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Subword complexes in Coxeter groups

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Abstract

Let (Π, Σ) be a Coxeter system. An ordered list of elements in Σ and an element in Π determine a *subword complex*, as introduced in Knutson and Miller (Ann. of Math. (2) (2003), to appear). Subword complexes are demonstrated here to be homeomorphic to balls or spheres, and their Hilbert series are shown to reflect combinatorial properties of reduced expressions in Coxeter groups. Two formulae for double Grothendieck polynomials, one of which appeared in Fomin and Kirillov (Proceedings of the Sixth Conference in Formal Power Series and Algebraic Combinatorics, DIMACS, 1994, pp. 183–190), are recovered in the context of simplicial topology for subword complexes. Some open questions related to subword complexes are presented.

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1. Introduction

We introduced subword complexes in [KM03] to elucidate the combinatorics of determinantal ideals and Schubert polynomials. In retrospect, however, they raise basic questions about the nature of reduced expressions in arbitrary Coxeter groups. For instance, given a fixed word— that is, an ordered list Q of simple reflections in a Coxeter group Π— what can be said about the set of all of its reduced subwords? In

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particular, given a fixed element $\pi \in \Pi$, what structure belongs to the set of subwords of Q that are reduced expressions for π?

The exchange axiom in Π answers this last question when Q is a list of $1 +$ length (π) simple reflections (see Lemma 3.5). The general answer, when Q and π are arbitrary, lies in properties of the subword complex $\Delta(Q, \pi)$, whose facets correspond (by definition) to reduced subwords of Q having product π.

Both the topology and combinatorics of $\Delta(Q, \pi)$ are governed by the exchange axiom in a strong sense. Our first main result, Theorem 3.7, says that subword complexes are homeomorphic to balls or spheres. The proof uses the fact that subword complexes are shellable, which was demonstrated in [KM03] by exhibiting an explicit vertex decomposition. The lurking exchange axiom surfaces here as the transition between adjacent facets across the codimension 1 face joining them.

Given their topological simplicity, the invariants of subword complexes necessarily derive from more refined combinatorial data, namely the links of all faces. Therefore we focus on homological aspects of Stanley–Reisner theory in Section 4, where we calculate the Hilbert series of face rings of subword complexes in Theorems 4.1 and 4.4. The exchange axiom here gives rise to the criterion for a face to lie in the boundary of a subword complex.

The structure theorem for subword complexes, Theorem 3.7, is reminiscent of fundamental results for Bruhat as well as weak orders, and also for finite distributive lattices. The topology of subword complexes looks similar to that of order complexes of intervals in the Bruhat order studied by Björner and Wachs [BW82], even though there seems to be little direct connection to subword complexes. Indeed, the simplicial complexes in [BW82] are by definition independent of the reduced expressions involved, although the lexicographic shellings there depend on such choices. In contrast, Björner [Bjö84, Section 3] concerns himself with intervals in weak orders, where the reduced expressions involved form the substance of the simplicial complexes, as they do for subword complexes. Björner proves that intervals in the weak order are homotopy equivalent to balls or spheres. Our results are geometrically somewhat stronger, in that we prove not just homotopy equivalence, but homeomorphism.

The comparison between subword complexes and order complexes of intervals in the weak order occurs most clearly when the word Q contains as a subword every reduced expression for π. In essence, the reduced expressions for π must be repeated often enough, and in enough locations inside Q, to make $\Delta(Q, \pi)$ homeomorphic to a manifold, whereas the set of reduced expressions for π—without repeats—only achieves homotopy equivalence. Some open questions in Section 6 are relevant here.

The plan of the paper is as follows. We review in Section 2 the shelling construction and its proof from [KM03], along with related definitions. Next, we prove the structure theorem in Section 3. Section 4 contains the Hilbert series calculation, which requires a review of Hochster's formula and the Alexander inversion formula [Mil00], the latter expressing the simple relation between the Hilbert series of a squarefree monomial ideal and that of its Alexander dual. In Section 5, we apply our Hilbert series formula in the context of symmetric groups from [KM03] to deduce two formulae for Grothendieck polynomials, one of which is due originally to Fomin and Kirillov [FK94]. Finally, we present some open problems in Section 6.

We note that the particular subword complexes in Example 5.1 were, in fact, the special cases that originally led us to define subword complexes in general. These special cases appear as the initial schemes of certain types of determinantal varieties ('matrix Schubert varieties'). The shellability for subword complexes proved in [KM03] and reviewed in Section 2 allowed us to give in [KM03] an independent proof of Cohen–Macaulayness for matrix Schubert varieties [Ful92], and therefore also for ordinary Schubert varieties in the flag manifold [Ram85].

2. Subword complexes

We deal with an arbitrary Coxeter system (Π, Σ) consisting of a Coxeter group Π and a set Σ of simple reflections minimally generating Π. See [Hum90] for background. In Section 5, we shall be particularly interested in an application where $\Pi = S_n$ is the symmetric group, and Σ consists of the adjacent transpositions $s_1, \ldots, s_{n-1}$, where s_i switches i and $i+1$. Here is our main definition, copied from [KM03, Definition 1.8.1].

Definition 2.1. A *word* of size m is an ordered sequence $Q = (\sigma_1, \ldots, \sigma_m)$ of elements of Σ. An ordered subsequence P of Q is called a *subword* of Q.

1. P *represents* $\pi \in \Pi$ if the ordered product of the simple reflections in P is a reduced decomposition for π.
2. P *contains* $\pi \in \Pi$ if some subsequence of P represents π.

The *subword complex* $\Delta(Q, \pi)$ is the set of subwords $Q \backslash P$ whose complements P contain π.

In other words, if $Q \backslash D$ is a facet of the subword complex $\Delta(Q, \pi)$, then the reflections in D give a reduced expression for π. Note that subwords of Q come with their embeddings into Q, so two subwords P and P' involving reflections at different positions in Q are unequal, even if the sequences of reflections in P and P' are equal.

Often we write Q as a string without parentheses or commas, and abuse notation by saying that Q is a word in Π, without explicit reference to Σ. Note that Q need not itself be a reduced expression. The following lemma is immediate from the definitions and the fact that all reduced expressions for $\pi \in \Pi$ have the same length.

Lemma 2.2. *$\Delta(Q, \pi)$ is a pure simplicial complex whose facets are the subwords $Q \backslash P$ such that $P \subseteq Q$ represents π.*

Example 2.3. Consider the subword complex $\Delta = \Delta(s_3 s_2 s_3 s_2 s_3, 1432)$ in $\Pi = S_4$. This $\pi = 1432$ has two reduced expressions, namely $s_3 s_2 s_3$ and $s_2 s_3 s_2$. Labeling the vertices of a pentagon with the reflections in $Q = s_3 s_2 s_3 s_2 s_3$ (in cyclic order), we find that the facets of Δ are the pairs of adjacent vertices. Therefore Δ is the pentagon.

Definition 2.4. Let Δ be a simplicial complex and $F \in \Delta$ a face.

1. The *deletion* of F from Δ is $\operatorname{del}(F, \Delta) = \{G \in \Delta \mid G \cap F = \emptyset\}$.
2. The *link* of F in Δ is $\operatorname{link}(F, \Delta) = \{G \in \Delta \mid G \cap F = \emptyset \text{ and } G \cup F \in \Delta\}$.

Δ is *vertex-decomposable* if Δ is pure and either (1) $\Delta = \{\emptyset\}$, or (2) for some vertex $v \in \Delta$, both $\operatorname{del}(v, \Delta)$ and $\operatorname{link}(v, \Delta)$ are vertex-decomposable. A *shelling* of Δ is an ordered list $F_1, F_2, \ldots, F_t$ of its facets such that $\bigcup_{j<i} \hat{F}_j \cap \hat{F}_i$ is a subcomplex generated by codimension 1 faces of F_i for each $i \leqslant t$, where $\hat{F}$ denotes the set of faces of F. We say that Δ is *shellable* if it is pure and has a shelling.

Vertex-decomposability can be seen as a sort of universal property. Indeed, suppose that $\mathscr{F}$ is a family of pure simplicial complexes in which every nonempty complex $\Delta \in \mathscr{F}$ has a vertex whose link and deletion both lie in $\mathscr{F}$. Then $\mathscr{F}$ consists of vertex-decomposable complexes. The set of vertex-decomposable complexes is the largest (hence universal) such family.

In the above definition, the empty set $\emptyset$ is a perfectly good face of Δ, representing the empty set of vertices; we set its dimension equal to -1. Thus $\Delta = \{\emptyset\}$ is a sphere of dimension -1, with reduced homology $\mathbb{Z}$ in dimension -1.

The notion of vertex-decomposability was introduced by Provan and Billera [BP79], who proved that it implies shellability. For the convenience of the reader, the proof of the next result is copied more or less verbatim from [KM03, Section 1.8].

Theorem 2.5. *Subword complexes $\Delta(Q, \pi)$ are vertex-decomposable, hence shellable.*

Proof. Supposing that $Q = (\sigma, \sigma_2, \sigma_3, \ldots, \sigma_m)$, it suffices to show that both the link and the deletion of σ from $\Delta(Q, \pi)$ are subword complexes. By definition, both consist of subwords of $Q' = (\sigma_2, \ldots, \sigma_m)$. The link is naturally identified with the subword complex $\Delta(Q', \pi)$. For the deletion, there are two cases. If $\sigma\pi$ is longer than π, then the deletion of σ equals its link because no reduced expression for π begins with σ. On the other hand, when $\sigma\pi$ is shorter than π, the deletion is $\Delta(Q', \sigma\pi)$. $\square$

Remark 2.6. Among the known vertex decomposable simplicial complexes are the dual greedoid complexes [BKL85], which include the matroid complexes. Although subword complexes strongly resemble dual greedoid complexes, the exchange axioms defining greedoids seem to be slightly stronger than the exchange axioms for facets of subword complexes imposed by Coxeter relations. In particular, the naïve ways to correspond subword complexes to dual greedoid complexes do not work, and we conjecture that they are not in general isomorphic to dual greedoid complexes.

To be precise, a collection M of subsets of a finite vertex set Q constitutes the *feasible subsets* of a *greedoid* when $\emptyset \in M$, and

> if X and Y are in M with $|X| > |Y|$, then there is some element $x \in X \setminus Y$ such that $Y \cup x$ lies in M.

The facets of the *dual greedoid complex* are then the complements in Q of the maximal elements (*bases*) in M.

There is a natural attempt at defining a greedoid whose dual complex is $\Delta(Q,\pi)$: namely, let $M(Q,\pi)$ be the collection of subwords of Q that are themselves *reduced* subwords of some $P \subseteq Q$ representing π. Thus an element $Y \in M(Q,\pi)$ is a sublist of Q such that (i) the ordered product of elements in Y has length $|Y|$, and (ii) there is some sublist $Z \subseteq Q$ such that $Y \cup Z$ is a reduced expression for π. However, this $M(Q,\pi)$ need not be a greedoid.

An easy nongreedoid example occurs when $\pi = 12543 = s_3s_4s_3 = s_4s_3s_4$ and Q is the reduced expression $s_4s_3s_2s_1s_4s_3s_2s_4s_3s_4$ for the long word in S_5:

$$\begin{array}{rcccccccccc}
Q = & s_4 & s_3 & s_2 & s_1 & s_4 & s_3 & s_2 & s_4 & s_3 & s_4, \\
X = & & s_3 & & & s_4 & s_3, & & & & \\
Y = & s_4 & & & & & & & & s_3, & \\
Z = & & & & & & & & & & s_4.
\end{array}$$

Moving any of the elements from X down to Y creates a nonreduced expression.

The reader is invited to find a general construction of greedoids making subword complexes into dual greedoid complexes; we conjecture that none exists. Note that any successful attempt will exclude the subword Y above from the feasible set.

3. Balls or spheres

Knowing now that subword complexes in Coxeter groups are shellable, we are able to prove a much more precise statement. Our proof technique requires a certain deformation of the group algebra of a Coxeter group. As we shall see in Lemma 3.4.1, the Demazure product in the following definition "detects" Bruhat order on arbitrary words by a subword condition, just like the ordinary product detects Bruhat order on reduced words by a subword condition.

Definition 3.1. Let R be a commutative ring, and $\mathscr{D}$ a free R-module with basis $\{e_\pi | \pi \in \Pi\}$. Defining a multiplication on $\mathscr{D}$ by

$$e_\pi e_\sigma = \begin{cases} e_{\pi\sigma} & \text{if length } (\pi\sigma) > \text{length } (\pi) \\ e_\pi & \text{if length } (\pi\sigma) < \text{length } (\pi) \end{cases} \tag{1}$$

for $\sigma \in \Sigma$ yields the *Demazure algebra* of (Π, Σ) over R. Define the *Demazure product* $\delta(Q)$ of the word $Q = (\sigma_1, \ldots, \sigma_m)$ by $e_{\sigma_1} \cdots e_{\sigma_m} = e_{\delta(Q)}$.

Example 3.2. Let $\Pi = S_n$ act on the polynomial ring $R[x_1, \ldots, x_n]$ by permuting the variables. Define the *Demazure operator* $\bar{\partial}_i$ for $i = 1, \ldots, n-1$ on a polynomial $f = f(x_1, \ldots, x_n)$ with coefficients in R by

$$\bar{\partial}_i(f) = \frac{x_{i+1}f - x_i(s_i \cdot f)}{x_{i+1} - x_i}.$$

Checking monomial by monomial in f reveals that the denominator divides the numerator, so this rational function is really a polynomial in $R[x_1, \dots, x_n]$. The algebra $\mathscr{D}$ is isomorphic to the algebra generated over R by the Demazure operators $\bar{\partial}_i$; hence the name 'Demazure algebra'. In this case, the fact that $\mathscr{D}$ is an associative algebra with given free R-basis follows from the easily verified fact that the Demazure operators satisfy the Coxeter relations.

Remark 3.3. The operators in Example 3.2 and the related 'divided difference' operators were introduced by Demazure [Dem74] and Bernstein–Gel′fand–Gel′fand [BGG73] for arbitrary Weyl groups. Their context was the calculation of the cohomology and K-theory classes of Schubert varieties in G/P via desingularization. The operators $\bar{\partial}_i$, which are frequently denoted in the literature by π_i, were called *isobaric divided differences* by Lascoux and Schützenberger [LS82b]. See Section 5 for the relation to Grothendieck polynomials, and [Mac91] for background on the algebra of divided differences.

In general, the fact that the equations in (1) define an associative algebra is the special case of [Hum90, Theorem 7.1] where all of the 'a' variables equal 1 and all of the 'b' variables are zero. Observe that the ordered product of a word equals the Demazure product if the word is reduced. Here are some basic properties of Demazure products, using '$\geqslant$' and '$>$' signs to denote the Bruhat partial order on Π, in which $\tau \geqslant \pi$ if some (and hence every) reduced word representing τ contains a subword representing π [Hum90, Section 5.9]. For notation in the proof and henceforth, we write $Q \backslash \sigma_i$ for the word of size $m-1$ obtained from $Q = (\sigma_1, \dots, \sigma_m)$ by omitting σ_i.

Lemma 3.4. *Let P be a word in Π and let $\pi \in \Pi$.*

1. *The Demazure product $\delta(P)$ is $\geqslant \pi$ if and only if P contains π.*
2. *If $\delta(P) = \pi$, then every subword of P containing π has Demazure product π.*
3. *If $\delta(P) > \pi$, then P contains a word T representing an element $\tau > \pi$ satisfying $|T| = \text{length}\,(\tau) = \text{length}\,(\pi) + 1$.*

Proof. If $P' \subseteq P$ and P' contains π, then P' contains $\delta(P')$ and $\pi = \delta(P) \geqslant \delta(P') \geqslant \pi$, proving part 2 from part 1. Choosing any $\tau \in \Pi$ such that $\text{length}\,(\tau) = \text{length}\,(\pi) + 1$ and $\pi < \tau \leqslant \delta(P)$ proves part 3 from part 1.

Now we prove part 1. Suppose $\pi' = \delta(P) \geqslant \pi$, and let $P' \subseteq P$ be the subword obtained by reading P in order, omitting any reflections along the way that do not increase length. Then P' represents π' by definition, and contains π because any reduced expression for π' contains a reduced expression for π.

If $T \subseteq P$ represents π, then use induction on $|P|$ as follows. Let $\sigma \in \Sigma$ be the last reflection in the list P, so $\delta(P)\sigma < \delta(P)$ by definition of Demazure product, and $\delta(P \backslash \sigma)$ equals either $\delta(P)$ or $\delta(P)\sigma$. If $\pi\sigma > \pi$ then $T \subseteq P \backslash \sigma$, so $\pi \leqslant \delta(P \backslash \sigma) \leqslant \delta(P)$ by induction. If $\pi\sigma < \pi$ and $T' \subset T$ represents $\pi\sigma$, then $T' \subseteq P \backslash \sigma$ and hence $\pi\sigma \leqslant \delta(P \backslash \sigma)$ by induction. Since $\pi\sigma < \pi$, we have $\pi\sigma \leqslant \delta(P \backslash \sigma) \Rightarrow \pi \leqslant \delta(P)$. $\square$

Lemma 3.5. *Let T be a word in Π and $\pi \in \Pi$ such that $|T| = \text{length}(\pi) + 1$.*

1. *There are at most two elements $\sigma \in T$ such that $T \backslash \sigma$ represents π.*
2. *If $\delta(T) = \pi$, then there are two distinct $\sigma \in T$ such that $T \backslash \sigma$ represents π.*
3. *If T represents $\tau > \pi$, then $T \backslash \sigma$ represents π for exactly one $\sigma \in T$.*

Proof. Part 1 is obvious if $|T| \leqslant 2$, so suppose there are elements $\sigma_1, \sigma_2, \sigma_3 \in T$ (in order of appearance) such that $T \backslash \sigma_i$ represents π for each $i = 1, 2, 3$. Writing $T = T_1\sigma_1T_2\sigma_2T_3\sigma_3T_4$, we find that

$$T_1T_2\sigma_2T_3\sigma_3T_4 = T_1\sigma_1T_2T_3\sigma_3T_4.$$

Canceling T_1 from the left and $T_3\sigma_3T_4$ from the right yields $T_2\sigma_2 = \sigma_1T_2$. It follows that $\pi = T_1\sigma_1T_2\sigma_2T_3T_4 = T_1\sigma_1\sigma_1T_2T_3T_4 = T_1T_2T_3T_4$, contradicting the hypothesis that $\text{length}(\pi) = |T| - 1$.

In part 2, $\delta(T) = \pi$ means there is some $\sigma \in T$ such that

(i) $T = T_1\sigma T_2$;
(ii) T_1T_2 represents π; and
(iii) $\tau_1 > \tau_1\sigma$, where T_1 represents τ_1.

Omitting some σ' from T_1 leaves a reduced expression for $\tau_1\sigma$ by (iii). It follows that $T \backslash \sigma'$ and $T \backslash \sigma$ both represent π.

Part 3 is the exchange condition. □

Lemma 3.6. *Suppose every codimension 1 face of a shellable simplicial complex Δ is contained in at most two facets. Then Δ is a topological manifold-with-boundary that is homeomorphic to either a ball or a sphere. The facets of the topological boundary of Δ are the codimension 1 faces of Δ contained in exactly one facet of Δ.*

Proof. See Björner et al. [BLSWZ99, Proposition 4.7.22]. □

Theorem 3.7. *The subword complex $\Delta(Q, \pi)$ is a either a ball or a sphere. A face $Q \backslash P$ is in the boundary of $\Delta(Q, \pi)$ if and only if P has Demazure product $\delta(P) \neq \pi$.*

Proof. That every codimension 1 face of $\Delta(Q, \pi)$ is contained in at most two facets is the content of part 1 in Lemma 3.5, while shellability is Theorem 2.5. This verifies the hypotheses of Lemma 3.6 for the first sentence of the theorem.

If $Q \backslash P$ is a face and P has Demazure product $\neq \pi$, then $\delta(P) > \pi$ by part 1 of Lemma 3.4. Choosing T as in part 3 of Lemma 3.4, we find by part 3 of Lemma 3.5 that $Q \backslash T$ is a codimension 1 face contained in exactly one facet of $\Delta(Q, \pi)$. Thus, using Lemma 3.6, we conclude that $Q \backslash P \subseteq Q \backslash T$ is in the boundary of $\Delta(Q, \pi)$.

If $\delta(P) = \pi$, on the other hand, part 2 of Lemmas 3.4 and 3.5 say that every codimension 1 face $Q \backslash T \in \Delta(Q, \pi)$ containing $Q \backslash P$ is contained in two facets of $\Delta(Q, \pi)$. Lemma 3.6 says each such $Q \backslash T$ is in the interior of $\Delta(Q, \pi)$, whence $Q \backslash P$ must itself be an interior face. □

Corollary 3.8. *The complex $\Delta(Q,\pi)$ is a sphere if $\delta(Q)=\pi$ and a ball otherwise.*

4. Hilbert series

Let us review some standard notions from Stanley–Reisner theory. Fix a field $\mathbf{k}$ and a set $\mathbf{z}=z_1,\dots,z_m$ of variables. Suppose Δ is a simplicial complex with m vertices, which we think of as corresponding to the simple reflections $\sigma_1,\dots,\sigma_m$ in the word Q. Recall that the *Stanley–Reisner ideal* of Δ is $I_\Delta=\langle \prod_{i\in D} z_i \mid D\notin\Delta\rangle$, the ideal generated by monomials corresponding to the (minimal) nonfaces of Δ. Equivalently,

$$I_\Delta=\bigcap_{D\in\Delta}\langle z_i \mid i\notin D\rangle$$

is an intersection of prime ideals for faces of Δ by an easy exercise. By definition, the *Hilbert series* $H(\mathbf{k}[\Delta];\mathbf{z})$ of the *Stanley–Reisner ring* $\mathbf{k}[\Delta]=\mathbf{k}[\mathbf{z}]/I_\Delta$ equals the sum of all monomials in $\mathbf{k}[\mathbf{z}]$ that lie outside I_Δ. Thus $H(\mathbf{k}[\Delta];\mathbf{z})$ is the sum of all monomials outside every one of the ideals $\langle z_i \mid i\notin D\rangle$ for faces $D\in\Delta$. This sum is over the monomials $\mathbf{z}^{\mathbf{b}}$ for $\mathbf{b}\in\mathbb{N}^m$ having support exactly D for some face $D\in\Delta$:

$$H(\mathbf{k}[\Delta];\mathbf{z})=\sum_{D\in\Delta}\prod_{i\in D}\frac{z_i}{1-z_i}=\sum_{D\in\Delta}\frac{\prod\limits_{i\in D}(z_i)\prod\limits_{i\notin D}(1-z_i)}{\prod_{i=1}^m(1-z_i)}. \tag{2}$$

In the special case where $\Delta=\Delta(Q,\pi)$ is a subword complex, the Stanley–Reisner ideal is the intersection $I_\Delta=\bigcap\langle z_i \mid \sigma_i\in P\rangle$ over subwords $P\subseteq Q$ such that P represents π. Now we are ready to state the main result of this section.

Theorem 4.1. *If Δ is the subword complex $\Delta(Q,\pi)$ and $\ell=$ length (π), then the Hilbert series of the Stanley–Reisner ring $\mathbf{k}[\Delta]$ is*

$$H(\mathbf{k}[\Delta];\mathbf{z})=\frac{\sum_{\delta(P)=\pi}(-1)^{|P|-\ell}(\mathbf{1}-\mathbf{z})^P}{\prod_{i=1}^m(1-z_i)},$$

where $(\mathbf{1}-\mathbf{z})^P=\prod_{\sigma_i\in P}(1-z_i)$, and the sum is over subwords $P\subseteq Q$.

The proof of Theorem 4.1 is after Proposition 4.3. First, we set about stating and proving the two results used in the proof of the theorem.

In general, if Γ is an arbitrary monomial ideal $J\subseteq\mathbf{k}[\mathbf{z}]$, or a quotient $\mathbf{k}[\mathbf{z}]/J$, then the Hilbert series of Γ (which is the sum of all monomials inside or outside J, respectively) has the form

$$H(\Gamma;\mathbf{z})=\frac{\mathcal{K}(\Gamma;\mathbf{z})}{\prod_{i=1}^m(1-z_i)},$$

and we call $\mathcal{K}(\Gamma;\mathbf{z})$ the *K-polynomial* or *Hilbert numerator* of Γ. It has the following direct interpretation in terms of $\mathbb{Z}^m$-graded homological algebra. Since Γ is $\mathbb{Z}^m$-graded, it has a minimal $\mathbb{Z}^m$-graded free resolution

$$0 \leftarrow \Gamma \leftarrow E_0 \leftarrow E_1 \leftarrow \cdots \leftarrow E_m \leftarrow 0, \qquad E_j = \bigoplus_{P \subseteq Q} \mathbf{k}[\mathbf{z}](-\deg \mathbf{z}^P)^{\beta_{j,P}}, \tag{3}$$

where $\beta_{j,P}$ is the jth *Betti number* of Γ in $\mathbb{Z}^m$-graded degree $\deg \mathbf{z}^P$. Then the K-polynomial of Γ is $\mathcal{K}(\Gamma;\mathbf{z}) = \sum_j (-1)^j \beta_{j,P} \cdot \mathbf{z}^P$.

Hochster's formula, which we shall state in (4), says how to calculate explicitly the Betti numbers of the *Alexander dual ideal* of I_Δ, which is defined by

$$I_\Delta^\star = \left\langle \prod_{i \notin D} z_i \mid D \in \Delta \right\rangle.$$

Note that the generators of $I_\Delta^\star$ are obtained by multiplying the variables in each prime component of I_Δ. Thus, for instance, when $\Delta = \Delta(Q,\pi)$ is a subword complex, we get

$$I_\Delta^\star = \langle \mathbf{z}^P \mid P \subseteq Q \quad \text{and} \quad P \text{ represents } \pi \rangle,$$

where $\mathbf{z}^P = \prod_{\sigma_i \in P} z_i$ for any subword $P \subseteq Q$. Hochster's formula [MP01, p. 45] now says that, in terms of reduced homology of $\Delta = \Delta(Q,\pi)$ over the field $\mathbf{k}$, the $\mathbb{Z}^m$-graded Betti numbers of $I_\Delta^\star$ over $\mathbf{k}[\mathbf{z}]$ are

$$\beta_{j,P} = \dim_{\mathbf{k}} \tilde{H}_{j-1}(\mathrm{link}(Q \backslash P, \Delta); \mathbf{k}). \tag{4}$$

Lemma 4.2. *If Δ is the subword complex $\Delta(Q,\pi)$ and $\ell = \mathrm{length}\,(\pi)$, then*

$$\mathcal{K}(I_\Delta^\star;\mathbf{z}) = \sum_{\substack{P \subseteq Q \\ \delta(P)=\pi}} (-1)^{|P|-\ell} \mathbf{z}^P$$

is the Hilbert numerator of the Alexander dual ideal.

Proof. Let $Q \backslash P \in \Delta$, so $P \subseteq Q$ contains π. By Theorem 3.7, either $\delta(P) \neq \pi$, in which case link $(Q \backslash P, \Delta)$ is contractible, or $\delta(P) = \pi$, in which case link $(Q \backslash P, \Delta)$ is a sphere of dimension

$$(\dim \Delta) - |Q \backslash P| = (|Q| - \ell - 1) - |Q \backslash P| = |P| - \ell - 1.$$

(Recall that a sphere of dimension -1 is taken to mean the empty complex $\{\emptyset\}$ having nonzero reduced homology in dimension -1.) Therefore $\tilde{H}_{j-1}\mathrm{link}\,(Q \backslash P, \Delta)$ is zero unless $\delta(P) = \pi$ and $j = |P| - \ell$, in which case the reduced homology has dimension 1. Now apply (4) to the formula $\mathcal{K}(\Gamma;\mathbf{z}) = \sum_j (-1)^j \beta_{j,P} \cdot \mathbf{z}^P$. $\square$

Lemma 4.2 helps us calculate the Hilbert series of $\mathbf{k}[\Delta]$ because the K-polynomials of the Stanley–Reisner ring $\mathbf{k}[\Delta]$ and the Alexander dual ideal $I_\Delta^\star$ are intimately related, as the next result demonstrates. Although it holds more generally for the "squarefree modules" of Yanagawa [Yan00], as shown in [Mil00, Theorem 4.36], we include an elementary proof of Proposition 4.3 because of its simplicity. A $\mathbb{Z}$-graded version was proved by Terai for squarefree ideals using some calculations involving f-vectors of simplicial complexes [Ter99, Lemma 2.3]. For notation, $\mathscr{K}(\Gamma; \mathbf{1}-\mathbf{z})$ is the polynomial obtained from $\mathscr{K}(\Gamma; \mathbf{z})$ by substituting $1 - z_i$ for each variable z_i.

Proposition 4.3 (Alexander inversion formula). *For any simplicial complex Δ we have* $\mathscr{K}(\mathbf{k}[\Delta]; \mathbf{z}) = \mathscr{K}(I_\Delta^\star; \mathbf{1}-\mathbf{z})$.

Proof. See (2) for the Hilbert series of $\mathbf{k}[\Delta]$. On the other hand, the Hilbert series of $I_\Delta^\star$ is the sum of all monomials $\mathbf{z}^\mathbf{b}$ divisible by $\prod_{i \notin D} z_i$ for some $D \in \Delta$:

$$H(I_\Delta^\star; \mathbf{z}) = \sum_{D\in\Delta} \prod_{i\notin D} \frac{z_i}{1-z_i} = \sum_{D\in\Delta} \frac{\prod_{i\notin D}(z_i)\prod_{i\in D}(1-z_i)}{\prod_{i=1}^m (1-z_i)}. \tag{5}$$

Now compare the last expressions of (2) and (5). □

Proof of Theorem 4.1. Lemma 4.2 gives the numerator of the Hilbert series of the Alexander dual ideal $I_\Delta^\star$, and Proposition 4.3 says how to recover the numerator of the Hilbert series of the Stanley–Reisner ring from that. □

Theorem 4.1 can be restated in a somewhat different form, grouping subwords with Demazure product π according to their lexicographically first reduced subwords for π. Given a reduced subword $D \subseteq Q$, say that $\sigma_i \in Q \backslash D$ is *absorbable* if the word $T = D \cup \sigma_i$ in Q has the properties: (i) $\delta(T) = \delta(D)$, and (ii) the unique reflection $\sigma_j \in D$ (afforded by Lemma 3.5.2) satisfying $\delta(T\backslash\sigma_j) = \delta(D)$ has index $j < i$.

Theorem 4.4. *If Δ is the subword complex $\Delta(Q, \pi)$ and* $\mathrm{abs}\,(D) \subseteq Q$ *is the set of absorbable reflections for each reduced subword $D \subseteq Q$, then $\mathbf{k}[\Delta]$ has K-polynomial*

$$\mathscr{K}(\mathbf{k}[\Delta]; \mathbf{z}) = \sum_{\text{facets} Q\backslash D} (\mathbf{1}-\mathbf{z})^D \mathbf{z}^{\mathrm{abs}(D)},$$

where $(\mathbf{1}-\mathbf{z})^D = \prod_{\sigma_i \in D}(1-z_i)$, *and* $\mathbf{z}^{\mathrm{abs}\,(D)} = \prod_{\sigma_i \in \mathrm{abs}\,(D)} z_i$.

Proof. Given a subword $P \subseteq Q$, say that P *simplifies* to $D \subseteq P$, and write $P \rightsquigarrow D$, if D is the lexicographically first subword of P with Demazure product $\delta(P)$. If P has Demazure product π and $P \rightsquigarrow D$, the subword $Q\backslash D$ is automatically a facet of Δ.

If we denote by $P_{\leqslant i}$ the initial string of reflections in P with index at most i, the simplification D is obtained from P by omitting any reflection $\sigma_i \in P$ such that

$\delta(P_{\leqslant i-1}) = \delta(P_{\leqslant i})$. Theorem 4.1 says that

$$\mathscr{K}(\mathbf{k}[\Delta]; \mathbf{z}) = \sum_{\text{facets} Q\setminus D} (\mathbf{1} - \mathbf{z})^D \sum_{P \rightsquigarrow D} (\mathbf{z} - \mathbf{1})^{P\setminus D}.$$

Now note that subwords P simplifying to D are (by definition of Demazure product) obtained by adding to D (at will) some of its absorbable reflections in Q. Therefore

$$\sum_{P \rightsquigarrow D} (\mathbf{z} - \mathbf{1})^{P\setminus D} = \prod_{\sigma_i \in \mathrm{abs}(D)} (1 + (z_i - 1)) = \mathbf{z}^{\mathrm{abs}\,(D)},$$

completing the proof. □

Remark 4.5. The Hilbert numerator as expressed in Theorem 4.4 looks more like one would expect from a shellable simplicial complex, using a version of [Sta96, Proposition 2.3] suitably enhanced for the fine grading. We believe the reason comes from the *facet adjacency graph* $\Gamma(Q, \pi)$ of $\Delta(Q, \pi)$, which by definition has the facets of $\Delta(Q, \pi)$ for vertices, while its edges are the interior *ridges* (codimension 1 faces) of $\Delta(Q, \pi)$. Two facets are adjacent if they share a ridge. Note that every interior ridge lies in exactly two facets by Lemma 3.6.

The facet adjacency graph $\Gamma(Q, \pi)$ can be oriented, by having each ridge $Q\setminus P$ point toward the facet $Q\setminus D$ whenever P simplifies to D. The resulting directed facet adjacency graph is acyclic—so its transitive closure is a poset—because the relation by ridges is a subrelation of lexicographic order. We believe that every linear extension of this poset gives a shelling order for $\Delta(Q, \pi)$. The shelling formulae we get for the K-polynomial will all be the same, namely the one in Theorem 4.4.

5. Combinatorics of Grothendieck polynomials

The Grothendieck polynomial $\mathcal{G}_w(\mathbf{x})$ in variables $x_1, \ldots, x_n$ and its "double" analogue $\mathcal{G}_w(\mathbf{x}, \mathbf{y})$ represent the classes of Schubert varieties in ordinary and equivariant K-theory of the flag manifold [LS82b]. Their algebraic definition will be recalled below. The goal of this section is to derive as special cases of Theorems 4.1 and 4.4 two formulae for Grothendieck polynomials. The first formula (Corollary 5.4) coincides with a special case of a formula discovered by Fomin and Kirillov [FK94]. It is the K-theoretic analogue of the Billey–Jockusch–Stanley formula for the Schubert polynomial $\mathfrak{S}_w(\mathbf{x})$ [BJS93,FS94], interpreted here for the first time in terms of simplicial topology. The second formula (Corollary 5.5) relates to other combinatorial models for Grothendieck polynomials in work of Lenart et al. [LRS03]. We begin with the example of subword complexes that pervades [KM03].

Example 5.1. Set $\Pi = S_{2n}$, and let

$$Q_{n\times n} = s_n s_{n-1} \ldots s_2 s_1 \;\; s_{n+1} s_n \ldots s_3 s_2 \;\; s_{n+2} s_{n+1} \ldots \;\; \ldots s_{n+2} s_{n+1} \;\; s_{2n-1} s_{2n-2} \ldots s_{n+1} s_n.$$

This is the *square word* from [KM03, Example 1.8.3], so named because the n^2 simple reflections in this list Q fill the $n \times n$ grid naturally by starting at the upper-right, continuing to the left, and subsequently reading each row from right to left, in turn. Observe that every occurrence of s_i in $Q_{n\times n}$ sits on the ith antidiagonal of the resulting square array.

Given $w \in S_n$ (not S_{2n}), the subword complex $\Delta = \Delta(Q_{n\times n}, w)$ plays a crucial role in the main theorems of [KM03]; see Proposition 5.3, below. For the Stanley–Reisner ring $\mathbf{k}[\Delta]$, we index the variables $\mathbf{z} = z_1, \ldots, z_{n^2}$ by their positions in the $n \times n$ grid, so $\mathbf{z} = \{z_{ij}\}_{i,j=1}^{n}$ with z_{11} at the upper-left and z_{1n} at the upper-right.

Definition 5.2. Let $w \in S_n$ be a permutation, and recall the Demazure operators $\bar{\partial}_i$ from Example 3.2. The *Grothendieck polynomial* $\mathcal{G}_w(\mathbf{x})$ is obtained recursively from the top one $\mathcal{G}_{w_0}(\mathbf{x}) := \prod_{i=1}^{n} (1 - x_i)^{n-i}$ via the recurrence

$$\mathcal{G}_{ws_i}(\mathbf{x}) = \bar{\partial}_i \mathcal{G}_w(\mathbf{x})$$

whenever length $(ws_i) <$ length (w). The *double Grothendieck polynomials* are defined by the same recurrence, but start from $\mathcal{G}_{w_0}(\mathbf{x}, \mathbf{y}) := \prod_{i+j \leqslant n} (1 - x_i y_j)$.

We use slightly different notation in Definition 5.2 than in [KM03, Definition 1.1.3]: the polynomial $\mathcal{G}_w(\mathbf{x}, \mathbf{y})$ here is obtained from the corresponding Laurent polynomial in [KM03] by setting each variable y_i^{-1} to y_i (the geometry in [KM03] required inverses). This alteration makes the notation more closely resemble that in [FK94], where their polynomial $\mathfrak{L}_w^{(-1)}(y, x)$ corresponds to what we call $\mathcal{G}_w(\mathbf{1} - \mathbf{x}, \mathbf{1} - \mathbf{y})$ here.

Grothendieck polynomials connect to subword complexes by part of the 'Gröbner geometry theorems' in [KM03]. In our context, they say the following.

Proposition 5.3 (Knutson and Miller [KM03, Theorems A and B]). *Suppose $w \in S_n$, and let $\Delta = \Delta(Q_{n\times n}, w)$ be the subword complex for the square word. Setting z_{ij} equal to $x_i y_j$ or to x_i in the Hilbert numerator $\mathcal{K}(\mathbf{k}[\Delta]; \mathbf{z})$ yields respectively the double Grothendieck polynomial $\mathcal{G}_w(\mathbf{x}, \mathbf{y})$ or the Grothendieck polynomial $\mathcal{G}_w(\mathbf{x})$.*

For notation, regard subwords $P \subseteq Q_{n\times n}$ as subsets of the $n \times n$ grid.

Corollary 5.4 (Fomin and Kirillov [FK94, Theorem 2.3, p. 190]). *If $w \in S_n$ and $Q_{n\times n}$ is the square word as in Example* 5.1, *then the double Grothendieck polynomial $\mathcal{G}_w(\mathbf{x}, \mathbf{y})$ satisfies*

$$\mathcal{G}_w(\mathbf{1} - \mathbf{x}, \mathbf{1} - \mathbf{y}) = \sum_{\substack{P \subseteq Q_{n\times n} \\ \delta(P) = w}} \prod_{(i,j) \in P} (-1)^{|P| - \ell} (x_i + y_j - x_i y_j),$$

where length $(w) = \ell$. *The version for single Grothendieck polynomials reads*

$$\mathcal{G}_w(\mathbf{1}-\mathbf{x}) = \sum_{\substack{P \subseteq Q_{n\times n} \\ \delta(P)=w}} (-1)^{|P|-\ell}\mathbf{x}^P, \qquad \textit{where} \qquad \mathbf{x}^P = \prod_{(i,j)\in P} x_i.$$

Proof. Apply Theorem 4.1 to the subword complex $\Delta = \Delta(Q_{n\times n}, w)$. Substituting $x_i y_j$ for z_{ij} as stipulated in Proposition 5.3 yields the double version after calculating $1-(1-x_i)(1-y_j) = x_i + y_j - x_i y_j$, while the single version follows trivially. $\square$

Corollary 5.4 can be rewritten in terms of absorbable reflections as in Theorem 4.4.

Corollary 5.5. *If* $w \in S_n$ *and* $\Delta(Q_{n\times n}, w)$ *is the square subword complex, then*

$$\mathcal{G}_w(\mathbf{x},\mathbf{y}) = \sum_{\substack{\text{facets} \\ Q_{n\times n}\setminus D}} \prod_{(i,j)\in D} (1-x_i y_j) \prod_{(i,j)\in \mathrm{abs}\,(D)} x_i y_j.$$

The version for single Grothendieck polynomials reads

$$\mathcal{G}_w(\mathbf{x}) = \sum_{\substack{\text{facets} \\ Q_{n\times n}\setminus D}} (\mathbf{1}-\mathbf{x})^D \mathbf{x}^{\mathrm{abs}\,(D)},$$

where $(\mathbf{1}-\mathbf{x})^D = \prod_{(i,j)\in D}(1-x_i)$ *and* $\mathbf{x}^{\mathrm{abs}\,(D)} = \prod_{(i,j)\in \mathrm{abs}\,(D)} x_i$.

Proof. Apply Proposition 5.3 to the result of Theorem 4.4 for $\Delta = \Delta(Q_{n\times n}, w)$. $\square$

Readers familiar with reduced pipe dreams (also called rc-graphs; see [KM03, Section 1.4] for an introduction) can see a geometric interpretation of absorbable reflections: given a reduced pipe dream D, an elbow tile is absorbable if the two pipes passing through it intersect in a crossing tile to its northeast. Thus the pipe dream D almost fails to be being a reduced pipe dream because of that elbow tile. Note that there must be exactly one reduced pipe dream with no absorbable elbow tiles (in [BB93] this is the 'bottom' rc-graph), because the constant term of the K-polynomial of any Stanley–Reisner ring—or indeed any non zero quotient of the polynomial ring—equals 1.

Here is a weird consequence of the Demazure product characterization of the Hilbert numerator $\mathcal{K}(\mathbf{k}[\Delta];\mathbf{z})$ for $\Delta = \Delta(Q_{n\times n}, w)$

Porism 5.6. *For each squarefree monomial* $\mathbf{z}^P$ *in the variables* $\mathbf{z} = (z_{ij})_{i,j=1}^{\infty}$, *there exists a unique permutation* $w \in S_\infty = \bigcup_n S_n$ *such that* $\mathbf{z}^P$ *appears with nonzero coefficient in the Hilbert numerator of the Alexander dual ideal* $I^\star_{\Delta(Q_{n\times n},w)}$ *for some* (*and hence all*) n *such that* $w \in S_n$. *The coefficient of* $\mathbf{z}^P$ *is* ± 1.

Proof. The permutation w in question is $\delta(P)$, by Lemma 4.2. $\square$

6. Open problems

The considerations in this paper motivate some questions concerning the combinatorics of reduced expressions in Coxeter groups.

Question 6.1. *Given an element $\pi \in \Pi$, what is the smallest size of a word in Σ containing every reduced expression for π as a subword*?

Note that a smallest size word containing all reduced expressions for π will not in general be unique. Indeed, even for the long word $w = 321 \in S_3$, there are two such: $s_1 s_2 s_1 s_2$ and $s_2 s_1 s_2 s_1$.

Question 6.1 asks for a measure of how far intervals in the weak order are from being subword complexes. Another measure would be provided by a solution to the following problem, which asks roughly how many faces must be added to order complexes of intervals in the weak order to get subword complexes. To be precise, let the *repetition number* $\text{repnum}\,(Q,\pi)$ be the largest number of times that a single reduced expression for π appears as a subword of Q.

Problem 6.2. *Describe the function sending* $\pi \mapsto \text{repnum}\,(\pi)$, *where* $\text{repnum}\,(\pi) = \min(\text{repnum}\,(Q) \mid Q$ *contains all reduced expressions for* $\pi)$.

Restricting to symmetric groups, for instance,

Question 6.3. *Is the function in Problem* 6.2 *bounded above on* $S_\infty = \bigcup_n S_n$? *If not, how does it grow*?

Given that subword complexes appeared naturally in the context of the geometry of Schubert varieties, it is natural to ask whether there are good geometric representatives for subword complexes.

Question 6.4. *Can any spherical subword complex be realized as a convex polytope*?

One could also take the opposite perspective, by starting with a simplicial sphere.

Problem 6.5. *Characterize those simplicial spheres realizable as subword complexes.*

Of course, in all of these problems it may be useful to try restricting to words in S_n.

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for Grothendieck polynomials in Corollary 5.5, which prompted us to include also Theorem 4.4.

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