European Journal of Combinatorics 30 (2009) 879–888

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

On the Lovász ϑ-number of almost regular graphs with application to Erdős–Rényi graphs

E. de Klerk [a], M.W. Newman [b], D.V. Pasechnik [c], R. Sotirov [a]

[a] *Department of Econometrics and Operations Research, Tilburg University, The Netherlands*

[b] *School of Mathematical Sciences at Queen Mary, University of London, UK*

[c] *School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore*

ARTICLE INFO

Article history:
Received 23 February 2007
Received in revised form
1 July 2008
Accepted 29 July 2008
Available online 17 October 2008

ABSTRACT

We consider k-regular graphs with loops, and study the Lovász ϑ-numbers and Schrijver ϑ'-numbers of the graphs that result when the loop edges are removed. We show that the ϑ-number dominates a recent eigenvalue upper bound on the stability number due to Godsil and Newman [C.D. Godsil and M.W. Newman. Eigenvalue bounds for independent sets, J. Combin. Theory B 98 (4) (2008) 721–734].

As an application we compute the ϑ and ϑ' numbers of certain instances of Erdős–Rényi graphs. This computation exploits the graph symmetry using the methodology introduced in [E. de Klerk, D.V. Pasechnik and A. Schrijver, Reduction of symmetric semidefinite programs using the regular *-representation, Math. Program. B 109 (2–3) (2007) 613–624].

The computed values are strictly better than the Godsil–Newman eigenvalue bounds.

1. Introduction

In this paper we study the Lovász ϑ-number [12] and Schrijver ϑ'-number [16] for classes of almost regular graphs, i.e. graphs that become regular if a 'small' number of loops are added to the edge set.

The purpose is to study upper bounds on the stability (independence) numbers of such graphs.

E-mail addresses: E.deKlerk@UvT.nl (E. de Klerk), M.Newman@qmul.ac.uk (M.W. Newman), dima@ntu.edu.sg (D.V. Pasechnik), R.Sotirov@UvT.nl (R. Sotirov).

0195-6698/$ – see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ejc.2008.07.022

Assume now that G is a k-regular graph with ℓ loops and adjacency matrix A, and let τ denote the smallest eigenvalue of A. Godsil and Newman [10] recently derived the following upper bound on $\alpha(G)$:

$$\alpha(G) \leq \frac{-\tau + \sqrt{\tau^2 + 4\left(\frac{k-\tau}{n}\right)\ell}}{2\left(\frac{k-\tau}{n}\right)}, \tag{1}$$

where n is the number of vertices, and $\alpha(G)$ is the stability number of G. Here, and throughout the paper, we use the convention that vertices with loops are allowed in a stable set.

For k-regular graphs without loops, i.e. if $\ell = 0$, (1) reduces to the well-known Hoffman–Delsarte eigenvalue bound; see [4] Section 3.3, or [3] page 115.

The Lovász ϑ-number is not defined for graphs with loops, but for the purpose of providing an upper bound on $\alpha(G)$ we simply delete the loop edges and compute the ϑ-number of the resulting graph. We will show that this ϑ-number, and therefore also the related Schrijver ϑ'-number, are upper bounded by the right-hand side of (1). This is a generalization of the well-known result that the ϑ-number bound is stronger than the Hoffman–Delsarte eigenvalue bound for k-regular graphs without loops.

In practice it is possible to compute ϑ and ϑ' for large graphs with symmetries, by using a methodology introduced in [8].

As an application we compute the ϑ and ϑ' numbers of certain instances of Erdős–Rényi graphs. The Erdős–Rényi graph $ER(q)$ is the graph whose vertices are the points of the projective plane $PG(2,q)$, with two vertices x and y adjacent if they are distinct and $x^{\mathrm{T}}y = 0$. The graph $ER(q)$ has q^2+q+1 vertices and can be made $(q+1)$-regular by adding $q+1$ loops. In the present work we restrict ourselves to q being an odd prime.

The $ER(q)$ graphs were first introduced in [2,5] as examples of graphs with many edges but no 4-cycle. They were further studied in [15,6,7,14,10].

Godsil and Newman [10] showed that, for $ER(q)$, the eigenvalue bound (1) becomes

$$\begin{aligned}\alpha(ER(q)) &\leq \frac{\sqrt{q} + \sqrt{q + 4(q+1)\frac{q+\sqrt{q}+1}{q^2+q+1}}}{2\frac{q+\sqrt{q}+1}{q^2+q+1}} \\ &= q^{3/2} - q + 2\sqrt{q} - 1/q + 3/q^2 + O\left(\frac{1}{q^3}\right).\end{aligned} \tag{2}$$

Recently, Mubayi and Williford [14] proved that

$$\alpha(ER(q)) \geq \frac{120}{73\sqrt{73}} q^{3/2} > 0.19239 q^{3/2},$$

which shows that the upper bound (2) is tight in terms of the dependence of its leading term on q.

In this paper, we apply the approach from [8] to compute the Lovász ϑ and Schrijver ϑ' numbers of $ER(q)$. We show that, for $q \leq 31$, odd and prime, the computed bounds are in fact *strictly* better than the eigenvalue bounds (2), although the differences are small.

Outline of the paper

The paper is organized in the following way. In Sections 2 and 3 we show that the ϑ-number dominates the Godsil–Newman eigenvalue bound (1). In Section 4 we define the Erdős–Rényi graph $ER(q)$ and review its properties. In Section 5 we provide basic facts on finite groups and regular $*$-representations of matrix algebras. In Section 6 we review how regular $*$-representations may be used to reduce the size of certain semidefinite programming problems, and apply this methodology to reduce the sizes of the semidefinite programming problems that define ϑ and ϑ'. Finally, in Section 7 we provide numerical results on the computation of $\vartheta(ER(q))$ and $\vartheta'(ER(q))$ for $q \leq 31$, odd, and prime.

Notation

We use $\mathrm{tr}(A)$ to denote the trace of a square matrix A. The space of symmetric matrices is denoted by:

$$\mathcal{S}_n := \{X \in \mathbb{R}^{n\times n} : X = X^{\mathrm{T}}\}.$$

For $A, B \in \mathcal{S}_n$, $A \succeq 0$ (resp. $A \succ 0$) denotes positive semidefiniteness (resp. positive definiteness), and $A \succeq B$ denotes $A - B \succeq 0$. The cone of $n \times n$ positive semidefinite matrices is denoted by

$$\mathcal{S}_n^+ := \{X \in \mathcal{S}_n : z^{\mathrm{T}}Xz \geq 0 \ \forall z \in \mathbb{R}^n\}.$$

For two matrices $A, B \in \mathcal{S}_n$, $A \geq B$, $(A > B)$ means $a_{ij} \geq b_{ij}$, $(a_{ij} > b_{ij})$ for all i, j. The vector of all ones is denoted by e and the matrix of all ones by J. We denote the Kronecker delta by δ_{ij}.

A graph with vertex set $V = \{1, \ldots, n\}$ and edge set E is denoted by $G = (V, E)$.

2. The maximum stable set problem, ϑ and ϑ'

Given a graph $G = (V, E)$ with adjacency matrix A, a subset $V' \subseteq V$ is called a stable set of G if the induced subgraph on V' contains no edges except loops. The *maximum stable set problem* is to find the stable set of maximum cardinality. The *stability number* $\alpha(G)$ is the cardinality of the largest stable set in the graph G.

The Lovász ϑ number

The Lovász ϑ-number, introduced in [12],

$$\left.\begin{array}{rl} \vartheta(G) := & \max \mathrm{tr}(JX) \\ & \text{s.t. } X_{ij} = 0, \quad \{i,j\} \in E \ (i \neq j) \\ & \quad \mathrm{tr}(X) = 1 \\ & \quad X \in \mathcal{S}_n^+, \end{array}\right\} \tag{3}$$

gives an upper bound on $\alpha(G)$.

The Schrijver ϑ' number

The Schrijver ϑ'-function [16] is defined as:

$$\begin{array}{rl} \vartheta'(G) := & \max \mathrm{tr}(JX) \\ & \text{s.t. } \mathrm{tr}(AX) = 0 \\ & \mathrm{tr}(X) = 1 \\ & \quad X \geq 0 \\ & \quad X \in \mathcal{S}_n^+. \end{array} \tag{4}$$

Clearly one has $\alpha(G) \leq \vartheta'(G) \leq \vartheta(G)$.

3. An eigenvalue bound and its relation to ϑ

Let $G = (V, E)$ be a k-regular graph with ℓ loops. Let A denote its adjacency matrix and $\tau < 0$ the smallest eigenvalue of A.

Godsil and Newman [10] derived the upper bound (1) on $\alpha(G)$ as follows. Let z be the characteristic vector of a maximum stable set of G, and assume that this stable set contains $\bar{\ell}$ loops.

Since $A - \tau I \succeq 0$, one has:

$$\left(z - \frac{\alpha(G)}{n}e\right)^{\mathrm{T}} (A - \tau I) \left(z - \frac{\alpha(G)}{n}e\right) \geq 0$$

which simplifies to

$$\left(\frac{k-\tau}{n}\right)\alpha(G)^2 + \tau\alpha(G) \leq \bar{\ell}.$$

Using $\bar{\ell} \leq \ell$, we obtain bound (1), and we reproduce it here for convenience:

$$\alpha(G) \leq \frac{-\tau + \sqrt{\tau^2 + 4\left(\frac{k-\tau}{n}\right)\ell}}{2\left(\frac{k-\tau}{n}\right)}.$$

We will show that $\vartheta(G)$ dominates the eigenvalue bound (1). To this end, consider the following formulation of the ϑ-number (see Lemma 2.17 in [13]):

$$\left.\begin{aligned} \vartheta(G) &= \max e^{\mathrm{T}} x \\ &\text{s.t.} \\ X - xx^{\mathrm{T}} &\succeq 0 \\ X_{ii} &= x_i \quad (i \in V) \\ X_{ij} &= 0 \quad (\{i,j\} \in E,\ i \neq j). \end{aligned}\right\} \tag{5}$$

Note that the first constraint implies that any feasible solution satisfies $x_i \in [0, 1] (i \in V)$.

Theorem 1. *Let $G = (V, E)$ be a connected, k-regular graph with ℓ loops. Let $\vartheta(G)$ be the Lovász ϑ-number of the graph obtained by removing the loop edges from E. One has*

$$\vartheta(G) \leq \frac{-\tau + \sqrt{\tau^2 + 4\left(\frac{k-\tau}{n}\right)\ell}}{2\left(\frac{k-\tau}{n}\right)}.$$

Proof. Let x, X denote an optimal solution of the ϑ formulation (5). Since

$$A - \tau I - \frac{k-\tau}{n} J \succeq 0,$$

one has

$$x^{\mathrm{T}} \left(A - \tau I - \frac{k-\tau}{n} J\right) x \geq 0.$$

Using $J = ee^{\mathrm{T}}$ and $e^{\mathrm{T}} x = \vartheta(G)$, this becomes

$$x^{\mathrm{T}} (A - \tau I) x \geq \frac{k-\tau}{n} \vartheta(G)^2.$$

We now use $X - xx^{\mathrm{T}} \succeq 0$ to find

$$\begin{aligned} x^{\mathrm{T}} (A - \tau I) x &= \operatorname{tr}\left((A - \tau I) xx^{\mathrm{T}}\right) \\ &\leq \operatorname{tr}\left((A - \tau I) X\right) \\ &\leq \ell - \tau \vartheta(G), \end{aligned}$$

where the last inequality is due to $\operatorname{tr}(AX) \leq \ell$ (since $X_{ii} = x_i \in [0, 1] (i \in V)$), and $\operatorname{tr}(X) = e^{\mathrm{T}} x = \vartheta(G)$. Thus we have obtained

$$\left(\frac{k-\tau}{n}\right) \vartheta(G)^2 + \tau \vartheta(G) - \ell \leq 0,$$

and the required result follows. □

In the next section we introduce the so-called Erdős–Rényi graphs, that form a class of 'almost regular' graphs. The aim is to illustrate the results of this section. In particular, we will show how to compute ϑ and ϑ' in an efficient manner for the Erdős–Rényi graphs, and will compare the results to the Godsil–Newman eigenvalue bound (1) for these graphs.

4. Erdős–Rényi graphs

Let V be a three-dimensional vector space over the finite field of order q, $GF(q)$, where q is an odd prime. There are q^2+q+1 one-dimensional subspaces of V: these are the *points* of $PG(2,q)$. There are q^2+q+1 two-dimensional subspaces of V: these are the *lines* of $PG(2,q)$. Each point may be represented by any non-zero vector in its one-dimensional subspace (which then spans that subspace). For background on projective planes, see [11].

The Erdős–Rényi graph $ER(q)$ is the graph whose vertices are the points of $PG(2,q)$, with two vertices x and y adjacent if they are distinct and $x^{\mathrm{T}}y=0$.

Consider the graph whose vertices are the points of $PG(2,q)$, with x and y adjacent if they are distinct and $x^{\mathrm{T}}My=0$, where

$$M=\begin{pmatrix}0&0&-1\\0&1&0\\-1&0&0\end{pmatrix}.$$

By the classification of bilinear forms over $GF(q)$ (see [11]), this graph is isomorphic to $ER(q)$. For convenience, we will use this definition of $ER(q)$ and let $\langle x,y\rangle := x^{\mathrm{T}}My$. The reason for the alternative definition is to avoid the problem of the distinguished cases of primes modulo 4.

Most vertices of $ER(q)$ have degree $q+1$ but there are $q+1$ vertices of degree q. These are known as *absolute vertices*, and are self-orthogonal (removing the word "distinct" from the definition of $ER(q)$ would make it regular, with loops). The absolute vertices form an independent set. There are $(q^2+q)/2$ vertices that are adjacent to exactly 2 absolute vertices each; these are the *external* vertices. The remaining $(q^2-q)/2$ vertices are adjacent to no absolute vertices; these are the *internal* vertices. See [15] for more details. We will denote the absolute, external, and internal vertices by $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{M}$, respectively. The automorphism group of $ER(q)$, for q an odd prime, is shown in [15] to be $P\mathcal{O}_3(q)$.

By the properties of ϑ, ϑ' and Theorem 1 we know that

$$\alpha(ER(q))\le\vartheta'(ER(q))\le\vartheta(ER(q))\le\frac{\sqrt{q}+\sqrt{q+4(q+1)\frac{q+\sqrt{q}+1}{q^2+q+1}}}{2\frac{q+\sqrt{q}+1}{q^2+q+1}},$$

where the last expression is the Godsil–Newman eigenvalue bound (2) for $ER(q)$.

Our goal is to use the automorphism group of $ER(q)$ to compute the ϑ and ϑ' numbers of $ER(q)$ in an efficient way. This is necessary since the computation of $\vartheta(ER(q))$ involves matrices of order $|V|=q^2+q+1$, and this computation is already problematic for relatively small values of q. To this end, we first review some facts from representation theory, and then review how they may be used to simplify the calculation of ϑ and ϑ' for symmetric graphs (like $ER(q)$). We will see that the values $\vartheta(ER(q))$ and $\vartheta'(ER(q))$ may actually be computed by solving semidefinite programs involving matrices of order $2q+11$.

5. Finite groups and regular *-representations

Let V be a finite set and S_V the group of all permutations of V. Let $\mathcal{G}$ be a finite group acting on V, and for each $g\in\mathcal{G}$ define $\pi_g: V\to V$ by $\pi_g(z)=g\cdot z$. Then $\pi_g\in S_V$, and $\phi:\mathcal{G}\to S_V$ given by $\phi_g:=\pi_g$ is a homomorphism, i.e. $\phi_{gg'}=\phi_g\phi_{g'}$ and $\phi_{g^{-1}}=\phi_g^{-1}$ for all $g,g'\in\mathcal{G}$.

The image ϕ_g of g under ϕ can be represented by the permutation matrix $P_g\in\mathbb{R}^{|V|\times|V|}$,

$$(P_g)_{x,y}:=\begin{cases}1 & \text{if } \phi_g(x)=y\\ 0 & \text{otherwise,}\end{cases}$$

for $x,y\in V$. The representation ϕ is orthogonal, i.e.

$$P_{gg'}=P_gP_{g'}\quad\text{and}\quad P_{g^{-1}}=P_g^{\mathrm{T}}.$$

In what follows, we will identify $\mathcal{G}$ with its representation $\{\phi_g\mid g\in\mathcal{G}\}$.

The *orbit* of an element $z \in V$ under the action of a group $\mathcal{G}$ is the set

$$\left\{\bar{x} : \bar{x} = \phi_g(z) \text{ for some } g \in \mathcal{G}\right\}.$$

Similarly the *orbit of a pair* $(x, y) \in V \times V$ under the action of a group $\mathcal{G}$ is the set

$$\left\{(\bar{x}, \bar{y}) : (\bar{x}, \bar{y}) = (\phi_g(x), \phi_g(y)) \text{ for some } g \in \mathcal{G}\right\}.$$

Recall that $x \in V$ and $y \in V$ either have the same orbits under the action of $\mathcal{G}$, or disjoint orbits.

The *centralizer ring* (or commutant) of the group $\mathcal{G}$ is defined as

$$\mathcal{A} = \left\{X \in \mathbb{R}^{|V|\times|V|} : XP_g = P_gX \quad \forall g \in \mathcal{G}\right\}.$$

The matrix *-algebra $\mathcal{A}$ has a basis of $0 - 1$ matrices

$$(B_k)_{ij} = \begin{cases} 1 & \text{if } (i, j) \text{ in orbit } k; \\ 0 & \text{otherwise} \end{cases} \qquad ((i, j) \in V \times V, k = 1, \ldots, d). \tag{6}$$

Also note that:

- $\sum_k B_k = J$;
- For each k there is a k^* (possibly $k^* = k$) with $B_k = B_{k^*}^{\mathrm{T}}$.

For what follows, we need to normalize the basis B_k, $k = 1, \ldots, d$:

$$D_k := \frac{1}{\sqrt{\mathrm{tr}(B_k^{\mathrm{T}}B_k)}} B_k, \quad k = 1, \ldots, d. \tag{7}$$

Note that

$$\mathrm{tr}(D_r^{\mathrm{T}}D_s) = \delta_{rs} \quad (r, s = 1, \ldots, d).$$

The *multiplication parameters* γ_{rs}^k are defined by

$$D_rD_s = \sum_{k=1}^{d} \gamma_{rs}^k D_k$$

for $r, s = 1, \ldots, d$. Note that:

$$\gamma_{rs}^k = \mathrm{tr}(D_{k^*}D_rD_s) \quad (k, r, s = 1, \ldots, d). \tag{8}$$

Now, for $k = 1, \ldots, d$ we define $d \times d$ matrices L_k:

$$(L_k)_{rs} := \gamma_{ks}^r, \quad r, s = 1, \ldots, d. \tag{9}$$

The matrices L_k form a basis as a vector space of a *faithful* representation of $\mathcal{A}$, say $\mathcal{A}'$, that is called the regular $*$-representation of $\mathcal{A}$.

Theorem 2 (*See e.g. [8]*). *The linear map $\varphi : D_k \to L_k$, $k = 1, \ldots, d$ defines a *-isomorphism from $\mathcal{A}$ to $\mathcal{A}'$.*

The following is a consequence of this theorem.

Corollary 3 (*[8]*). *Let $x \in \mathbb{R}^d$. One has*

$$\sum_{k=1}^{d} x_k D_k \succeq 0 \Longleftrightarrow \sum_{i=1}^{d} x_k L_k \succeq 0.$$

6. Exploiting symmetry in semidefinite programs

We now show how to use the ideas from the previous section to reduce the size of certain semidefinite programs, and subsequently apply this to the semidefinite programming formulations of the ϑ and ϑ' numbers. The methodology we will describe is essentially due to [8], where it was used to bound crossing numbers of complete bipartite graphs. The idea of using representation theory to reduce the size of certain semidefinite programs dates back to Schrijver [16], and there is a recent survey on the topic by Gatermann and Parrilo [9].

Assume that the following semidefinite programming problem is given

$$\min_{X\succeq 0}\left\{\mathrm{tr}(A_0X) : \mathrm{tr}(A_kX) = b_k\ k = 1,\ldots,m\right\}, \tag{10}$$

where the matrices $A_i \in \mathcal{S}_n$ $(i = 0,\ldots,m)$ and the vector $b \in \mathbb{R}^m$ are given. Assume further that there is a finite group $\mathcal{G}$ such that the associated Reynolds operator

$$R(X) := \frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} P_g^{\mathrm{T}}XP_g, \quad X \in \mathbb{R}^{n\times n}$$

maps the feasible set of (10) into itself and leaves the objective value invariant, i.e.

$$\mathrm{tr}(A_0R(X)) = \mathrm{tr}(A_0X) \quad \text{if } X \text{ is a feasible solution of (10).}$$

Since the Reynolds operator is a projection onto the centralizer ring of $\mathcal{G}$, maps the convex feasible set into itself, and preserves the objective values of feasible solutions, we may restrict the optimization to feasible solutions in the centralizer ring of $\mathcal{G}$. As explained in the previous section, we may obtain a normalized basis D_i $(i = 1,\ldots,d)$ of the centralizer ring via (6) and (7), by determining the orbits of pairs under the action of $\mathcal{G}$.

In other words, we may restrict our attention to feasible solutions of (10) of the form $X = \sum_{i=1}^{d} x_iD_i$ for some $x \in \mathbb{R}^d$.

From Corollary 3 it follows that the SDP problem (10) can be formulated as

$$\min_{x\in\mathbb{R}^d}\left\{\sum_{i=1}^{d} x_i\mathrm{tr}(A_0D_i) : \sum_{r=1}^{d} x_r\mathrm{tr}(A_kD_r) = b_k\ \forall k, \sum_{r=1}^{d} x_rL_r \succeq 0\right\}, \tag{11}$$

where the L_r's are defined in (9).

We assume that the numbers $\mathrm{tr}(A_kD_r)$ may be computed beforehand, so that problem (11) only involves $d\times d$ data matrices (i.e. the L_r matrices) as opposed to $n\times n$ matrices (i.e. the matrices D_r). Thus we may have a considerable reduction of the size of the matrices to which we apply semidefinite programming.

If problem (10) has the additional constraint $X \geq 0$, then its reformulation is identical to (11) except for the additional requirement $x \geq 0$.

Application to ϑ and ϑ'

We now reformulate the SDP problem (3) that defines the ϑ-number using the technique described above. The following lemma explains how problem (3) fits in the general setting.

Lemma 4. *Let $G = (V, E)$ be given and denote $\mathcal{G} := \mathrm{Aut}(G)$ and $n = |V|$. If X is a feasible point of (3), then*

$$R(X) = \frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} P_g^{\mathrm{T}}XP_g, \quad X \in \mathbb{R}^{n\times n}$$

is also a feasible point with the same objective value.

Proof. Assume X is a feasible point of (3). By the definition of $\mathrm{Aut}(G)$, one has

$$R(X)_{ij} = 0 \quad \text{if } \{i,j\} \in E,$$

since $X_{ij} = 0$ for all $(i, j) \in E$. Moreover,

$$\operatorname{tr}(JR(X)) = \frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} \operatorname{tr}(JP_g^{\mathrm{T}} X P_g) = \frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} \operatorname{tr}(P_g J P_g^{\mathrm{T}} X) = \operatorname{tr}(JX),$$

since $P_g J P_g^{\mathrm{T}} = J$. Similarly, $\operatorname{tr}(R(X)) = \operatorname{tr}(X) = 1$. Finally, one has $R(X) \in \mathcal{S}_n^+$ since $X \in \mathcal{S}_n^+$. $\square$

Thus the general 'symmetry reduction' methodology applies to problem (3), and we have the following theorem.

Theorem 5. *Let $G = (V, E)$ be given and denote $\mathcal{G} := \operatorname{Aut}(G)$. Denote the number of orbits of $V \times V$ under the action of $\mathcal{G}$ by d, and the length of orbit k by l_k $(k = 1, \ldots, d)$. Finally denote the set of orbits of identical pairs $(v, v) \in V \times V$ under the action of $\mathcal{G}$ by $\Pi_{\mathcal{G}}$.*

One has

$$\vartheta(G) = \max_{x\in\mathbb{R}^d} \sum_{k=1}^{d} x_k\sqrt{l_k}$$

subject to

$$x_k = 0 \quad \text{if orbit } k \text{ intersects } E \ (k = 1, \ldots, d)$$

$$\sum_{k\in\Pi_{\mathcal{G}}} \sqrt{l_k}x_k = 1$$

$$\sum_{k=1}^{d} x_k L_k \succeq 0,$$

where the $d \times d$ matrices L_k $(k = 1, \ldots, d)$ are constructed from the orbit matrices B_k $(k = 1, \ldots, d)$ via (7)–(9).

Proof. The proof follows from (11), after writing problem (3) in the standard SDP form (10) (with maximization instead of minimization). We omit the details. $\square$

We may replace $\vartheta(G)$ by $\vartheta'(G)$ in the statement of the theorem by simply adding the additional constraint $x \geq 0$.

7. Numerical results

In this section we give numerical results for computing ϑ and ϑ' for the Erdős–Rényi graph $ER(q)$, for all odd, prime values $q \leq 31$. As discussed in the previous section, the first step is to construct the orbits of pairs of vertices under the action of the automorphism group of the graph $ER(q)$.

There are $d = 2q + 11$ such orbits for q odd and prime; the details may be found in the appendix to this paper. Thus we may formulate the $d = 2q+11$ matrices B_k $(k = 1, \ldots, d)$ that are associated with the orbits of pairs. After normalizing the matrices B_k $(k = 1, \ldots, d)$, we use (8) to obtain the matrices L_k $(k = 1, \ldots, d)$. Finally, we solve the SDP problems described in Section 6 to obtain $\vartheta(ER(q))$ and $\vartheta'(ER(q))$.

Note that, for given q, the Schrijver ϑ'-function in the form (4) is an SDP problem with a matrix variable of order $q^2 + q + 1$ and $O(q^4)$ sign constraints. For $q > 17$, say, solving such an SDP problem is difficult. However, using the regular $*$-representation, we reduce this to obtain a problem that involves matrices of order $2q + 11$ only. Thus it is possible to obtain $\vartheta'(ER(q))$ for the values of q listed in the table by interior-point methods in a couple of seconds on a standard pc.

In Table 1 we present our numerical results. All computations were done using the semidefinite programming software SeDuMi [17] and Matlab 6.5. In the first column we give the order q of

Table 1
Bounds for the stability number of the graph $ER(q)$

q	α(ER(q))	$\vartheta'(ER(q))$	$\vartheta(ER(q))$	(2)
3	5	5.00	5.00	5.56
5	10	10.07	10.09	10.56
7	15	15.74	15.82	16.73
11	29	31.09	31.29	32.05
13	38	40.51	40.52	41.03
17	n.a.	60.22	60.42	61.29
19	n.a.	71.30	71.49	72.49
23	n.a.	96.2400	96.2408	96.86
29	n.a.	136.98	137.07	137.91
31	n.a.	151.70	151.95	152.71

the projective plane which defines the Erdős–Rényi graph; the second column lists known stability numbers (due to J. Williford, private communication); in the third column we give the computed values for the Schrijver ϑ'-number, and in the fourth column the values of the Lovász theta number for $ER(q)$. In the last column we give the eigenvalue bound (2) from [10].

Note that the $\vartheta(ER(q))$ bounds are strictly better than the eigenvalue bounds (2), but the differences between the bounds are small. In six cases the bound $\lfloor\vartheta'(ER(q))\rfloor$ improves on the bound from (2) (rounded down), but in all these cases the difference is only 1. Also note that $\lfloor\vartheta(ER(q))\rfloor$ gives the same bound as $\lfloor\vartheta'(ER(q))\rfloor$ in all cases except $q = 29$.

Appendix. Constructing the orbits of pairs of Aut($ER(q)$)

In this appendix we give the details of the orbits of pairs of vertices under the action of Aut($ER(q)$).

With reference to the notation introduced in Section 4, there are exactly three orbits of vertices of $ER(q)$: $\mathcal{R}$, $\mathcal{L}$, and $\mathcal{M}$.

The absolute vertices are exactly the vertices x such that $\langle x, x\rangle = 0$. Due to our choice of M, for the external vertices $\langle x, x\rangle$ is a square and for the internal vertices $\langle x, x\rangle$ is a non-square. So we may scale the external vertices so that $\langle x, x\rangle = 1$ and the internal vertices so that $\langle x, x\rangle = g$, where g is some generator of the multiplicative group of the field.

(There is an abuse of notation here: we are using x to represent both a one-dimensional subspace and a particular vector in that subspace.)

If one uses the more "standard" choice of $M = I$, the external vertices would have norm alternating between square and non-square according to q (mod 4), and the opposite for the internal vertices.

We will now compute the orbits of the automorphism group of $ER(q)$ on the pairs of vertices. (See also [1], where they derive the parameters of the association schemes on the external and internal vertices, which can be used to read off the orbits for $\mathcal{L} \times \mathcal{L}$ and $\mathcal{M} \times \mathcal{M}$.)

There are of course three diagonal orbits on pairs, corresponding to the three orbits on vertices:

- $\{(x, x) : x \in \mathcal{R}\}$
- $\{(x, x) : x \in \mathcal{L}\}$
- $\{(x, x) : x \in \mathcal{M}\}$.

For a pair of distinct vertices (x, y), let X be the matrix whose columns are x and y, and let $A := X^{\mathrm{T}}MX$. Similarly, for (x', y') we define X' and A'. Assume (x, y) and (x', y') are in the same orbit. Then $X' = mXd$ for some $m \in P\mathcal{O}_3(q)$ and some non-singular diagonal matrix d (as $P\mathcal{O}_3(q)$ acts on 1-subspaces, we may need to rescale to achieve our normalization, hence d). Now

$$X' = mXd \iff X'^{\mathrm{T}}MX' = dX^{\mathrm{T}}m^{\mathrm{T}}MmXd \iff A' = dAd. \quad (12)$$

The diagonal elements of A are either 0, 1, or g (according to the type of x and y) and must be identical to the diagonal elements of A'. Our task is then to classify such matrices A under the equivalence suggested by Eq. (12).

If x is absolute then all pairs (x, y) where y is of fixed type and $\langle x, y\rangle \neq 0$ are in the same orbit; this can be seen from

$$\begin{pmatrix} 0 & b \\ b & c \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & c \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}.$$

Recalling that for absolute vertices adjacency means equality, and that absolute vertices are never adjacent to internal ones, we have the following orbits on pairs of distinct vertices:

- $\{(x, y) : x \in \mathcal{R}, y \in \mathcal{R}, x \neq y\}$
- $\{(x, y) : x \in \mathcal{R}, y \in \mathcal{L}, \langle x, y\rangle = 0\}$
- $\{(x, y) : x \in \mathcal{R}, y \in \mathcal{L}, \langle x, y\rangle \neq 0\}$
- $\{(x, y) : x \in \mathcal{R}, y \in \mathcal{M}\}$.

(There are of course two analogous orbits in $\mathcal{L} \times \mathcal{R}$, and one in $\mathcal{M} \times \mathcal{R}$.)

If neither vertex is absolute then the diagonal entries of d are constrained to be ± 1, and we have the following orbits on pairs of distinct vertices:

- $\{(x, y) : x \in \mathcal{L}, y \in \mathcal{L}, \langle x, y\rangle = 0\}$
- $\{(x, y) : x \in \mathcal{L}, y \in \mathcal{L}, \langle x, y\rangle = \pm g^t\}, t = 0, 1, 2, \ldots, \frac{q-3}{2}$
- $\{(x, y) : x \in \mathcal{M}, y \in \mathcal{M}, \langle x, y\rangle = 0\}$
- $\{(x, y) : x \in \mathcal{M}, y \in \mathcal{M}, \langle x, y\rangle = \pm g^t\}, t = 0, 2, \ldots, \frac{q-3}{2}$
- $\{(x, y) : x \in \mathcal{L}, y \in \mathcal{M}, \langle x, y\rangle = 0\}$
- $\{(x, y) : x \in \mathcal{L}, y \in \mathcal{M}, \langle x, y\rangle = \pm g^t\}, t = 0, 1, 2, \ldots, \frac{q-3}{2}$,

(similarly for orbits in $\mathcal{M} \times \mathcal{L}$). Note that it can be shown that there are no internal vertices x, y with $\langle x, y\rangle = g$.

In total there are $2q + 11$ orbits of pairs and they form a basis for the centralizer ring of $\text{Aut}(ER(q))$, for q odd and prime.

References

[1] E. Bannai, S. Hao, S.-Y. Song, Character tables of the association schemes of finite orthogonal groups acting on the nonisotropic points, J. Combin. Theory Ser. A 54 (2) (1990) 164–200.
[2] W.G. Brown, On graphs that do not contain a Thomsen graph, Canad. Math. Bull. 9 (1966) 281–285.
[3] D.M. Cvetković, M. Doob, H. Sachs, Spectra of Graphs, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1980.
[4] P. Delsarte, An algebraic approach to the association schemes of coding theory, Philips Res. Rep. Suppl. 10 (1973) vi+97.
[5] P. Erdős, A. Rényi, V.T. Sós, On a problem of graph theory, Studia Sci. Math. Hungar. 1 (1966) 215–235.
[6] Z. Füredi, Graphs without quadrilaterals, J. Combin. Theory Ser. B 34 (2) (1983) 187–190.
[7] Z. Füredi, On the number of edges of quadrilateral-free graphs, J. Combin. Theory Ser. B 68 (1) (1996) 1–6.
[8] E. de Klerk, D.V. Pasechnik, A. Schrijver, Reduction of symmetric semidefinite programs using the regular *-representation, Math. Program. B 109 (2–3) (2007) 613–624.
[9] K. Gatermann, P.A. Parrilo, Symmetry groups, semidefinite programs, and sum of squares, J. Pure Appl. Algebra 192 (2004) 95–128.
[10] C.D. Godsil, M.W. Newman, Eigenvalue bounds for independent sets, J. Combin. Theory B 98 (4) (2008) 721–734.
[11] J.W.P. Hirschfeld, Projective geometry over finite fields, Clarendon Press, Oxford, 1998.
[12] L. Lovász, On the Shannon capacity of a graph, IEEE Trans. Inform. Theory 25 (1979) 1–7.
[13] L. Lovász, A. Schrijver, Cones of matrices and set-functions and 0–1 optimization, SIAM J. Optim. 1 (2) (1991) 166–190.
[14] D. Mubayi, J. Williford, On the independence number of the Erdős–Rényi and projective norm graphs and a related hypergraph, J. Graph Theory 56 (2) (2007) 113–127.
[15] T.D. Parsons, Graphs from projective planes, Aequationes Math. 14 (1–2) (1976) 167–189.
[16] A. Schrijver, A comparison of the Delsarte and Lovász bounds, IEEE Trans. Inform. Theory 25 (1979) 425–429.
[17] J.F. Sturm, Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones, Optim. Methods Softw. 11–12 (1999) 625–653.