Direct trajectory optimization based on a mapped Chebyshev pseudospectral method

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Abstract In view of generating optimal trajectories of Bolza problems, standard Chebyshev pseudospectral (PS) method makes the points’ accumulation near the extremities and rarefaction of nodes close to the center of interval, which causes an ill-condition of differentiation matrix and an oscillation of the optimal solution. For improvement upon the difficulties, a mapped Chebyshev pseudospectral method is proposed. A conformal map is applied to Chebyshev points to move the points closer to equidistant nodes. Condition number and spectral radius of differentiation matrices from both methods are presented to show the improvement. Furthermore, the modification keeps the Chebyshev pseudospectral method’s advantage, the spectral convergence rate. Based on three numerical examples, a comparison of the execution time, convergence and accuracy is presented among the standard Chebyshev pseudospectral method, other collocation methods and the proposed one. In one example, the error of results from mapped Chebyshev pseudospectral method is reduced to 5% of that from standard Chebyshev pseudospectral method.

KEYWORDS Chebyshev approximation; Conformal shift; Interpolation; Optimization; Pseudospectral method; Trajectory

1. Introduction

For a trajectory optimization with dynamic system described by differential equations, the optimal methods generally fall into two categories, direct and indirect methods. Indirect methods, mostly with the application of Pontryagin’s minimum principle, result in more accurate overall solution than direct methods. However, better convergence and programming for computers make direct numerical methods to be applied to a number of practical trajectory optimization problems. Direct solution methods have been used extensively in the domain of trajectory optimization. Betts¹ has presented several general direct and indirect numerical methods, which can be further broadly classified into either shooting methods or collocation methods. Pseudospectral (PS) method falls into the category of collocation method, which uses global orthogonal Lagrange polynomials to approximate the state and control variables, while the nodes are selected as the roots of the derivative of the named polynomial, such as the Legendre–Gauss–Lobatto (LGL)² and the Chebyshev–Gauss–Lobatto (CGL)³ points. In recent years, direct trajectory optimization with pseudospectral methods has a great development and extensive application. Based on the Chebyshev technique, Vlassenbroeck and van Dooren⁴,⁵ successfully transform nonlinear optimal control problems into systems of algebraic or transcendental expressions in the Cheb-
shev coefficients. The state and control variables are determined by solving a set of linear equations in the Chebyshev coefficients with application of Pontryagin’s maximum principle. And in Ref. 4, Fahroo and Ross employed 4th-degree classic Lagrange polynomial approximations for the state and control variables with values of these CGL points as the expansion coefficients to convert the trajectory optimization problem into a nonlinear parameter optimization problem and with nonlinear programming (NLP) solvers, which yielded the numerical solution with high degree of accuracy. Benson et al. 6 used Gauss pseudospectral method to solve nonlinear optimal control problems. Qi et al. 7 considered a pseudospectral method to compute optimal controls and proved that a sequence of solutions to the pseudospectral discretized constrained problem converges to the optimal solution of the continuous-time optimal control problem under numerically verifiable conditions. Moreover, in 2011, Darby et al. 8 presented a hp framework for pseudospectral method using adaptive mesh on direct trajectory optimization. But in one mesh interval, without increasing the degree of the polynomial, a method for improvement of the global pseudospectral method is not introduced in the literature.

As is well-known in approximation theory, polynomial interpolation between CGL points is trigonometric interpolation of even functions between equidistant points, which has great properties: fast convergence for very smooth functions and small operator norm. But it also has some drawbacks mentioned in Ref. 9. Under the condition of nodes’ concentration at the extremities of the interval of interpolation, three main difficulties are presented:

1) ill-conditioning of the derivatives near the extremities;
2) bad distribution of the information over the interval;
3) mediocre approximation of functions with shocks close to the center, where points are scarcer.

Bayliss and Turkel10 suggested to react to these difficulties by a conformal shift of the nodes toward the equidistant position. The transfer preserves exponential convergence and markedly lessens the difficulties. In 1993, for overcoming time step restriction in some resolution of partial differential equation (PDE) problems, Kosloff and Tal-Ezer11 provided a concrete conformal map to be adapted for reforming the distribution of the Chebyshev points in the interval. For solving PDEs, the mapped pseudospectral method has been applied successfully. 12 But in the domain of optimal control problem, the shift has not been used to improve the pseudospectral method for trajectory optimization. And an optimal process is attached to find the most optimal solution in the feasible region. Some mapping functions are not appropriate for the trajectory optimization. In Refs. 13, 14, the parameters of conformal shift were decided by the function forms or the singularities which are indefinite for trajectory optimization problems.

In this paper, we apply a Chebyshev pseudospectral method with conformal shift suggested by Kosloff and Tal-Ezer. Moreover, barycentric Lagrange interpolation 15 is substituted for the classic Lagrange interpolation. One notable advantage of Chebyshev pseudospectral method is the high degree of accuracy that pseudospectral approximations offer. The modification in this paper reserves the superiority and improves the stability. In Section 2, a general trajectory optimization problem is defined. Then, Section 3 describes the details of the mapped Chebyshev pseudospectral method. In Section 4, three numerical applications are provided.

2. Trajectory optimization problem

Trajectory optimization problem is defined in the region \( t_0 \leq t \leq t_f \), where the independent variable \( t \) is time. Within the region, the dynamics of the system are described by a set of variables defined by

\[
X = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}
\]

which includes state variables \((x \in \mathbb{R}^m)\) and control variables \((u \in \mathbb{R}^r)\). They are applied to minimizing a Bolza cost function of

\[
J = M(x(t_k), t_f) + \int_{t_0}^{t_f} L(x(t), u(t), t) dt
\]

where \( L: \mathbb{R}^m \times \mathbb{R}^r \times \mathbb{R} \rightarrow \mathbb{R}, M: \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R} \).

Typically, the dynamic constraints of the problem are defined as follows:

\[
f_i \leq f(x(t), u(t), t) \leq f_u
\]

and if the upper and lower limitations are both equal to zero, the formulation can be described as differential algebraic equations (DAEs):

\[
f(x(t), u(t), t) = 0
\]

Additionally, with the \( df/dx \) being nonsingular, Eq. (4) can be transformed into the form of ordinary differential equations (ODEs):

\[
x(t) = f(x(t), u(t), t)
\]

Furthermore, initial conditions at time \( t_0 \) are defined by

\[
\psi_0 \leq \psi(x(t_0), u(t_0)) \leq \psi_0
\]

where \( \psi_0: \mathbb{R}^m \times \mathbb{R}^r \times \mathbb{R} \rightarrow \mathbb{R}^0 \), and \( \psi_0, \psi_0 \in \mathbb{R}^0 \). And at the final time, terminal conditions are described as

\[
\psi_f(x(t_f), u(t_f)) \leq \psi_f
\]

where \( \psi_f: \mathbb{R}^m \times \mathbb{R}^r \times \mathbb{R} \rightarrow \mathbb{R}^0 \), and \( \psi_0, \psi_0 \in \mathbb{R}^0 \). In addition, the solution must satisfy the algebraic path constraints of

\[
g_i \leq g(x(t), u(t), t) \leq g_u
\]

where \( g_i: \mathbb{R}^m \times \mathbb{R}^r \times \mathbb{R} \rightarrow \mathbb{R}^r \), and \( g_0, g_0 \in \mathbb{R}^r \) represent the lower and upper bounds of path constraints. Besides, the simple bounds on the state and control variables are

\[
\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \leq \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}
\]

Because the CGL points all lie in interval \( J = [-1, 1] \) while the time points distribute on \( J = [t_0, t_1] \), a linear map is utilized between \( I \) and \( J \),

\[
t_k = \frac{t_f - t_0}{2} (t_k + 1) + t_0
\]

\[
= \left[(t_f - t_0) t_k + (t_f + t_0)\right]/2 \quad (k = 0, 1, \cdots, N)
\]

where \( t_k \in I \) and stands for the CGL points, and \( N \) is the total number. Consequently, Eqs. (2), (3) and Eqs. (6)-(8) can be rewritten as
which applies two polynomials to approximate the numerator and denominator, respectively. Fornberg\textsuperscript{16} has illustrated the relationship between the rates of convergence for polynomial interpolation of analytic functions and the poles of functions. But rational interpolation results in more computation compared with barycentric interpolation. Consequently, barycentric formula is adopted for the improvement of numerical stability of interpolation compared to the classic Lagrange interpolation.

Barycentric interpolation formula is described as\textsuperscript{18}

\[
r_N(t) = \sum_{k=0}^{N} \frac{\omega_k}{t-t_k} f_k \left( \sum_{j=0}^{N} \frac{\omega_j}{t-t_j} \right)
\]

where \( f_k = f(t_k) \), which is the function’s value at \( t_k \), \( \omega_k \) is defined as barycentric weight,

\[
\omega_k = \prod_{j \neq k} \frac{t_k - t_j}{t_k - t_j}
\]

where

\[
l(t) = (t-t_0)(t-t_1) \cdots (t-t_N)
\]

Higham\textsuperscript{19} gave an error analysis of the evaluation of the interpolating polynomial using barycentric interpolation and showed that Eq. (18) is unconditionally stability. Additionally, the barycentric formula is forward stable for any set of interpolation points with a small Lebesgue constant. And with a special series of nodes, such as Chebyshev points, the weights can be transformed to a simple form.

Chebyshev points of the second kind are picked out as the collocation nodes,

\[
t_k = \cos\left(\pi k/N\right) \quad (i = 0, 1, \ldots, N)
\]

Based on Chebyshev points of the second kind, for which one simply has\textsuperscript{20}

\[
\omega_k = (-1)^k \delta_k, \quad \delta_k = \begin{cases} 1/2 & k = 0, N \\ 1 & \text{Otherwise} \end{cases}
\]

And the time histories of status and control are approximated as

\[
\begin{align*}
X^N(t) &= \sum_{k=0}^{N} x_k l_k(t) \\
U^N(t) &= \sum_{k=0}^{N} u_k l_k(t)
\end{align*}
\]

where

\[
\begin{align*}
l_k(t) &= \frac{\omega_k}{t-t_k} \left( \sum_{j=0}^{N} \frac{\omega_j}{t-t_j} \right) \\
X_k &= x(t_k) = x(\tau(t_k)) \\
u_k &= u(t_k) = u(\tau(t_k))
\end{align*}
\]

3. Mapped Chebyshev pseudospectral method

Pseudospectral method applies orthogonal functions as the basic function classes to approximate arbitrary functions including periodic and nonperiodic ones. And three requirements for the methods are figured out in Ref.\textsuperscript{16}. One of the most important factors is that the approximations \( \sum_{k=0}^{N} a_k \phi_k(x) \) of \( \phi(x) \) must converge rapidly, where \( \phi(x) \) denotes the function approximated by the orthogonal functions of \( \phi_k(x) \). Considering the numerical computation, the coefficients \( a_k \) should be easily determined by \( \delta_k \), and the relationship between them are defined as

\[
\frac{d\phi_k(x)}{dx} = \sum_{k=0}^{N} a_k \phi_k(x) = \sum_{k=0}^{N} b_k \phi_k(x)
\]

For reduction of the computing consumption, it should be fast to convert between coefficients \( a_k, k = 0, 1, \ldots, N \), and the values for the sum \( \phi(x) \) at some set of nodes \( x_i, i = 0, 1, \ldots, N \).

Polynomial interpolation of functions based on the Chebyshev nodes is well-known to provide approximations with nearly uniform accuracy over \([-1, 1]\) and it satisfies all the requirements. Interpolation at the Chebyshev nodes has a smaller Lebesgue constant compared to interpolation using Legendre nodes and is far superior to the disastrous one for equi-spaced interpolation. Moreover, when Lagrange interpolation is adapted, the coefficients are just the function value at the corresponding points. In addition, by adopting Chebyshev nodes and barycentric formula, the analytical expression of differentiation matrix can be achieved.

But the Chebyshev points’ accumulation near the extremities and nodes scatter close to the center of interval cause an ill-condition of differentiation matrix and an oscillation of the optimal solution. To overcome the difficulties, conformal point shifts are introduced. By the conformal map’s properties, the pseudospectral method based on Chebyshev nodes also satisfies the requirements. The method is described in details as follows.

3.1. Barycentric formula

Barycentric interpolation is one kind of Lagrange interpolation but with barycentric weights and beautiful symmetry. Besides, it can fall into the category of rational interpolation\textsuperscript{9,17} which applies two polynomials to approximate the numerator and denominator, respectively. Fornberg\textsuperscript{16} has illustrated the relationship between the rates of convergence for polynomial interpolation of analytic functions and the poles of functions. But rational interpolation results in more computation compared with barycentric interpolation. Consequently, barycentric formula is adopted for the improvement of numerical stability of interpolation compared to the classic Lagrange interpolation.

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where \( f_k = f(t_k) \), which is the function’s value at \( t_k \), \( \omega_k \) is defined as barycentric weight,

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\]

where

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And the time histories of status and control are approximated as

\[
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U^N(t) &= \sum_{k=0}^{N} u_k l_k(t)
\end{align*}
\]

where

\[
l_k(t) = \frac{\omega_k}{t-t_k} \left( \sum_{j=0}^{N} \frac{\omega_j}{t-t_j} \right) \\
X_k &= x(t_k) = x(\tau(t_k)) \\
u_k &= u(t_k) = u(\tau(t_k))
\]

3.2. Conformal shift

Following Kosloff and Tal-Ezer, the application of a conformal map can improve upon the difficulties mentioned in Section 1.
by moving the points closer to equidistant. To get the series of
points closer to equidistant, consider, beside the \(t\)-space in which
\( f \) is to be approximated, another space, with variable \( y \in J = [-1, 1] \), and let \( g \) be a conformal map form \( D_1 \) containing \( I \) to a domain \( D \) containing \( J \). Therefore, a series of new interpolation points is defined on \( J, y_k = g(t_k), k = 0, 1, \ldots, N \), so that the
Eq. (23) can be transformed into the expression of

\[
\begin{align*}
\tilde{x}^k(y_k) &= \sum_{l=0}^{N} \tilde{x}_l \delta_k(g^{-1}(y_k)) \\
\tilde{u}^k(y_k) &= \sum_{l=0}^{N} \tilde{u}_l \delta_k(g^{-1}(y_k))
\end{align*}
\]

(26)

where

\[
\begin{align*}
\tilde{x}_k &= \tilde{x}(y_k) := x(g^{-1}(y_k)) = x(t_k) \\
\tilde{u}_k &= \tilde{u}(y_k) := u(g^{-1}(y_k)) = u(t_k)
\end{align*}
\]

(27)

The conformal map \( g \) is defined as\(^{11}\)

\[
g(t) = \frac{\arcsin(x(t))}{\arcsin(x)} \quad x \in (0, 1)
\]

(28)

Fig. 1 presents a series of nodes’ distribution based on different
values of \( x \) and the map defined by Eq. (28). When \( x \to 0 \), \( y_k \to t \). Conversely, \( y_k \) becomes closer to equidistant
as \( x \to 1 \) and the cumulate nodes near the extremities move
towards the center. Through the refurbishment of the center with
nodes, the shocks and oscillations will be abated.

Kosloff and Tal-Ezer suggested several choices of \( x \), in particular

\[
x = 2/(t + 1), \quad t = e^{-1/N}
\]

(29)

where \( e \) is the precision for numerical computation. And therefore
the mapping function is not affected by the form of problems.

Baltensperger et al.\(^{21}\) have proved that

\[
|\tilde{x}^N(y) - x(y)| = O(e^{-N})
\]

(30)
can be obtained uniformly for all \( y \in [-1, 1] \). Even when \( x = 0.999 \), which results in an equal-spaced distribution, the
interpolant approximation based on Chebyshev points with
conformal shift presents a high degree of convergence.

3.3. Differentiation matrix

For barycentric formula Eq. (23), the first order derivatives of
\( x \) and \( u \) can be approximated as

\[
\begin{align*}
\tilde{x}^N(t) &= \sum_{k=0}^{N} \tilde{x}_k \delta_k(t) \\
\tilde{u}^N(t) &= \sum_{k=0}^{N} \tilde{u}_k \delta_k(t)
\end{align*}
\]

(31)

And Schneider and Werner\(^{22}\) have given a very elegant general
formula for the derivatives of rational interpolants in the form of
barycentric formula. In a simple style, the first-order derivatives
are presented as follows:

\[
\tilde{l}_k(t_j) = \begin{cases} \frac{\omega_k/\omega_j}{t_j - t_k} & k \neq j \\ \sum_{j \neq k} \tilde{l}_k(t_j) & k = j \end{cases}
\]

(32)

The advantages of Eq. (32) from the view of numerical stability are discussed in Refs.\(^{23,24}\). What we have just achieved
with the aid of Lagrange interpolation formulas is the computation of the entries of what are commonly known as first-order
differentiation matrix,

\[
D_{ch(k)} = \tilde{l}_k(t)
\]

(33)

where \( D_{ch(k)} \) is the \((j, k)\)th entry of matrix \( D_{ch} \). The matrix \( D_{ch} \)
is very ill-conditioned, with eigenvalues scattering in the left
side of the complex plane. While most of the eigenvalues grow
like \( O(N) \), a few of them are \( O(N^2) \). These extreme eigenvalues
are the reason for the severe stability condition.

In addition, to lessen the difficulties of classic Chebyshev
pseudo-spectral method, the conformal map, illustrated by
Eq. (28), is introduced and the approximation is transformed from
\( t\)-space to \( y\)-space by the conformal map. By taking derivative of Eq. (26) with \( y \), the first-order derivatives of state
and control variables are

\[
\begin{align*}
\frac{\partial \tilde{x}^N(y) / \delta y} &= \sum_{k=0}^{N} \tilde{x}_k \delta_k(g^{-1}(y)) / \delta y \\
\frac{\partial \tilde{u}^N(y) / \delta y} &= \sum_{k=0}^{N} \tilde{u}_k \delta_k(g^{-1}(y)) / \delta y
\end{align*}
\]

(34)

where using multiple derivative technique, the derivatives of \( l_k \)
are for \( y \)

\[
\frac{\delta l_k(g^{-1}(y))}{\delta y} = \frac{\delta g^{-1}(y)}{\delta y} \frac{\delta l_k}{\delta t} = \frac{1}{g'(t)} \frac{\delta l_k}{\delta t}
\]

(35)

And the derivative of \( g \) utilized is given by

\[
g'(t) = \frac{x}{\arcsin x} \frac{1}{\sqrt{1 - (x t)^2}}
\]

(36)

Consequently, first-order differentiation matrix \( D_{ch} \), is summarized as

\[
D_{ch(k)} = \frac{\delta l_k(g^{-1}(y_j))}{\delta y} = \frac{1}{g'(t_j)} \frac{\arcsin x \sqrt{1 - (x t)^2}}{x} l_k(t_j)
\]

(37)

where \( D_{ch(k)} \) is the \((j, k)\)th entry of matrix \( D_{ch} \).

Table 1 presents the comparison between the two differentiation
operators on the condition number and spectral radius \( \rho \), where \( x \) is given by Eq. (29). Additionally, time step restriction
can be observed by

\[
\frac{\Delta t_m}{\Delta t_{ch}} \approx \rho(D_{ch}) / \rho(D_m)
\]

(38)

For \( N = 125 \), the restriction of mapped algorithm is almost seven times larger than that of a standard Chebyshev
method.
From Table 1, it can be seen that an improvement on the differentiation matrix’s ill-condition is provided by the conformal shift. Fig. 2 shows the details of the elements’ value in the differentiation matrices and illustrates that by the conformal shift, the elements in the diagonal line of the differentiation matrix distribute fairly.

### 3.4. Integral scheme

For integral section of Bolza trajectory optimization problem, Clenshaw–Curtis quadrature scheme is adopted to discretize the integral part to a finite sum, and the scheme is presented as

$$
\int_{-1}^{1} p(t)dt = \sum_{k=0}^{N} p(t_k) \sigma_k
$$

where $p$ is an arbitrary function for variable $t$ and $\sigma_k$ weight for every point. With conformal map, Eq. (39) can be transformed into

$$
\int_{-1}^{1} p(y)dy = \int_{-1}^{1} p(g(t))g'(t)dt = \sum_{k=0}^{N} p(g(t_k))g'(t_k) \sigma_k
$$

And for even $N$, the weights $\sigma_k$ are

$$
\sigma_0 = \sigma_N = 1/(N^2 - 1) \quad (41)
$$

$$
\sigma_s = \sigma_{N-s} = \frac{4}{N} \sum_{j=0}^{N/2} \frac{1}{1 - 4j^2} \cos \frac{2\pi js}{N} \quad (s = 1, 2, \ldots, \frac{N}{2}) \quad (42)
$$

While $N$ is odd, the weights satisfy

$$
\sigma_0 = \sigma_N = 1/N^2 \quad (43)
$$

$$
\sigma_s = \sigma_{N-s} = \frac{4}{N} \sum_{j=0}^{(N-1)/2} \frac{1}{1 - 4j^2} \cos \frac{2\pi js}{N} \quad (s = 1, 2, \ldots, \frac{N-1}{2}) \quad (44)
$$

In Eqs. (42) and (44), the double prime in the summations indicates that the first and the last elements have to be halved.

### 3.5. Nonlinear programming problem

The trajectory optimization problem, with the mapped Chebyshev points of second kind, is converted into a nonlinear programming problem which is to find the value of variables at the nodes $y_k$,

$$
\hat{X} = [\hat{x}_0 \quad \hat{x}_1 \quad \ldots \quad \hat{x}_N], \quad \hat{U} = [\hat{u}_0 \quad \hat{u}_1 \quad \ldots \quad \hat{u}_N]
$$

which are the approximate coefficients in Eq. (26) and final time $t_f$ to minimize the discretized cost function of

$$
J^*(\hat{X}, \hat{U}, \tau_f, t_0) = M(\hat{x}_N, \hat{x}_0, \tau_f, t_0) + \frac{\tau_f - t_0}{2} \sum_{k=0}^{N} L_k g_k^2 \sigma_k
$$

where

$L_k = L(\hat{x}_k, \hat{u}_k, \tau_k)$ subject to

$$
f_i \leq f \left( \frac{2}{\tau_f - t_0}, \frac{\delta \hat{x}_k}{\delta y}, \hat{x}_k, \hat{u}_k, \hat{t}_k \right) \leq f_u
$$

$$
\psi_{ul} \leq \psi(\hat{x}_0, \hat{u}_0, \tau_0) \leq \psi_{ul}
$$

$$
\psi_{vl} \leq \psi(\hat{x}_N, \hat{u}_N, \tau_N) \leq \psi_{v0}
$$

$$
g_i \leq g(x_k, \hat{u}_k, \tau_k) \leq g_u
$$

$$
\begin{bmatrix} x_i \\ u_i \end{bmatrix} \leq \begin{bmatrix} \hat{x}_k \\ \hat{u}_k \end{bmatrix} \leq \begin{bmatrix} x_u \\ u_u \end{bmatrix}
$$

where $k = 0, 1, \ldots, N$. And Eq. (47) in some conditions can be simplified into the forms of DAEs (Eq. (4)) and ODEs (Eq. (5)).
4. Numerical examples

In this section, three numerical instances are presented to demonstrate the improvement based on mapped Chebyshev pseudospectral method and comparisons are illuminated between mapped Chebyshev pseudospectral method, standard Chebyshev pseudospectral method and other collocation methods. All computations are executed by the software, MATLAB, on a portable computer with a 2 GHz processor and 2 GB of RAM. And additionally, the function of MATLAB’s optimization toolbox, fmincon with SQP algorithm is utilized as the nonlinear programming solver.

4.1. Example 1

A minimum-energy problem with a second-order state variable inequality constraints is considered. It is taken from Ref. 27 and Bryson and Ho presented the analytic solution in the literature, so the absolute accuracy can be obtained. The problem is to find the control variable \( u \) to minimize cost function of

\[
J = 0.5 \int_0^1 a^2(t) \, dt
\]

The dynamic constraints are

\[
x' = v, \quad v' = a
\]

while initial and terminal conditions are

\[
x(0) = x(1) = 0, \quad v(0) = -v(1) = 1
\]

And due to the different path constraints of

\[
x(t) \leq q
\]

The results are quite different from each other. For \( 0 < q < 1/6 \), the path stays on the constraint boundary for a finite time. Overall, solution with this constraint is

\[
\begin{align*}
a &= \begin{cases} 
-\frac{2}{3q} \left( 1 - \frac{t}{3q} \right) & 0 \leq t \leq 3q \\
0 & 3q < t \leq 1 - 3q \\
-\frac{2}{3q} \left( 1 - \frac{1-t}{3q} \right) & 1 - 3q < t \leq 1
\end{cases}
\end{align*}
\]

\[
v = \begin{cases} 
\left( 1 - \frac{t}{3q} \right)^2 & 0 \leq t \leq 3q \\
0 & 3q < t \leq 1 - 3q \\
\left( 1 - \frac{1-t}{3q} \right)^2 & 1 - 3q < t \leq 1
\end{cases}
\]

(57)

\[
x = \begin{cases} 
q \left[ 1 - \left( 1 - \frac{t}{3q} \right)^2 \right] & 0 \leq t \leq 3q \\
q \left[ 1 - \left( 1 - \frac{1-t}{3q} \right)^2 \right] & 1 - 3q < t \leq 1
\end{cases}
\]

(58)

\[
J = \frac{4}{3q}
\]

(59)

And in this paper, \( q \) is chosen to be 1/36, so the value of analytic cost function \( J_{\text{ana}} \) equals 16. To get numerical solution, the mapped Chebyshev pseudospectral method is used for discretization of the problem and converts it into a NLP problem, which is to find the variables

\[
\begin{align*}
\tilde{X} &= [\tilde{x}_0 \; \tilde{x}_1 \; \cdots \; \tilde{x}_N] \\
\tilde{V} &= [\tilde{v}_0 \; \tilde{v}_1 \; \cdots \; \tilde{v}_N] \\
A &= [a_0 \; \tilde{a}_1 \; \cdots \; \tilde{a}_N]
\end{align*}
\]

and minimize the cost function of

\[
\tilde{J} = \frac{1}{4} \sum_{k=0}^N \tilde{a}_k^2 g(t_k)m_k
\]

subject to the equality constraints of

\[
2 \tilde{D} \tilde{X} = \tilde{V}, \quad 2 \tilde{D} \tilde{V} = \tilde{A}
\]

and inequality constraints of

\[
\tilde{x}_k = x(g(t_k)) \leq 1/36
\]

(63)

The initial and terminal conditions are

\[
\tilde{x}_0 = \tilde{x}_N = 0, \quad \tilde{v}_0 = -\tilde{v}_N = 1
\]

(64)

Tables 2 and 3 list the optimal results and the errors of mapped Chebyshev pseudospectral method, classic Chebyshev pseudospectral method and Simpson collocation method compared with the analytic solution. The solutions are based on 31 nodes, so \( N = 30 \). In respect of conformal map, \( z \) is defined by Eq. (28) and with a general machine precision \( 10^{-6} \).

**Table 2** Comparison of optimal results from two methods.

| Method            | \( J \)     | \( |J - J_{\text{ana}}| \) | \( |J - J_{\text{ana}}|/J_{\text{ana}} \) | CPU time/s |
|-------------------|------------|-------------------------|---------------------------------|-----------|
| Simpson           | 23.721565  | 7.721565                | 0.4826                          | 0.2719    |
| Standard Chebyshev PS | 15.914243 | 0.085757                | 5.3598 \times 10^{-3}           | 5.6093    |
| Mapped Chebyshev PS | 16.004602 | 0.004602                | 2.8763 \times 10^{-4}           | 3.5468    |

**Table 3** Comparison of error with the two methods.

<table>
<thead>
<tr>
<th>Method</th>
<th>( |X - X_{\text{ana}}|<em>{L</em>\infty} )</th>
<th>( |v - v_{\text{ana}}|<em>{L</em>\infty} )</th>
<th>( |a - a_{\text{ana}}|<em>{L</em>\infty} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simpson</td>
<td>( 1.4745 \times 10^{-2} )</td>
<td>6.5515 \times 10^{-2}</td>
<td>7.0209</td>
</tr>
<tr>
<td>Standard Chebyshev PS</td>
<td>( 4.4861 \times 10^{-3} )</td>
<td>2.1372 \times 10^{-2}</td>
<td>4.2823 \times 10^{-1}</td>
</tr>
<tr>
<td>Mapped Chebyshev PS</td>
<td>( 6.6696 \times 10^{-5} )</td>
<td>3.5825 \times 10^{-3}</td>
<td>1.6127 \times 10^{-2}</td>
</tr>
</tbody>
</table>
\( x = 0.902588 \). In order to compare the errors of two methods, \( L_\infty \text{ norm } || \cdot ||_{\infty} = \max | \cdot | \), is introduced to evaluate the difference between the optimal and analytic solutions. Moreover, the execution time of the optimal progress on computer is obtained by MATLAB relative command `cputime`.

The time histories of variables are presented in Fig. 3, where “+” indicates the solution of mapped Chebyshev pseudospectral method, “o” is for the one of Chebyshev pseudospectral method and “Δ” stands for the results of Simpson collocation method, while solid line represents the analytic solution. In Fig. 3, we can notice an obvious difference between analytical solution and the one by standard Chebyshev pseudospectral method, especially for the time history of \( x \). And the results of Simpson collocation method have the same situation and even a less accuracy.

According to information in Tables 2 and 3, it is evident that for the same number of Chebyshev nodes, the mapped Chebyshev pseudospectral method has a better performance on accuracy and spends less CPU time in the same computing condition to complete optimization than the classic Chebyshev pseudospectral method does. The Simpson collocation method can rapidly obtain the optimal results, but the optimal trajectory is far away from the analytical one. For the classic pseudospectral method, the obvious difference between analytic solution is mainly caused by nodes scatter close to the center of interval. With an uniform distribution of nodes, the mapped method improves on the ill-condition of differentiation matrix, the accuracy and calculation time.

4.2. Example 2

A hyper-sensitive problem is presented.\(^{28,29}\) A characteristic three-segment structure constructs the solution of a completely hyper-sensitive problem, and the optimal trajectory is analogous to an optimal airport-airport route for a transport aircraft.\(^{30}\) So the solution to hyper-sensitive problem is important and this problem is extremely difficult to solve using indirect methods. While Jain and Tsiotras\(^{31}\) solved this problem using multi-resolution techniques for a higher density of nodes in the take-off and landing segments, a set of Chebyshev points naturally accumulated near the extremities. However, in this example, with the direct standard Chebyshev pseudospectral method, when the points accumulate in the take-off and landing segments, ill-conditioning of the differentiation matrix and oscillation in the two phases becomes more serious. Using the mapped Chebyshev pseudospectral method, this ill-conditioning and the solution’s oscillation are markedly lessened.

Consider that the problem is to minimize

\[
J = \int_0^{10000} \left( h^2(t) + c^2(t) \right) \mathrm{d}t
\]

subject to

\[
\dot{h} = -h^3 + c
\]

and the bound constraints of

\[
h(0) = 1, \quad h(10000) = 1.5
\]

By the two pseudospectral methods, the problem is discretized and transferred into a nonlinear optimal question. The cost function is

\[
J_n = \frac{1}{2} \sum_{k=0}^{N} (\tilde{h}_k^2 + \tilde{c}_k^2) g(t_k) e_k
\]

Dynamic constraints and bound constraints are

\[
2D\tilde{H} = -\tilde{H}^T + \tilde{C}, \quad \tilde{h}_0 = 1, \quad \tilde{h}_N = 1.5
\]

where

\[
\tilde{H} = \begin{bmatrix} \tilde{h}_0 & \tilde{h}_1 & \cdots & \tilde{h}_N \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} \tilde{c}_0 & \tilde{c}_1 & \cdots & \tilde{c}_N \end{bmatrix}
\]
Fig. 4 shows time histories of variables based on 101 nodes for both methods. Upon the standard Chebyshev pseudospectral method, an obvious oscillation happens to the time history of $c$ near the left extremity (shown in Fig. 4(e)) while the result of mapped Chebyshev pseudospectral method performs more smoothly. Likewise, for time history of variable $h$, the same situation takes place near the right boundary (Fig. 4(f)).

Fig. 5 presents that, with a large number of nodes, near the extremities, the optimal results based on standard Chebyshev pseudospectral method start to shake, and along with the increase of point numbers, the oscillation amplitude becomes more severe. In Fig. 5(b) and (d), the optimal variables at the same point but just with different quantity of nodes do not converge to the same value. However, the mapped method
The oscillation and is even based on a considerably big number of nodes.

4.3. Example 3

To demonstrate the practical applicability of the method, a benchmark problem for trajectory optimization methods is finally considered, which is to optimize a reentry trajectory of an Apollo type vehicle.\textsuperscript{32} Eqs. (71)–(74) describe the equations of motion.

\begin{align}
\dot{v} = & -\frac{S}{2m} \rho v^2 C_D(u) - \frac{g \sin \gamma}{(1 + \xi)^2} \\
\dot{\gamma} = & \frac{S}{2m} \rho v C_L(u) + \frac{v \cos \gamma}{R(1 + \xi)} - \frac{g \cos \gamma}{v(1 + \xi)^2} \\
\dot{\xi} = & \frac{v}{R} \sin \gamma \\
\dot{\zeta} = & \frac{v}{1 + \xi} \cos \gamma
\end{align}

where \( v \) is velocity, \( \gamma \) flight-path angle, \( \xi = h/R \) the normalized altitude and \( \zeta \) the distance on the Earth’s surface of the trajectory of an Apollo type vehicle. One control variable, angle of attack \( u \), decides the lift and drag by the following equations:

\begin{align}
C_D = & C_{D_0} + C_{D_L} \cos u, \quad C_{D_0} = 0.88, \quad C_{D_L} = 0.52 \\
C_L = & C_{L_0} \sin u, \quad C_{L_0} = -0.505
\end{align}

the air density is supposed to satisfy \( \rho = \rho_0 \exp(-\beta R \zeta) \), and the values of constants are

\begin{align*}
R = & 63.7139 \times 10^5 \text{ m}, \quad S/m = 4448 \times 10^{-5} \text{ m}^2/\text{kg}, \\
\rho_0 = & 1.2250 \text{ kg/m}^3, \quad g = 0.2989 \times 10^{-4} \text{ m/s}^2, \\
\beta = & 1/0.0716 \times 10^{-5} \text{ m}^{-1}
\end{align*}

The cost function denotes the total stagnation point convective heating per unit area, which is given by

\[ J = \int_0^T 10v^3 \sqrt{\rho} dt \]

According to Ref.\textsuperscript{32}, the reentry problem contains either a control variable constraint or a state variable constraint. The former situation is considered and the constraints of the initial position are

\[ \text{(77)} \]
\begin{align*}
 v(0) &= 0.10668 \times 10^7 \text{ m/s} \\
 \gamma(0) &= -5.75^\circ \\
 \zeta(0) &= 1.2192/R \left( h(0) = 121920 \text{ m} \right) \\
 \zeta(0) &= 0 \text{ m} 
\end{align*}

In this case, an ability limitation of control variable \( u \) is applied.
\begin{equation}
|u| \leq u_{\text{max}}, \quad u_{\text{max}} > 0
\end{equation}

The final time \( t_f \) is unspecified and is described as a parameter to attend the optimization. And at the final time, the terminal constraints are as follows:
\begin{align*}
 v(t_f) &= 0.00503 \times 10^7 \text{ m/s} \\
 \zeta(t_f) &= 0.230215/R \left( h(t_f) = 23021.5 \text{ m} \right) \\
 \zeta(t_f) &= 15.7554 \times 10^5 \text{ m}
\end{align*}

\( \gamma \) at the terminal time is not constrained. We solve this problem with \( u_{\text{max}} = 16^\circ \) mentioned by Pesch.32

Figs. 6–8 illustrate the time histories of variables for mapped Chebyshev pseudospectral method and the time history of \( u \) for Chebyshev pseudospectral method. An improvement is presented for the angle of attack \( u \) in Fig. 8.

Table 4 illustrates a comparison between the two methods with difference numbers of nodes. For the standard method, with the increase of nodes, the SAK increases a little, while the mapped method’s SAK value reduces. Though the other items in the table seem almost the same, from the SAK, we can get the idea that the oscillation becomes more serious for the standard method. In Table 4, SAK is the sum of all the dynamic constraints, MEBC means the maximum error at the boundary conditions, PREC means precision.

With the Chebyshev pseudospectral method, the optimal solution of \( u \) has a shake, while a better performance and less shake are presented by the mapped method. With the standard Chebyshev pseudospectral method, due to the ill-condition of differentiation matrix caused by points’ accumulation near the extremities and the big slope of control variable before 75 s, an intensive oscillation happens to the segment. In the middle segment, nodes scatter substitutes for the accumulation of points near the left extremity. Therefore, the shake is weakened. And near the right extremity, the slope of \( u \) almost equals zero, so the oscillation is prevented in the segment. Fig. 9 presents that even reducing the number of nodes applied to the optimization cannot improve the oscillation.
5. Conclusions

A mapped Chebyshev pseudospectral method has been used to generate the solution of trajectory optimization. In view of the standard Chebyshev pseudospectral method, the points’ accumulation near the extremities and nodes scatter close to the center of interval cause an ill-condition of differentiation matrix and an oscillation of the optimal solution. A conformal shift is adopted to rearrange Chebyshev points so that the drawbacks from the polynomial approximation based on standard Chebyshev points can be markedly lessened. From Examples 1 and 2, according to the improvement on the ill-condition of the differentiation matrix, the calculative time for the optimal progress is significantly reduced. The Example 2 demonstrates the application of the new method on the problems with constraints in the form of DAEs. Moreover, though new method has almost the same accuracy with standard Chebyshev pseudospectral method, the distribution of rearranged points overcomes the ill-condition of differentiation matrix and weakens the oscillation near the extremities.

In conclusion, it can be said that the proposed method keeps the accuracy of the Chebyshev pseudospectral methods but also improves the stability and the efficient of trajectory optimization as well.
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References