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On the Hyers–Ulam Stability of the Functional Equations That Have the Quadratic Property

Soon-Mo Jung*

Mathematics Section, College of Science and Technology, Hong-Ik University,

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The Hyers–Ulam stability of the quadratic functional equation (1) on a restricted domain shall be investigated, and the result shall be applied to the study of an asymptotic behavior of that equation. Furthermore, the Hyers–Ulam stability problems of another quadratic equation (4) (on a restricted domain) shall also be treated under the approximately even (or odd) condition, and some asymptotic behaviors of the quadratic mappings and the additive mappings shall be investigated. © 1998 Academic Press

Key Words: Hyers–Ulam stability; quadratic functional equation.

1. INTRODUCTION

In 1940, S. M. Ulam [17] raised a question concerning the stability of homomorphisms:

> Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h: G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$ then a homomorphism $H: G_1 \to G_2$ exists with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

The case of approximately additive mappings was solved by D. H. Hyers under the assumption that G_1 and G_2 are Banach spaces (see [9]). Taking this fact into account, the additive functional equation $f(x + y) = f(x) + f(y)$ is said to have Hyers–Ulam stability on (G_1, G_2). This terminology is also applied to the case of other functional equations. For a more detailed definition of such terminology, one can refer to [7, 10].

* E-mail: smjung@wow.hongik.ac.kr.

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Throughout the paper, let X and Y be a real normed space and a real Banach space, respectively. A mapping $f: X \to Y$ is called quadratic if f satisfies the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1}$$

for all $x, y \in X$. A functional $f: X \to \mathbb{R}$ is quadratic if and only if there exists a symmetric biadditive mapping $B: X^2 \to \mathbb{R}$ such that $f(x) = B(x, x)$, and this B is unique (see [1]).

F. Skof was the first author to treat the Hyers–Ulam stability of the quadratic equation (see [15]). Thereafter, many authors investigated the stability problem of that equation in various settings [2–6, 8, 15, 16].

In 1992, S. Czerwik proved in [4] a Hyers–Ulam–Rassias stability theorem on the quadratic equation which contains the following theorem as a particular case:

THEOREM 1. *Let $\delta \geq 0$ be fixed. If a mapping $f: X \to Y$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \delta \tag{2}$$

for all $x, y \in X$, then there exists a unique quadratic mapping $Q: X \to Y$ such that

$$\|f(x) - Q(x)\| \leq (1/2)\delta$$

for all $x \in X$. Moreover, if f is measurable or if $f(tx)$ is continuous in t for each fixed $x \in X$, then $Q(tx) = t^2 Q(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

It should be noted that the results of Skof [15] and Czerwik [4] are immediate consequences of a stability theorem of Borelli and Forti [2] for a wide class of functional equations.

In Section 2, the Hyers–Ulam stability of the quadratic equation (1) on a restricted domain shall be investigated. Furthermore, an interesting asymptotic behavior of that equation shall be proved. More precisely, we prove that a mapping $f: X \to Y$ is quadratic if and only if

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \to 0 \qquad \text{as } \|x\| + \|y\| \to \infty. \tag{3}$$

We note that besides the asymptotic condition (3) no additional condition is assumed. In Section 3, the Hyers–Ulam stability problem of another quadratic equation

$$f(x+y+z) + f(x) + f(y) + f(z) = f(x+y) + f(y+z) + f(z+x) \tag{4}$$

under the approximately even (or odd) condition shall be treated. The solution of the functional equation (4) where the range is a field of characteristic 0 has been determined by Kannappan (see [11]). In particular, if a mapping $f\colon X \to \mathbb{R}$ satisfies the functional equation (4), then there are an additive mapping $A\colon X \to \mathbb{R}$ and a biadditive mapping $B\colon X^2 \to \mathbb{R}$ such that f is given by $f(x) = A(x) + B(x,x)$. In the last section, the Hyers–Ulam stability of (4) on a restricted domain shall be proved under the approximately even (odd) condition, and some asymptotic properties of the quadratic mappings and the additive mappings shall be investigated.

2. STABILITY OF EQ. (1) ON A RESTRICTED DOMAIN

The Hyers–Ulam stability of the quadratic functional equation (1) on a restricted domain is investigated, and the result is applied to the study of an interesting asymptotic behavior of that equation.

THEOREM 2. *Let $d > 0$ and $\delta \geq 0$ be given. Assume that a mapping f: $X \to Y$ satisfies the inequality (2) for all $x, y \in X$ with $\|x\| + \|y\| \geq d$. Then there exists a unique quadratic mapping $Q\colon X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq (7/2)\,\delta \tag{5}$$

for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$, then $Q(tx) = t^2 Q(x)$ for all $x \in X$ and all $t \in \mathbb{R}$.

Proof. Assume $\|x\| + \|y\| < d$. If $x = y = 0$, then we choose a $z \in X$ with $\|z\| = d$. Otherwise, let $z = (1 + d/\|x\|)x$ for $\|x\| \geq \|y\|$ or $z = (1 + d/\|y\|)y$ for $\|x\| < \|y\|$. Clearly, we see

$$\begin{gathered} \|x - z\| + \|y + z\| \geq d, \qquad \|x + z\| + \|y + z\| \geq d, \\ \|y + z\| + \|z\| \geq d, \qquad \|x\| + \|y + 2z\| \geq d, \qquad \|x\| + \|z\| \geq d. \end{gathered} \tag{6}$$

From (2), (6), and the relation

$$\begin{aligned} f(x+y) &+ f(x-y) - 2f(x) - 2f(y) \\ &= f(x+y) + f(x-y-2z) - 2f(x-z) - 2f(y+z) \\ &\quad + f(x+y+2z) + f(x-y) - 2f(x+z) - 2f(y+z) \\ &\quad - 2f(y+2z) - 2f(y) + 4f(y+z) + 4f(z) \\ &\quad - f(x+y+2z) - f(x-y-2z) + 2f(x) + 2f(y+2z) \\ &\quad + 2f(x+z) + 2f(x-z) - 4f(x) - 4f(z), \end{aligned}$$

we get

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \le 7\delta. \tag{7}$$

Obviously, the inequality (7) holds true for all $x, y \in X$. According to (7) and Theorem 1, there exists a unique quadratic mapping $Q\colon X \to Y$ which satisfies the inequality (5) for all $x \in X$. Our last assertion is trivial in view of Theorem 1. ∎

Define $B = \{(x,y) \in X^2\colon \|x\| < d, \|y\| < d\}$ for some $d > 0$. In view of the fact that $\{(x,y) \in X^2\colon \|x\| + \|y\| \ge 2d\} \subset X^2 \setminus B$, the following corollary is a direct consequence of Theorem 2.

COROLLARY 3. *Assume that a mapping $f\colon X \to Y$ satisfies the inequality* (2) *for some $\delta \ge 0$ and for all $(x,y) \in X^2 \setminus B$. Then there exists a unique quadratic mapping $Q\colon X \to Y$ satisfying the inequality* (5) *for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$, then $Q(tx) = t^2 Q(x)$ for all $x \in X$ and all $t \in \mathbb{R}$.*

Skof [14] has proved an asymptotic property of the additive mappings. It is a natural thing to expect such a property also for the quadratic equation (1).

COROLLARY 4. *A mapping $f\colon X \to Y$ is quadratic if and only if the asymptotic condition* (3) *holds true.*

Proof. Due to the asymptotic condition (3), there exists a sequence (δ_n) monotonically decreasing to 0 such that

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \le \delta_n \tag{8}$$

for all $x, y \in X$ with $\|x\| + \|y\| \ge n$. Hence, it follows from (8) and Theorem 2 that there exists a unique quadratic mapping $Q_n\colon X \to Y$ such that

$$\|f(x) - Q_n(x)\| \le (7/2)\,\delta_n \tag{9}$$

for all $x \in X$. Let $l, m \in \mathbb{N}$ satisfy $m \ge l$. Obviously, by (9) we obtain

$$\|f(x) - Q_m(x)\| \le (7/2)\,\delta_m \le (7/2)\,\delta_l$$

for all $x \in X$, since (δ_n) is a monotonically decreasing sequence. The uniqueness of Q_n implies $Q_m = Q_l$. Hence, by letting $n \to \infty$ in (9), we conclude that f is quadratic. The reverse assertion is trivial. ∎

3. STABILITY OF EQ. (4)

The Hyers–Ulam stability of another quadratic equation (4) is proved under a suitable condition by using ideas from the papers of Rassias [13] and Hyers [9]. We first prove the following lemma:

LEMMA 5. *Assume that a mapping f: $X \to Y$ satisfies the following inequality*:

$$\|f(x+y+z)+f(x)+f(y)+f(z) \\ -f(x+y)-f(y+z)-f(z+x)\| \leq \delta \qquad (10)$$

for some $\delta \geq 0$ and for all $x, y, z \in X$. It then holds that

$$\left\| f(x) - \frac{2^n+1}{2^{2n+1}} f(2^n x) + \frac{2^n-1}{2^{2n+1}} f(-2^n x) \right\| \leq 3\delta \sum_{k=1}^{n} 2^{-k} \qquad (11)$$

for all $x \in X$ and $n \in \mathbb{N}$.

Proof. If we replace x, y, and z in (10) by 0, we get $\|f(0)\| \leq \delta$. Putting $x = y = -z$ in (10) yields

$$\|3f(x)+f(-x)-f(2x)\| \leq 3\delta. \qquad (12)$$

By substituting $-x$ for x in (12), we obtain

$$\|3f(-x)+f(x)-f(-2x)\| \leq 3\delta. \qquad (13)$$

We use induction on n to prove our lemma. By (12) and (13), we have

$$\begin{aligned} &\|f(x)-(3/8)f(2x)+(1/8)f(-2x)\| \\ &\quad \leq (3/8)\|3f(x)+f(-x)-f(2x)\| \\ &\qquad +(1/8)\|-3f(-x)-f(x)+f(-2x)\| \\ &\quad \leq (3/2)\delta, \end{aligned}$$

which proves the validity of the inequality (11) for $n = 1$. Now, assume that the inequality (11) holds true for some $n \in \mathbb{N}$. By using (12), (13), and the following relation:

$$\begin{aligned} f(x) &- \frac{2^{n+1}+1}{2^{2n+3}} f(2^{n+1}x) + \frac{2^{n+1}-1}{2^{2n+3}} f(-2^{n+1}x) \\ &= f(x) - \frac{2^n+1}{2^{2n+1}} f(2^n x) + \frac{2^n-1}{2^{2n+1}} f(-2^n x) \\ &\quad + \frac{2^{n+1}+1}{2^{2n+3}} \left[3f(2^n x)+f(-2^n x)-f(2^{n+1}x)\right] \\ &\quad - \frac{2^{n+1}-1}{2^{2n+3}} \left[3f(-2^n x)+f(2^n x)-f(-2^{n+1}x)\right], \end{aligned}$$

we can easily verify the inequality (11) for $n+1$. This ends the proof. ∎

In the following theorem, the Hyers–Ulam stability of Eq. (4) is proved under the approximately even condition.

THEOREM 6. *Assume a mapping $f: X \to Y$ satisfies the system of inequalities*

$$\begin{aligned} &\|f(x+y+z)+f(x)+f(y)+f(z) \\ &\qquad -f(x+y)-f(y+z)-f(z+x)\| \le \delta, \\ &\qquad\qquad \|f(x)-f(-x)\| \le \theta \end{aligned} \tag{14}$$

for some $\delta, \theta \ge 0$ and for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q: X \to Y$ which satisfies (4) *and the inequality*

$$\|f(x)-Q(x)\| \le 3\delta \tag{15}$$

for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$, then $Q(tx) = t^2 Q(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

Proof. It follows from (11) and the second condition in (14) that

$$\left\|f(x) - 2^{-2n} f(2^n x)\right\| \le 3\delta \sum_{k=1}^{n} 2^{-k} + \frac{2^n - 1}{2^{2n+1}}\theta. \tag{16}$$

By (16) we have

$$\begin{aligned} &\left\|2^{-2n} f(2^n x) - 2^{-2m} f(2^m x)\right\| \\ &\quad = 2^{-2m}\left\|2^{-2(n-m)} f(2^{n-m}\cdot 2^m x) - f(2^m x)\right\| \\ &\quad \le 2^{-2m}\left[3\delta \sum_{k=1}^{n-m} 2^{-k} + \frac{2^{n-m}-1}{2^{2(n-m)+1}}\theta\right] \end{aligned} \tag{17}$$

for $n \ge m$. Since the right-hand side of the inequality (17) tends to 0 as m tends to ∞, the sequence $\{2^{-2n} f(2^n x)\}$ is a Cauchy sequence. Therefore, we may apply a direct method to the definition of Q. Define

$$Q(x) = \lim_{n\to\infty} 2^{-2n} f(2^n x)$$

for all $x \in X$ [9, 13]. From the first condition in (14), it follows that

$$\begin{aligned} &\|Q(x+y+z)+Q(x)+Q(y)+Q(z) \\ &\qquad -Q(x+y)-Q(y+z)-Q(z+x)\| \le 2^{-2n}\delta \end{aligned}$$

for all $x, y, z \in X$ and for all $n \in \mathbb{N}$. Therefore, by letting $n \to \infty$ in the last inequality, it is clear that Q is a solution of Eq. (4). Analogously, by the second condition in (14), we can show that Q is even. By putting $z = -y$ in (4) and by taking account of $Q(0) = 0$, we see that Q as an even solution of Eq. (4) is quadratic. According to (16), the inequality (15) holds true.

Now, let $T\colon X \to Y$ be another quadratic mapping which satisfies Eq. (4) and the inequality (15). Obviously, we have

$$Q(2^n x) = 4^n Q(x) \qquad \text{and} \qquad T(2^n x) = 4^n T(x)$$

for all $x \in X$ and $n \in \mathbb{N}$. Hence, it follows from (15) that

$$\begin{aligned} \|Q(x) - T(x)\| &= 4^{-n}\|Q(2^n x) - T(2^n x)\| \\ &\le 4^{-n}(\|Q(2^n x) - f(2^n x)\| + \|f(2^n x) - T(2^n x)\|) \\ &\le 6\delta/4^n \end{aligned}$$

for all $x \in X$ and $n \in \mathbb{N}$. By letting $n \to \infty$ in the preceding inequality, we immediately see the uniqueness of Q. The proof of the last assertion in the theorem goes through in the same way as that of [4, Theorem 1]. ∎

Remark 1. From the direct combination of the inequalities in (14), it follows that the mapping $f\colon X \to Y$ in Theorem 6 satisfies the inequality

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \le \delta + \theta + \|f(0)\| \le 2\delta + \theta.$$

According to Theorem 1, there is a unique quadratic mapping $Q\colon X \to Y$ such that

$$\|f(x) - Q(x)\| \le \delta + (1/2)\theta.$$

We see that the last inequality contains a θ term which appeared as the upper bound for the second inequality in (14). The advantage of the inequality (15) in Theorem 6 compared to the last inequality is that the right-hand side of (15) contains no θ term.

Similarly, as in the proof of Theorem 6, the Hyers–Ulam stability for Eq. (4) under the approximately odd condition is proved.

THEOREM 7. *Assume a mapping $f\colon X \to Y$ satisfies the system of inequalities*

$$\begin{aligned} &\|f(x+y+z) + f(x) + f(y) + f(z) \\ &\qquad - f(x+y) - f(y+z) - f(z+x)\| \le \delta, \qquad (18) \\ &\|f(x) + f(-x)\| \le \theta \end{aligned}$$

for some $\delta, \theta \geq 0$ *and for all* $x, y, z \in X$. *Then there exists a unique additive mapping* $F\colon X \to Y$ *satisfying the inequality*

$$\|f(x) - F(x)\| \leq 3\delta \tag{19}$$

for all $x \in X$.

Proof. From (11) and the second condition in (18), we get

$$\|f(x) - 2^{-n}f(2^n x)\| \leq 3\delta \sum_{k=1}^{n} 2^{-k} + \frac{2^n - 1}{2^{2n+1}}\theta. \tag{20}$$

The sequence $\{2^{-n}f(2^n x)\}$ is a Cauchy sequence because, for $n \geq m$,

$$\begin{aligned} &\|2^{-n}f(2^n x) - 2^{-m}f(2^m x)\| \\ &\quad = 2^{-m}\|2^{-(n-m)}f(2^{n-m}\cdot 2^m x) - f(2^m x)\| \\ &\quad \leq 2^{-m}\left[3\delta \sum_{k=1}^{n-m} 2^{-k} + \frac{2^{n-m} - 1}{2^{2(n-m)+1}}\theta\right] \\ &\quad \to 0 \qquad \text{as } m \to \infty. \end{aligned}$$

Now, define

$$F(x) = \lim_{n\to\infty} 2^{-n}f(2^n x)$$

for all $x \in X$. Similarly, as in the proof of Theorem 6, due to (18), we see that the mapping F satisfies Eq. (4) and is odd. By putting $z = -y$ in (4) and considering the oddness of F and letting $u = x + y$, $v = x - y$, we get

$$2F\left(\frac{u+v}{2}\right) = F(u) + F(v).$$

According to [12], since $F(0) = 0$, the mapping F is additive. The validity of the inequality (19) follows directly from (20) and the definition of F.

Now, let $G\colon X \to Y$ be another additive mapping which satisfies (19). It then follows from (19) that

$$\begin{aligned} \|F(x) - G(x)\| &= 2^{-n}\|F(2^n x) - G(2^n x)\| \\ &\leq 2^{-n}(\|F(2^n x) - f(2^n x)\| + \|f(2^n x) - G(2^n x)\|) \\ &\leq 6\delta/2^n \end{aligned}$$

for all $x \in X$ and $n \in \mathbb{N}$. This implies the uniqueness of F. ∎

Remark 2. The approximately even condition in (14) guarantees the "quadratic" property of Q, whereas the approximately odd condition in (18) guarantees the "additive" behavior of F.

4. STABILITY OF EQ. (4) ON A RESTRICTED DOMAIN

In this section, we investigate the Hyers–Ulam stability problems of the functional equation (4) on a restricted domain and apply the results to the study of some asymptotic behaviors of the quadratic mappings and the additive mappings. First, we prove some theorems on the Hyers–Ulam stability of the functional equation (4) on a restricted domain under the approximately even condition.

THEOREM 8. *Let $d > 0$, $\delta, \theta \geq 0$ be given. Assume that a mapping $f\colon X \to Y$ satisfies the first inequality in* (14) *for all $x, y, z \in X$ with $\|x\| + \|y\| + \|z\| \geq d$ and the second inequality in* (14) *for all $x \in X$. Then there exists a unique quadratic mapping $Q\colon X \to Y$ which satisfies Eq.* (4) *and the inequality*

$$\|f(x) - Q(x)\| \leq 21\delta \tag{21}$$

for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$, then $Q(tx) = t^2 Q(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

Proof. Assume $\|x\| + \|y\| + \|z\| < d$. Choose a $t \in X$ with $\|t\| \geq 2d$. Clearly, we see

$$\begin{gathered} \|x\| + \|y + t\| + \|z - t\| \geq d, \qquad \|x\| + \|y\| + \|t\| \geq d, \\ \|z\| + \|x\| + \|-t\| \geq d, \qquad \|x + t\| + \|x - t\| + \|z - t\| \geq d, \\ \|x\| + \|x\| + \|z - t\| \geq d, \qquad \|x + z - t\| + \|t\| + \|-t\| \geq d. \end{gathered} \tag{22}$$

From the first inequality in (14), (22), and the relation

$$\begin{aligned} f(x+y+z) &+ f(x) + f(y) + f(z) \\ -f(x+y) &- f(y+z) - f(z+x) + f(0) \\ &= f(x+y+z) + f(x) + f(y+t) + f(z-t) \\ &\quad -f(x+y+t) - f(y+z) - f(z+x-t) \\ &\quad +f(x+y+t) + f(x) + f(y) + f(t) \\ &\quad -f(x+y) - f(y+t) - f(x+t) \\ &\quad +f(z+x-t) + f(z) + f(x) + f(-t) \\ &\quad -f(z+x) - f(x-t) - f(z-t) \\ &\quad +f(2x+z-t) + f(x+t) + f(x-t) + f(z-t) \\ &\quad -f(2x) - f(x+z-2t) - f(x+z) \\ &\quad -f(2x+z-t) - f(x) - f(x) - f(z-t) \\ &\quad +f(2x) + f(x+z-t) + f(x+z-t) \\ &\quad -f(x+z-t) - f(x+z-t) - f(t) \\ &\quad -f(-t) + f(x+z) + f(0) + f(x+z-2t), \end{aligned}$$

we get

$$\|f(x+y+z)+f(x)+f(y)+f(z) \\ -f(x+y)-f(y+z)-f(z+x)\| \le 6\delta + \|f(0)\| \le 7\delta. \quad (23)$$

Obviously, the inequality (23) holds for all $x, y, z \in X$. On account of (23) and Theorem 6, there exists a unique quadratic mapping $Q: X \to Y$ which satisfies Eq. (4) for all $x, y, z \in X$ and the inequality (21) for all $x \in X$. The last assertion of the theorem follows immediately from Theorem 6. ▮

Define $S = \{(x, y, z) \in X^3: \|x\| < d, \|y\| < d, \|z\| < d\}$ for some $d > 0$. The fact that $\{(x, y, z) \in X^3: \|x\| + \|y\| + \|z\| \ge 3d\} \subset X^3 \setminus S$ implies that the following corollary is an immediate consequence of Theorem 8.

COROLLARY 9. *Suppose a mapping $f: X \to Y$ satisfies the first inequality in* (14) *for all $(x, y, z) \in X^3 \setminus S$ and the second inequality in* (14) *for all $x \in X$. Then there exists a unique quadratic mapping $Q: X \to Y$ which satisfies Eq.* (4) *and the inequality* (21) *for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$, then $Q(tx) = t^2 Q(x)$ for all $x \in X$ and $t \in \mathbb{R}$.*

The proof of the following corollary is quite similar to that of Corollary 4. The only significant difference between them is that in the proof of the next corollary we make use of Theorem 8 instead of Theorem 2. We omit the proof.

COROLLARY 10. *A mapping $f: X \to Y$ is quadratic and satisfies Eq.* (4) *if and only if*

$$\|f(x+y+z)+f(x)+f(y)+f(z) \\ -f(x+y)-f(y+z)-f(z+x)\| \to 0$$

as $\|x\| + \|y\| + \|z\| \to \infty$ and there exists a constant $\theta \ge 0$ such that f satisfies the second inequality in (14) *for all $x \in X$.*

Now, theorems on the Hyers–Ulam stability of the functional equation (4) on a restricted domain are investigated under the approximately odd condition. We can easily prove the following theorem and corollaries using methods similar to those that have been applied to the proofs of Theorem 8 and Corollaries 9 and 10. Hence, we omit the proofs.

THEOREM 11. *Let $d > 0$, $\delta, \theta \ge 0$ be given. Assume that a mapping $f: X \to Y$ satisfies the first inequality in* (18) *for all $x, y, z \in X$ with $\|x\| + \|y\| + \|z\| \ge d$ and the second inequality in* (18) *for all $x \in X$. Then there exists a unique additive mapping $F: X \to Y$ satisfying the inequality*

$$\|f(x) - F(x)\| \le 21\delta \quad (24)$$

for all $x \in X$.

COROLLARY 12. *Suppose a mapping* $f\colon X \to Y$ *satisfies the first inequality in* (18) *for all* $(x, y, z) \in X^3 \setminus S$ *and the second inequality in* (18) *for all* $x \in X$. *Then there exists a unique additive mapping* $F\colon X \to Y$ *satisfying the inequality* (24) *for all* $x \in X$.

COROLLARY 13. *A mapping* $f\colon X \to Y$ *is additive if and only if*

$$\| f(x+y+z) + f(x) + f(y) + f(z) - f(x+y) - f(y+z) - f(z+x) \| \to 0$$

as $\|x\| + \|y\| + \|z\| \to \infty$ *and there exists a constant* $\theta \geq 0$ *such that* f *satisfies the second inequality in* (18) *for all* $x \in X$.

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