lambda$ such that $\beta(\alpha) = \gamma$ for each $\alpha \in S$. Then $(x(\gamma), x]_{lX} \cap X \subset \bigcap_{\alpha\in\kappa} U_\alpha$, because $\{U_\alpha:\ \alpha \in \kappa\}$ is monotone decreasing.

Case 2: $\lambda > \kappa$. Put $\delta = \sup_{\alpha\in\kappa}\beta(\alpha)$. Then we obtain $(x(\delta), x]_{lX} \cap X \subset \bigcap_{\alpha\in\kappa} U_\alpha$. Similarly, we can get

Claim 2. There is a $y_1 \in lX$ with $y_1 > x$ such that $[x, y_1)_{lX} \cap X \subset \bigcap_{\alpha\in\kappa} U_\alpha$. Therefore, $(y_0, y_1)_{lX} \cap X$ is an open neighborhood of x contained in $\bigcap_{\alpha\in\kappa} U_\alpha$.

The "if" part: Assume there is an $x \in X$ with $0\text{-}\mathrm{cf}\, x = \kappa$. Fix a 0-normal sequence $\langle x(\alpha) : \alpha \in \kappa\rangle$ for x. Define $U_\alpha = (x(\alpha), \rightarrow)_{lX} \cap X$ for each $\alpha \in \kappa$. Then it is straightforward to show that $\{U_\alpha\colon \alpha \in \kappa\}$ is a monotone decreasing collection of open neighborhoods of x, and that $\bigcap_{\alpha\in\kappa} U_\alpha$ is not a neighborhood of x because $\sup_{\alpha\in\kappa} x(\alpha) = x$. It follows from Proposition 3.3 that π is not closed. The case of $1\text{-}\mathrm{cf}\, x = \kappa$ is similar. □

Immediately we have:

Corollary 4.3. *Let X be a $<\kappa$-paracompact GO-space. Then $X \times \kappa$ is normal if and only if the projection $\pi : X \times \kappa \to X$ is closed.*

Corollary 4.4. *Let X be a GO-space. Then $X \times \omega_1$ is normal if and only if the projection $\pi : X \times \omega_1 \to X$ is closed.*

5. On free sequences

Finally, we deal with the connection between the normality of $X \times \kappa$ and the non-existence of free sequences of length κ in X.

Proposition 5.1. *Let X be a regular Lindelöf space. If $X \times \kappa$ is normal, then X has no free sequence of length κ.*

Proof. Assume that there is a free sequence $\{x_\alpha\colon \alpha \in \kappa\}$ in X. By Lemma 3.5, there is a subsequence of length κ with no complete accumulation point. This contradicts the Lindelöfness of X. □

Proposition 5.2. *Let X be a $<\kappa$-paracompact GO-space. If X has no free sequence of length κ, then $X \times \kappa$ is normal.*

Proof. Assume $X \times \kappa$ is not normal. By Lemma 4.1, there are an $x \in X$ and an $i \in 2$ such that $i\text{-}\mathrm{cf}\, x = \kappa$. We may assume $i = 0$. Fix a 0-normal sequence $\langle x(\alpha) : \alpha \in \kappa\rangle$ for x. If $A = \{\alpha \in \kappa\colon x(\alpha) \in X\}$ is unbounded in κ, then $\{x(\alpha)\colon \alpha \in A - \mathrm{Lim}(A)\}$ is a free sequence of length κ. If A is bounded in κ, then take a $\beta \in \kappa$ such that $A \subset \beta$. For each $\alpha \geqslant \beta$, we can take a $y(\alpha) \in (x(\alpha), x(\alpha+1))_{lX} \cap X$ by [5, Lemma 2.1]. By renumbering the sequence $\{y(\alpha)\colon \alpha \geqslant \beta\}$ as $\{y'(\gamma)\colon \gamma \in \kappa\}$ where $y'(\gamma) = y(\beta + \gamma)$, we have a free sequence $\{y'(\alpha)\colon \alpha \in \kappa\}$ of length κ. □

The following example shows that, in the above Proposition 5.2, the normality in $X \times \kappa$ need not ensure the nonexistence of a free sequence of length κ.

Example 5.3. Let $X = \kappa$ with the order topology. Then X is a countably compact GO-space. As is well known, $X \times \kappa = \kappa^2$ is normal. But the set of all nonlimit ordinals less than κ is a free sequence (with the increasing order) in X of length κ.

In connection with these considerations, we have the following problems.

Problem 5.4. If X is a Lindelöf space without a free sequence of length ω_1, then is $X \times \omega_1$ normal?

Problem 5.5. If X is a normal, countably compact space without a free sequence of length ω_1, then is $X \times \omega_1$ normal?

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