ELSEVIER
Journal of Computational and Applied Mathematics 53 (1994) 225–241

JOURNAL OF
COMPUTATIONAL AND
APPLIED MATHEMATICS

Moments of Dirichlet splines and their applications to hypergeometric functions

Edward Neuman [a,*], Patrick J. Van Fleet [b,1]

[a] *Department of Mathematics, Southern Illinois University at Carbondale, Carbondale, IL 62901-4408, United States*
[b] *Department of Mathematics, Vanderbilt University, Nashville, TN 37240, United States*

Received 2 July 1992; revised 27 November 1992

Abstract

Dirichlet averages of multivariate functions are employed for a derivation of basic recurrence formulas for the moments of multivariate Dirichlet splines. An algorithm for computing the moments of multivariate simplex splines is presented. Applications to hypergeometric functions of several variables are discussed.

Keywords: Dirichlet spline; Simplex spline; Dirichlet average; R-hypergeometric function; Moment generating function; Bernstein polynomial; Bézier polynomial; Confluent hypergeometric function; Lauricella polynomial; Appell and Lauricella functions

1. Introduction

In [6], Curry and Schoenberg have pointed out that univariate B-splines can be constructed from volumes of slices of convex polyhedra. An extension of this idea to the case of multivariate splines is due to de Boor [11]. Since the geometric construction is too complicated to be used in numerical computations, some basic recurrence formulas for these functions have been found (see [5,7,8,13,14,17–19,22,23]). Multivariate B-splines (also called simplex splines) have been studied extensively over the past thirteen years by many researchers. These functions have been found useful for some applications of data fitting, computer aided geometric design and mathematical statistics. In [29] the author addressed some new problems where the simplex splines could play a prominent role.

* Corresponding author. e-mail: ga3856@siucvmb.bitnet.
[1] Present address: Department of Mathematics, Sam Houston State University, Huntsville, TX 77340, United States. e-mail: mth_pvf@shsu.edu.

SSDI 0377-0427(93)E0002-U

Further generalizations of multivariate simplex splines appear in [10,19]. Here and thereafter we call these functions Dirichlet splines. Our choice of terminology is motivated by the fact that the distributional definition of the Dirichlet splines (see (2.2)) involves the Dirichlet density function. It has been demonstrated that this class of splines is well designed for some problems of mathematical statistics (see [10,19] and references therein). Applications to the theory of multivariate convex functions are reported in [26,28].

Recently we have noticed that there is a simple relationship between univariate Dirichlet splines and some special functions such as R-hypergeometric functions and confluent hypergeometric functions (see [27]). In this paper, we present some results for multivariate Dirichlet splines together with their applications to special functions of several variables.

The outline of this paper is as follows. In Section 2 we introduce notation and definitions which will be used throughout the sequel. In Section 3 we give a definition and basic properties of a Dirichlet average of a multivariate function. Also, we give a recurrence relation for these averages. In Section 4 we give two results which play a crucial role in our subsequent considerations. Moments of the class of splines under discussion together with two moment generating functions are presented in Section 5. Therein, we also give an algorithm for computing the moments of multivariate simplex splines. Applications to hypergeometric functions, including Appell's F_4 and Lauricella's F_B, are discussed in Section 6. In the same section, we give a recurrence formula, two generating functions and an inequality for Lauricella polynomials.

2. Notation and definitions

Let us introduce some notation and definitions which will be used throughout the sequel. By $x, y, \ldots$ we denote elements of Euclidean space $\mathbb{R}^s$; $s \geqslant 1$, i.e., $x = (x_1, \ldots, x_s)$. Superscripts are used to number vectors. The inner product (or dot product) of $x, y \in \mathbb{R}^s$ is denoted by $x \cdot y = \sum_{k=1}^{s} x_k y_k$. For a given set $X \subset \mathbb{R}^s$ the symbols $[X]$ and $\mathrm{vol}_s(X)$ mean the convex hull of X and the s-dimensional Lebesgue measure, respectively. We use standard multi-index notation, i.e., for $\beta \in \mathbb{Z}_+^s$, $|\beta| = \beta_1 + \cdots + \beta_s$, $\beta! = \beta_1! \cdots \beta_s!$, $\alpha \leqslant \beta$ means $\alpha_1 \leqslant \beta_1, \ldots, \alpha_s \leqslant \beta_s$, $\alpha \in \mathbb{Z}_+^s$, $x^\beta = x_1^{\beta_1} x_2^{\beta_2} \cdots x_s^{\beta_s}$. For $r \in \mathbb{Z}_+$ and $\beta \in \mathbb{Z}_+^s$ with $|\beta| = r$, the multinomial coefficient $\binom{r}{\beta}$ is defined in the usual way:

$$\binom{r}{\beta} = \frac{r!}{\beta!}.$$

By

$$E_n = \left\{ t = (t_1, \ldots, t_n) \in \mathbb{R}^n \colon t_j \geqslant 0, \text{ for all } j, \sum_{j=1}^{n} t_j \leqslant 1 \right\},$$

we denote the standard n-simplex. Let $\mathbb{R}_>$ represent the set of all positive real numbers and let $b = (b_0, \ldots, b_n) \in \mathbb{R}_>^{n+1}$. The Dirichlet density function on E_n, denoted by ϕ_b, is given by

$$\phi_b(t) = \Gamma(c) \prod_{i=0}^{n} [\Gamma(b_i)]^{-1} t_i^{b_i - 1}, \tag{2.1}$$

where $t \in E_n$, $t_0 = 1 - t_1 - \cdots - t_n$ and $c = b_0 + \cdots + b_n$.

For $X=\{x^0,\ldots,x^n\}\subset\mathbb{R}^s$, $n\geqslant s\geqslant 1$, with $\mathrm{vol}_s([X])>0$, the multivariate Dirichlet spline $M(\cdot\mid b;\ X)$ is defined by requiring that

$$\int_{\mathbb{R}^s} f(x)M(x\mid b;\ X)\,\mathrm{d}x=\int_{E_n} f(Xt)\phi_b(t)\,\mathrm{d}t \tag{2.2}$$

holds for all $f\in C_0(\mathbb{R}^s)$, the space of all multivariate continuous functions on $\mathbb{R}^s$ with compact support (see [10,19]). Here $\mathrm{d}x=\mathrm{d}x_1\ldots\mathrm{d}x_s$, $\mathrm{d}t=\mathrm{d}t_1\ldots\mathrm{d}t_n$, $Xt=\sum_{i=0}^n t_i x^i$. When $b_0=\cdots=b_n=1$, $\phi_b(t)=n!$, and the corresponding spline becomes a simplex spline. The latter spline will be denoted by $M(\cdot\mid X)$. When $s=1$, we will write $m(\cdot\mid b;\ Z)$ instead of $M(\cdot\mid b;\ X)$, where $Z=\{z_0,\ldots,z_n\}\subset\mathbb{R}$. In this case (2.2) becomes

$$\int_{\mathbb{R}} h(u)m(u\mid b;\ Z)\,\mathrm{d}u=\int_{E_n} h(Zt)\phi_b(t)\,\mathrm{d}t,\quad h\in C_0(\mathbb{R}). \tag{2.3}$$

3. Dirichlet averages

The purpose of this section is two-fold. We give a definition of Dirichlet averages of multivariate functions. Next we prove a recurrence formula for these averages. This result has an immediate application in Section 5. For the reader's convenience, let us recall a definition of the Dirichlet average of a univariate function $h\in C_0(\mathbb{R})$. Assume that the set $Z=\{z_0,\ldots,z_n\}\subset\mathbb{R}$ is such that $\min\{z_j\colon 0\leqslant j\leqslant n\}<\max\{z_j\colon 0\leqslant j\leqslant n\}$. For $b\in\mathbb{R}^{n+1}_>$, the Dirichlet average of h, denoted by $H(b;\ Z)$, is given by

$$H(b;\ Z)=\int_{E_n} h(Zt)\phi_b(t)\,\mathrm{d}t \tag{3.1}$$

(see [3]). Comparison with (2.3) yields

$$H(b;\ Z)=\int_{\mathbb{R}} h(u)m(u\mid b;\ Z)\,\mathrm{d}u. \tag{3.2}$$

We list below some elementary properties of $H(b;\ Z)$.

(i) A vanishing parameter b_i can be omitted along with the corresponding variable z_i (see [3, (6.3-3)]).

(ii) $H(b_0,\ldots,b_n;\ z_0,\ldots,z_n)$ is symmetric in indices $0, 1,\ldots,n$ (see [3, Theorem 5.2-3]).

(iii) Equal variables can be replaced by a single variable if the corresponding parameters are replaced by their sum (see [3, Theorem, 5.2-4]).

We now introduce the Dirichlet average of $f\in C_0(\mathbb{R}^s)$. For $X\subset\mathbb{R}^s$ with $\mathrm{vol}_s([X])>0$ and $b\in\mathbb{R}^{n+1}_>$, $n\geqslant s\geqslant 1$, the Dirichlet average of f, denoted by $\mathscr{F}(b;\ X)$, is given by

$$\mathscr{F}(b;\ X)=\int_{E_n} f(Xt)\phi_b(t)\,\mathrm{d}t, \tag{3.3}$$

where Xt and ϕ_b have the same meaning as in Section 2. Comparison with (2.2) shows that

$$\mathscr{F}(b;\ X)=\int_{\mathbb{R}^s} f(x)M(x\mid b;\ X)\,\mathrm{d}x. \tag{3.4}$$

It is clear that the properties (i)–(iii) are also valid for the average $\mathscr{F}$. In particular, property (iii), when applied to $M(\cdot \mid b;\, X)$, yields

$$M(\cdot \mid b;\, X) = M\Big(\cdot \mid \underbrace{x^0,\dots,x^0}_{b_0 \text{ times}},\dots,\underbrace{x^n,\dots,x^n}_{b_n \text{ times}}\Big), \tag{3.5}$$

provided the b's are positive integers (see also [10,19]). The spline on the right-hand side of (3.5) is a multivariate simplex spline with coalescent knots (see [14] for a detailed analysis of this class of splines).

Before we state and prove the first result of this section, let us introduce more notation. By e_j, $0 \leqslant j \leqslant n$, we denote the jth coordinate vector in $\mathbb{R}^{n+1}$. For $f \in C_0^1(\mathbb{R}^s)$ define a function f_j as follows:

$$f_j(x) = D_{x^j - x} f(x). \tag{3.6}$$

Here $D_y f$ denotes the directional derivative of f in the direction $y \in \mathbb{R}^s$, i.e.,

$$D_y f(x) = \sum_{k=1}^{s} y_k \frac{\partial}{\partial x_k} f(x).$$

We are now ready to prove the following.

Theorem 3.1 (Carlson [4]). *Let* $X = \{x^0,\dots,x^n\} \subset \mathbb{R}^s$, $n \geqslant s \geqslant 1$, *be such that* $\mathrm{vol}_s([X]) > 0$. *Further, let* $f \in C_0^1(\mathbb{R}^s)$ *and let the vector* $b \in \mathbb{R}_>^{n+1}$ *be such that* $b_j \geqslant 1$ *for some* $0 \leqslant j \leqslant n$. *Then the identity*

$$(c-1)\mathscr{F}(b;\, X) = (c-1)\mathscr{F}(b - e_j;\, X) + \mathscr{F}_j(b;\, X) \tag{3.7}$$

is valid. Here $\mathscr{F}_j$ *denotes the Dirichlet average of the function* f_j.

Remark. The proof presented below bears no resemblance to what was done in [4, Theorem 3]. In this paper, the author has established (3.7) using generalized Euler–Poisson partial differential equations.

Proof. In order to establish the identity (3.7), we employ the following one:

$$(c-1)H(b;\, Z) = (c-1)H(b - e_j;\, Z) + H_j(b;\, Z). \tag{3.8}$$

Here H_j stands for the Dirichlet average of the function $h_j(u) = (z_j - u)h'(u)$, $h \in C_0^1(\mathbb{R})$. The relation (3.8) readily follows from [3, (5.6-13)]. Application of (3.2) to (3.8) yields

$$(c-1)\int_{\mathbb{R}} h(u) m(u \mid b;\, Z)\, \mathrm{d}u = (c-1)\int_{\mathbb{R}} h(u) m\big(u \mid b - e_j;\, Z\big)\, \mathrm{d}u + \int_{\mathbb{R}} (z_j - u) h'(u) m(u \mid b;\, Z)\, \mathrm{d}u. \tag{3.9}$$

We will lift (3.9) to the case of multivariate functions. To this aim we shall employ the formula

$$\int_{\mathbb{R}} h(u) m(u \mid b;\, Z)\, \mathrm{d}u = \int_{\mathbb{R}^s} h(\lambda \cdot x) M(x \mid b;\, X)\, \mathrm{d}x, \tag{3.10}$$

where now $Z=\{\lambda\cdot x^0,\ldots,\lambda\cdot x^n\}$, $\lambda\in\mathbb{R}^s\setminus\{0\}$, and $h(\lambda\cdot x)$ is a ridge function (or plane wave). Since the proof of (3.10) is similar to that presented in [23, p.496], we omit further details. Application of (3.10) to (3.9) yields

$$(c-1)\int_{\mathbb{R}^s} h(\lambda\cdot x)M(x\mid b;\ X)\ \mathrm{d}x=(c-1)\int_{\mathbb{R}^s} h(\lambda\cdot x)M(x\mid b-e_j;\ X)\ \mathrm{d}x$$
$$+\int_{\mathbb{R}^s}(\lambda\cdot x^j-\lambda\cdot x)h'(\lambda\cdot x)M(x\mid b;\ X)\ \mathrm{d}x. \qquad (3.11)$$

We appeal now to the denseness of ridge functions (these functions form a dense subset in $C_0^1(\mathbb{R}^s)$) to conclude that the above identity is valid for any multivariate function $f\in C_0^1(\mathbb{R}^s)$, see [20]. Substituting $h(\lambda\cdot x)=f(x)$ into (3.11), we obtain the assertion and the proof is completed. □

Corollary 3.2. *Along with the hypotheses of Theorem* 3.1, *assume that for some* $0\leqslant i,\ j\leqslant n$, $1\leqslant k\leqslant s$, *that* $x_k^i\neq 0$, $x_k^j\neq 0$, *and* $b_i\geqslant 1$, $b_j\geqslant 1$. *Then*,

$$(c-1)\left[\mathscr{F}(b-e_j;\ X)-\mathscr{F}(b-e_i;\ X)\right]+\mathscr{F}_j(b;\ X)-\mathscr{F}_i(b;\ X)=0 \qquad (3.12)$$

and

$$(c-1)(x_k^i-x_k^j)\mathscr{F}(b;\ X)=(c-1)\left[x_k^i\mathscr{F}(b-e_j;\ X)-x_k^j\mathscr{F}(b-e_i;\ X)\right]$$
$$+x_k^i\mathscr{F}_j(b;\ X)-x_k^j\mathscr{F}_i(b;\ X). \qquad (3.13)$$

Remark. Eqs. (3.12) and (3.13) are generalizations of [3, Exercise 5.9-6] with the latter being an extension of Zill's identity for R-hypergeometric functions.

Since the proof of (3.12) and (3.13) follows the lines introduced in [3, p.305], we omit further details.

4. Auxiliary results

Our first result reads as follows.

Proposition 4.1. *Let* $p(x)$ *be an affine function on* $\mathbb{R}^s$. *Then*,

$$p(x)M(x\mid b;\ X)=\sum_{i=0}^{n} w_i p(x^i)M(x\mid b+e_i;\ X), \qquad (4.1)$$

provided the splines $M(x\mid b+e_i;\ X)$, $0\leqslant i\leqslant n$, *are continuous at* $x\in\mathbb{R}^s$. *Here*, $w_i=b_i/c$, $i=0,1,\ldots,n$.

Proof. We need the following identity for the Dirichlet density function [3, (4.4-8)]:

$$t_i\phi_b(t)=w_i\phi_{b+e_i}(t),\quad t\in E_n,\ 0\leqslant i\leqslant n. \qquad (4.2)$$

Since $t_0 + \cdots + t_n = 1$, (4.2) gives

$$\phi_b(t) = \sum_{i=0}^{n} w_i \phi_{b+e_i}(t).$$

Multiplying both sides by $f(Xt)$ and next integrating over E_n, we obtain by virtue of (2.2),

$$\int_{\mathbb{R}^s} f(x) M(x \mid b;\ X)\ \mathrm{d}x = \int_{\mathbb{R}^s} f(x) \sum_{i=0}^{n} w_i M(x \mid b + e_i;\ X)\ \mathrm{d}x.$$

Hence, (4.1) follows when $p(x) = 1$. To complete the proof, we utilize (4.2) again. Multiplying both sides by x_l^i and next summing over i, we obtain

$$[(Xt)_l]\phi_b(t) = \sum_{i=0}^{n} w_i x_l^i \phi_{b+e_i}(t).$$

Here $(Xt)_l$ denotes the lth component of Xt. This leads to the following integral relation:

$$\int_{\mathbb{R}^s} f(x) x_l M(x \mid b;\ X)\ \mathrm{d}x = \int_{\mathbb{R}^s} f(x) \left[\sum_{i=0}^{n} w_i x_l^i M(x \mid b + e_i;\ X) \right] \mathrm{d}x,$$

which proves (4.1) when $p(x) = x_l$, $1 \leqslant l \leqslant s$. $\square$

Micchelli [22] gave a different proof of (4.1) for simplex splines. For this class of splines, identity (4.1) is called the "degree elevating formula". A special case of (4.1) appears in [14].

For our further aims, we recall a definition of the R-hypergeometric function in the real case. Let the set $Z = \{z_0, \ldots, z_n\} \subset \mathbb{R}^{n+1}$ be such that $0 \notin [Z]$. Further, let $b \in \mathbb{R}_{>}^{n+1}$. The R-hypergeometric function $R_{-a}(b;\ Z)$, $a \in \mathbb{R}$, is given by

$$R_{-a}(b;\ Z) = \int_{E_n} (Zt)^{-a} \phi_b(t)\ \mathrm{d}t \tag{4.3}$$

(see [3]). When $-a \in \mathbb{N}$, the restriction $0 \notin [Z]$ can be dropped. Comparison with (3.1) shows that the R_{-a} is the Dirichlet average of the power function u^{-a}. It is worth mentioning that the Gauss hypergeometric function ${}_2F_1$, Lauricella's hypergeometric function F_D, the Gegenbauer polynomials and the elliptic integrals in the Legendre form can all be represented in terms of the function R_{-a}. Combining (4.3) and (2.3) gives

$$R_{-a}(b;\ Z) = \int_{\mathbb{R}} u^{-a} m(u \mid b;\ Z)\ \mathrm{d}u. \tag{4.4}$$

For later use, let us record a very useful formula for R-hypergeometric functions (see [3, Theorem 6.8-3])

$$R_{-a}(b;\ Z) = \prod_{j=0}^{n} z_j^{-b_j} R_{a-c}(b;\ Z^{-1}), \tag{4.5}$$

where $Z^{-1} := \{z_0^{-1}, \ldots, z_n^{-1}\}$, $z_j > 0$, for all j, $c \neq 0, -1, -2, \ldots$. This important result is commonly referred to as Euler's transformation.

We close this section with the following proposition.

Proposition 4.2 *Let $a \in \mathbb{R}$ and let the vector $\lambda \in \mathbb{R}^s$ be such that $\lambda \cdot x^j < 1$, $j = 0, \ldots, n$. Then,*

$$\int_{\mathbb{R}^s} (1 - \lambda \cdot x)^{-a} M(x \mid b;\, X)\, \mathrm{d}x = R_{-a}(b;\, Y), \tag{4.6}$$

where

$$Y = 1 - \lambda \cdot X = \{(1 - \lambda \cdot x^0), \ldots, (1 - \lambda \cdot x^n)\}. \tag{4.7}$$

Proof. Substituting $Z = Y$ into (4.4), we obtain

$$R_{-a}(b;\, Y) = \int_{\mathbb{R}} u^{-a} m(u \mid b;\, 1 - \lambda \cdot X)\, \mathrm{d}u = \int_{\mathbb{R}} (1 - u)^{-a} m(u \mid b;\, \lambda \cdot X)\, \mathrm{d}u$$
$$= \int_{\mathbb{R}^s} (1 - \lambda \cdot X)^{-a} M(x \mid b;\, X)\, \mathrm{d}x.$$

In the last step we have used (3.10). □

When $a = c$, (4.6) becomes Watson's identity (see [31])

$$\int_{\mathbb{R}^s} (1 - \lambda \cdot x)^{-c} M(x \mid b;\, X)\, \mathrm{d}x = \prod_{j=0}^{n} (1 - \lambda \cdot x^j)^{-b_j}. \tag{4.8}$$

The above identity follows by applying (4.5) to the right-hand side of (4.6) and using $R_0 = 1$.

An alternative proof of (4.8) appears in [10] (see also [19] for some comments concerning this identity).

5. Moments of multivariate Dirichlet splines

A motivation for the investigation of the moments of Dirichlet splines has its origin in two mathematical disciplines. It is well known that the spline $M(\cdot \mid b;\, X)$ is a probability density function on $\mathbb{R}^s$. We feel that the results of this section can be applied to some problems in mathematical statistics. A second area of possible applications is the theory of special functions. We have already mentioned that some important special functions can be represented by the R-hypergeometric functions. For particular values of the a-parameter and the b-parameters in (4.4), this function becomes a complete symmetric function. For particular values of the z-variables, (4.4) gives an integral formula for the q-binomial coefficients (Gaussian polynomials) (see [25] for more details).

In this section, we derive recurrence formulas for the moments of multivariate Dirichlet splines. Also, we discuss implementation of these results in the case when $b = (1, \ldots, 1) \in \mathbb{R}^{n+1}$. For related results when $s = 1$, see [24]. We employ the multi-index notation introduced in Section 2.

For $\beta \in \mathbb{R}^s$, we define the moment of order $|\beta|$, $|\beta| = \beta_1 + \cdots + \beta_s$, of $M(\cdot \mid b;\, X)$ as follows:

$$m_\beta(b;\, X) = \int_{\mathbb{R}^s} x^\beta M(x \mid b;\, X)\, \mathrm{d}x, \tag{5.1}$$

provided $0_s \notin [X]$, 0_s being the origin in $\mathbb{R}^s$. When $\beta \in \mathbb{Z}_+^s$, this restriction is nonessential. In the case of the simplex spline, we shall omit the vector b and write $m_\beta(X)$ instead of $m_\beta(b;\, X)$. Also, let d_l stand for the lth coordinate vector in $\mathbb{R}^s$.

We are now ready to state and prove the following.

Theorem 5.1. *Let the weights* $w_0, \dots, w_n$ *be the same as in Proposition* 4.1. *Then,*

$$m_\beta(b;\, X) = \sum_{i=0}^{n} w_i m_\beta(b + e_i;\, X), \tag{5.2}$$

$$m_{\beta+d_l}(b;\, X) = \sum_{i=0}^{n} w_i x_l^i m_\beta(b + e_i;\, X), \tag{5.3}$$

for all $l = 1, 2, \dots, s$. *Moreover, if* $\operatorname{vol}_s([X]) > 0$ *and* $b_j \geqslant 1$, *for some* $0 \leqslant j \leqslant n$, *then*

$$(c + |\beta| - 1) m_\beta(b;\, X) = (c-1) m_\beta(b - e_j;\, X) + \sum_{l=1}^{s} \beta_l x_l^j m_{\beta-d_l}(b;\, X). \tag{5.4}$$

If for some $0 \leqslant i,\ j \leqslant n$, $1 \leqslant k \leqslant s$, $x_k^i \neq 0$, $x_k^j \neq 0$, *and* $b_i \geqslant 1$, $b_j \geqslant 1$, *then*

$$(c-1)\left[m_\beta(b - e_j;\, X) - m_\beta(b - e_i;\, X)\right] + \sum_{k=1}^{s} \beta_k (x_k^j - x_k^i) m_{\beta-d_k}(b;\, X) = 0 \tag{5.5}$$

and

$$(c + |\beta| - 1)(x_k^i - x_k^j) m_\beta(b;\, X) = (c-1)\left[x_k^i m_\beta(b - e_j;\, X) - x_k^j m_\beta(b - e_i;\, X)\right] + \sum_{l=1}^{s} \beta_l W_{k,l} m_{\beta-d_l}(b;\, X), \tag{5.6}$$

where

$$W_{k,l} = \det \begin{bmatrix} x_k^i & x_k^j \\ x_l^i & x_l^j \end{bmatrix}.$$

Remark. When $b_j = 1$, formula (5.4) holds true provided $n > s$.

Proof. In order to establish the recursions (5.2) and (5.3), we substitute $p(x) = 1$ and $p(x) = x_l$ respectively, into (4.1) and next integrate over $\mathbb{R}^s$. For the proof of (5.4), we utilize formula (3.7) with $f(x) = x^\beta$. The resulting equation, together with (3.6) and (5.1), yields the assertion. Formulas (5.5) and (5.6) follow immediately from Corollary 3.2 with $f(x) = x^\beta$. □

We now give two moment generating functions. The first generating function involves the confluent hypergeometric function S. Following [3, (5.8-1)], we define

$$S(b;\, Z) = \int_{E_n} \exp(Zt) \phi_b(t)\, \mathrm{d}t, \tag{5.7}$$

$b \in \mathbb{R}_>^{n+1}$, $Z = \{z_0, \dots, z_n\}$. Use of (3.1) and (3.2) gives

$$S(b;\, Z) = \int_{\mathbb{R}} \exp(u) m(u \mid b;\, Z)\, \mathrm{d}u.$$

Letting $Z=\lambda\cdot X=\{\lambda\cdot x^0,\dots,\lambda\cdot x^n\}$, $\lambda\in\mathbb{R}^s\setminus\{0\}$, and next using (3.10), we arrive at

$$S(b;\ \lambda\cdot X)=\int_{\mathbb{R}^s}\exp(\lambda\cdot x)M(x\mid b;\ X)\,\mathrm{d}x. \tag{5.8}$$

To obtain the first moment generating function, we expand $\exp(\lambda\cdot x)$ into a power series. Applying the multinomial theorem to powers of $\lambda\cdot x$ and next integrating the corresponding power series one term at a time, we obtain by virtue of (5.8) and (5.1),

$$S(b;\ \lambda\cdot X)=\sum\frac{\lambda^j}{j!}m_j(b;\ X), \tag{5.9}$$

where the summation extends over all multi-indices $j\in\mathbb{Z}_+^s$.

It is worth mentioning that the hypergeometric function $S(b;\ \lambda\cdot X)$ can be expressed as a divided difference of $\exp(z)$ provided that $b_0,\dots,b_n\in\mathbb{Z}_+$. We have

$$k!\ S(b;\ \lambda\cdot X)=[\lambda\cdot x^0(b_0),\dots,\lambda\cdot x^n(b_n)]\mathrm{e}^z, \tag{5.10}$$

where $k=b_0+\cdots+b_n-1$. Here the symbol $\lambda\cdot x^i(b_i)$ means that the knot $\lambda\cdot x^i$ is repeated b_i times. Formula (5.10) readily follows from (5.8), (5.7) and the Hermite–Genocchi formula for divided differences.

A second generating function is given by

$$R_{-a}(b;\ Y)=\sum\lambda^j\frac{(a,|j|)}{j!}m_j(b;\ X), \tag{5.11}$$

$|\lambda\cdot x^i|<1$, for all i, where the summation extends over all multi-indices $j\in\mathbb{Z}_+^s$. Here, $a\in\mathbb{R}$, the set Y is given in (4.7) and $(a,\ l)$ stands for the Appell symbol, i.e., $(a,\ 0)=1$, $(a,\ l)=a(a+1)\cdots(a+l-1)$, $l\in\mathbb{N}$. In order to establish (5.11), we expand $(1-\lambda\cdot x)^{-a}$ into a power series and next utilize the multinomial theorem to obtain

$$(1-\lambda\cdot x)^{-a}=\sum_{l=0}^{\infty}(a,\ l)\sum_{|j|=l}\frac{\lambda^j}{j!}x^j. \tag{5.12}$$

To complete the proof, we substitute (5.12) into (4.6) and next integrate term by term. Applications of (5.9) and (5.11) are discussed in the next section.

We shall now turn our attention to the case of multivariate simplex splines. To this end, let $\beta\in\mathbb{Z}_+^s$. In the case under discussion, the formulas (5.3) and (5.4) of Theorem 5.1 take the form

$$m_{\beta+d_l}(X)=\frac{1}{n+1}\sum_{i=0}^{n}x_l^im_\beta(X^i), \tag{5.13}$$

$$(n+|\beta|)m_\beta(X)=n[m_\beta(X_j)]+\sum_{l=1}^{s}\beta_lx_l^jm_{\beta-d_l}(X), \tag{5.14}$$

$1\leqslant j\leqslant n$, $1\leqslant l\leqslant s$. Here $X^i=X\cup\{x^i\}$ and $X_i=X\setminus\{x^i\}$, $0\leqslant i\leqslant n$. The set X^i appears on the right-hand side of (5.13) because of (3.5). A closer look at (5.14) shows that the recursion is in two directions. That is, given $X=\{x^0,\dots,x^n\}\subset\mathbb{R}^s$, $n<s$, to compute $m_\beta(X)$, we need the moment of order $|\beta|$ for the knot set consisting of one less vector than X, and also s moments of order $|\beta|-1$.

Define a set $\overline{X}_k=(x^0,\dots,x^k)$, $k=s, s+1,\dots,n$, and note $m_\beta(\overline{X}_k)=1$ when $|\beta|=0$, $k=s, s+1,\dots,n$. To employ (5.14), we must precompute certain moments of the form $m_\beta(\overline{X}_s)$, $|\beta|>0$, and $m_{d_l}(\overline{X}_k)$, $k=s,\dots,n$. Let us note that

$$m_{d_l}(\overline{X}_k)=\frac{1}{k+1}\sum_{j=0}^{k}x_l^j$$

follows immediately from the defining equation (2.2).

In order to compute the moments $m_\beta(\overline{X}_s)$, $|\beta|>0$, we first introduce some new notation and next appeal to the proposition that follows.

Let $t=(t_0, t_1,\dots,t_n)\in\mathbb{R}^{n+1}$ with $(t_1,\dots,t_n)\in E_n$ and $t_0=1-\Sigma_{i=1}^n t_i$. Also, let $l=(l_0,\dots,l_n)\in\mathbb{Z}_+^{n+1}$, with $m=|l|$. Then we define the Bernstein polynomial by

$$B_l^m(t)=\binom{m}{l}t^l.$$

For given coefficients $\{p_l\}$, $|l|=m$, we shall call any polynomial of the form

$$q(t)=\sum_{|l|=m}p_lB_l^m(t)$$

a Bézier polynomial. It is well known that any such q may be stably and efficiently evaluated using deCasteljau's algorithm (see [9,12]).

Proposition 5.2 illustrates that we may indeed incorporate deCasteljau's algorithm when evaluating $m_\beta(\overline{X}_s)$.

Proposition 5.2. *Let* $n\geqslant s\geqslant 1$, $\beta\in\mathbb{Z}_+^s$, *and* $X=\{x^0,\dots,x^n\}\subset\mathbb{R}_+^s$ *with* $\operatorname{vol}_s([X])>0$. *Let* $y^i=(x_i^0,\dots,x_i^n)\in\mathbb{R}_+^{n+1}$ *and set* $g_i=|y^i|$, $i=1,\dots,s$. *Then,*

$$m_\beta(X)=\frac{g^\beta n!}{(|\beta|+n)!}\sum_{|k_1|=\beta_1}B_{k_1}^{\beta_1}(\tilde{y}^1)\cdots\sum_{|k_s|=\beta_s}B_{k_s}^{\beta_s}(\tilde{y}^s)\eta!. \tag{5.15}$$

Here, $k_i=(k_{i0},\dots,k_{in})\in\mathbb{Z}_+^{n+1}$, $\eta!=\eta_0!\cdots\eta_n!$, *with* $\eta_j=\Sigma_{i=1}^s k_{ij}$, $0\leqslant j\leqslant n$, $\tilde{y}^i=(1/g_i)y^i$, $i=1,\dots,s$.

Proof. We use the defining relation (2.2) to write

$$\begin{aligned}m_\beta(X)&=n!\int_{E_n}\prod_{j=1}^{s}\left(t_0x_j^0+\cdots+t_nx_j^n\right)^{\beta_j}\,\mathrm{d}t\\&=n!\sum_{|k_1|=\beta_1}\binom{\beta_1}{k_1}(y^1)^{k_1}\cdots\sum_{|k_s|=\beta_s}\binom{\beta_s}{k_s}(y^s)^{k_s}\int_{E_n}t^\eta\,\mathrm{d}t,\end{aligned}$$

where η, y^i, k_i, $i=1,\dots,s$, are given in Proposition 5.2. Using [3, 4.3-4] to simplify the integral in the above identity, we have

$$m_\beta(X)=n!\sum_{|k_1|=\beta_1}\binom{\beta_1}{k_1}(y^1)^{k_1}\cdots\sum_{|k_s|=\beta_s}\binom{\beta_s}{k_s}(y^s)^{k_s}\frac{\eta!}{(|\beta|+n)!}. \tag{5.16}$$

Now $\tilde{y}^j$, $j = 1, \dots, s$, given in Proposition 5.2 can be viewed as the barycentric coordinates of some point in $\mathbb{R}^n$. Inserting $\tilde{y}^j$'s into (5.16) and scaling by g^β gives the desired result. □

Thus all moments of the form $m_\beta(\bar{X}_s)$ can be expressed as a nested sum of Bézier polynomials and subsequently may be evaluated using deCasteljau's algorithm. It should be noted that while both deCasteljau's algorithm and (5.14) are possible candidates for the task of computing $m_\beta(\bar{X}_k)$, $k = s+1, \dots, n$, the latter scheme requires the evaluation of fewer terms at each recursion step and is thus the preferred choice. In order to summarize the procedure for evaluating the moments of simplex splines in Algorithm 5.3, we introduce $d_0 = (0, \dots, 0) \in \mathbb{R}^s$.

Algorithm 5.3. Given $X = \{x^0, \dots, x^n\} \subset \mathbb{R}^s$, $n \geqslant s$, and $\beta \in \mathbb{Z}_+^s$ with $|\beta| > 1$, this algorithm generates the moment $m_\beta(X)$ of the simplex spline $M(\cdot \mid X)$.

(1) $\alpha := d_0$.

(2) For $k = s$ to n

$\quad m_\alpha(\bar{X}_k) = 1$.

(3) For $|\alpha| = 1$ to $|\beta|$, $\alpha \in \mathbb{Z}_+^s$, $\alpha \leqslant \beta$

Use (5.15) to express $m_\alpha(\bar{X}_s)$ in terms of Bézier polynomials and evaluate using deCasteljau's algorithm.

(4) For $k = s+1$ to n

$\quad$ For $\alpha \in \mathbb{Z}_+^s$, $\alpha \leqslant \beta$

$\quad\quad$ Compute $m_\alpha(\bar{X}_k)$ using (5.14).

We close this section with a remark that this algorithm is numerically stable if $x^j > 0$ for all $j = 0, 1, \dots, n$.

6. Applications to hypergeometric functions

In this section we demonstrate a relationship between Dirichlet splines and an important class of hypergeometric functions of several variables. We will deal mainly with Appell's F_4 and Lauricella's F_B. The link between these classes of functions is provided by another integral average which is commonly referred to as a double Dirichlet average (see [1] for more details). Throughout the sequel the double Dirichlet average of a continuous univariate function h will be denoted by $\mathscr{H}$.

Let $X \in \mathbb{R}^{s \times (n+1)}$, $n \geqslant s \geqslant 1$. Further, let $u = (u_1, \dots, u_s)$ be an ordered s-tuple of nonnegative numbers with $u_1 + \cdots + u_s = 1$, and similarly $v = (v_0, \dots, v_n)$. We define

$$u \cdot Xv = \sum_{i=1}^{s} \sum_{j=0}^{n} u_i x_i^j v_j,$$

where x_i^j stands for the ith component of the jth column of X. Let h be a continuous function on $I = [\text{Min } x_i^j, \text{Max } x_i^j]$. In order to avoid trivialities, we will assume that I has a nonempty interior. For $b = (b_1, \dots, b_s) \in \mathbb{R}_>^s$ and $d = (d_0, \dots, d_n) \in \mathbb{R}_>^{n+1}$, let [1, p.421]

$$\mathscr{H}(b; X; d) = \int_{E_n} \int_{E_{s-1}} h(u \cdot Xv) \phi_b(u) \phi_d(v) \, du \, dv,$$

$\mathrm{d}u = \mathrm{d}u_2 \ldots \mathrm{d}u_s$, $\mathrm{d}v = \mathrm{d}v_1 \ldots \mathrm{d}v_n$. Here ϕ_b and ϕ_d are the Dirichlet densities on E_{s-1} and E_n, respectively (see (2.1)). It is known that for $b \in \mathbb{R}^s_>$,

$$\mathscr{H}(b;\, X;\, d) = \int_{E_{s-1}} H(d;\, u \cdot X)\phi_b(u)\, \mathrm{d}u \tag{6.1}$$

(see [1, (2.8)]). In (6.1) H stands for the single Dirichlet average of h (see (3.1)), $u \cdot X = \{u \cdot x^0, \ldots, u \cdot x^n\}$, $x^0, \ldots, x^n$ are the columns of X.

We are in a position to state and prove the following theorem.

Theorem 6.1. *Let* $d \in \mathbb{R}^{n+1}_>$ *and let the vector* $b \in \mathbb{R}^s$ *be such that* $c \neq 0, -1, \ldots$, $c = b_1 + \cdots + b_s$. *If* $\mathrm{vol}_s([X]) > 0$, *then*

$$\mathscr{H}(b;\, X;\, d) = \int_{[X]} M(x \mid d;\, X) H(b;\, x)\, \mathrm{d}x, \tag{6.2}$$

$x = (x_1, \ldots, x_s)$, $\mathrm{d}x = \mathrm{d}x_1 \ldots \mathrm{d}x_s$.

Proof. In order to establish (6.2), assume for the moment that $b \in \mathbb{R}^s_>$. Application of (3.2) and (3.10) to (6.1) gives

$$\mathscr{H}(b;\, X;\, d) = \int_{E_{s-1}} \left[\int_{[X]} h(u \cdot x) M(x \mid d;\, X)\, \mathrm{d}x \right] \phi_b(u)\, \mathrm{d}u.$$

Interchanging the order of integration and next using (3.1), we obtain the assertion provided $b \in \mathbb{R}^s_>$. This restriction can be dropped because the average H can be continued analytically in the b-parameters, provided that $c \neq 0, -1, \ldots$ (see [3, Theorem 6.3-7]). This completes the proof. □

Before we state a corollary of Theorem 6.1, let us introduce more notation. For $h(z) = z^{-a}$, $a \in \mathbb{R}$, the double Dirichlet average of h will be denoted by $\mathscr{R}_{-a}$ (cf. [1]).

Corollary 6.2 (Carlson [4]). *Let* $d \in \mathbb{R}^{n+1}_>$, $b \in \mathbb{R}^s$, *and let the matrix* X *be such that* $0_s \notin [X]$. *Then,*

$$m_{-b}(d;\, X) = \mathscr{R}_{-c}(b;\, X;\, d), \tag{6.3}$$

where $m_{-b}(d;\, X)$ *stands for the moment of order* $-c$ *of the Dirichlet spline* $M(\cdot \mid d;\, X)$.

Proof. Apply [3, (6.6-5)]

$$R_{-c}(b;\, X) = \prod_{i=1}^{s} x_i^{-b_i} \tag{6.4}$$

to (6.2) with $h(t) = t^{-c}$. □

Hereafter, we will deal with the hypergeometric functions and polynomials of several variables. Appell's hypergeometric function F_4 is defined by the double power series [3, Example 6.3-5]

$$F_4(\alpha, \beta; \gamma, \delta; x_1, x_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(\alpha, i+j)(\beta, i+j)}{(\gamma, i)(\delta, i)\, i!\, j!} x_1^i x_2^j,$$

$\alpha, \beta, \gamma, \delta \in \mathbb{R}$, $\gamma, \delta \neq 0, -1, \ldots$, $|x_1|^{1/2} + |x_2|^{1/2} < 1$. The integral formula [2, p.963]

$$F_4(\alpha, \beta; \gamma, \delta; x_1(1-x_2), x_2(1-x_1)) = \int_0^1 R_{-\alpha}(d_0, d_1, d_2; u \cdot x^0, u \cdot x^1, u \cdot x^2)\phi_b(u)\, du \tag{6.5}$$

provides the analytic continuation of the F_4-series to the region Λ defined by

$$\Lambda = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1 < 1,\ x_2 < 1,\ x_1 + x_2 < 1\}.$$

In (6.5), $b = (\beta, \gamma - \beta)$, $d_0 = \gamma + \delta - \alpha - 1$, $d_1 = \alpha + \beta - \gamma - \delta + 1$, $d_2 = \delta - \beta$, $\phi_b(u)$ is the Dirichlet density on E_1 and x^0, x^1, and x^2 are the columns of X, where

$$X = \begin{bmatrix} (1-x_1)(1-x_2) & 1-x_1-x_2 & 1-x_1 \\ 1-x_2 & 1-x_2 & 1 \end{bmatrix}. \tag{6.6}$$

Corollary 6.3. *Let* $d = (d_0, d_1, d_2) \in \mathbb{R}^3_>$ *and let* $b = (\beta, \gamma - \beta) \in \mathbb{R}^2$. *If* $\mathrm{vol}_2([X]) > 0$, *then*

$$F_4(\alpha, \beta; \gamma, \delta; x_1(1-x_2), x_2(1-x_1)) = \int_{[X]} M(y \mid d; X) R_{-\alpha}(b; y)\, dy, \tag{6.7}$$

$y = (y_1, y_2)$, $dy = dy_1\, dy_2$. *Here*, $R_{-\alpha}$ *is the single Dirichlet average of* $h(z) = z^{-\alpha}$ *and the matrix X is given in* (6.6).

Proof. Apply (6.1) to (6.5) and next use (6.2). □

A special case of (6.7) is

$$F_4(\alpha, \beta; \alpha, \delta; x_1(1-x_2), x_2(1-x_1)) = m_{-b}(d; X),$$

where now $b = (\beta, \alpha - \beta)$ and $d = (\delta - 1, \beta - \delta + 1, \delta - \beta)$. This follows immediately from (6.7) and (6.4).

We will now deal with Lauricella's F_B function and Lauricella polynomials. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$, $\gamma \in \mathbb{R}$, $\gamma \neq 0, -1, \ldots,$ and let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, with $|x_i| < 1$, for all i. Following [21], we define

$$F_B(\alpha, \beta; \gamma; x) = \sum \frac{(\alpha, k)(\beta, k)}{(\gamma, |k|)\, k!} x^k, \tag{6.8}$$

where the summation extends over all multi-indices $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n_+$. In (6.8) we employ multi-index notation introduced in Section 2. Also,

$$(\alpha, k) = \prod_{i=1}^{n} (\alpha_i, k_i).$$

(β, k) is defined in an analogous manner. When $n = 1$, F_B becomes Gauss' ${}_2F_1$ function.

Corollary 6.4. *Let* $d=(\beta,\gamma-|\beta|)\in\mathbb{R}_{>}^{n+1}$ *and let*

$$X=\begin{bmatrix}1-x_1 & 1 & \cdots & 1 & 1\\ 1 & 1-x_2 & \cdots & 1 & 1\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 1 & 1 & \cdots & 1-x_n & 1\end{bmatrix}. \tag{6.9}$$

Then,

$$F_B(\alpha,\beta;\gamma;x)=m_{-\alpha}(d;X), \tag{6.10}$$

provided that $x_i<1$ *for all* i.

Proof. In the stated domain the entries of X are positive. Thus $0_n\notin[X]$. In order to establish (6.10), we utilize [1, (5.11)] to obtain

$$F_B(\alpha;\beta;\gamma;x)=\mathscr{R}_{-c}(\alpha;X;d),$$

where now $c=\alpha_1+\cdots+\alpha_n$. This in conjunction with (6.3) gives the assertion. □

Lauricella polynomials $L_j(x)$, $j\in\mathbb{Z}_+^n$, $x\in\mathbb{R}^n$, are defined in the following way:

$$L_j(x)=F_B(-j,\beta;\gamma;x).$$

These polynomials play an important role in the study of coherent states (cf. [16]). On account of (6.10),

$$L_j(x)=m_j(d;X), \tag{6.11}$$

where the vector d and the matrix X are the same as in Corollary 6.4.

Two generating functions for the polynomials under discussion can be derived from (5.9) and (5.11). Let $\lambda=(\lambda_1,\ldots,\lambda_n)$, $e=(1,\ldots,1)$, λ, $e\in\mathbb{R}^n$, and let $d=(\beta,\gamma-|\beta|)\in\mathbb{R}^{n+1}$. Then,

$$\exp(\lambda\cdot e)S(d;-\lambda_1x_1,\ldots,-\lambda_nx_n,0)=\sum\frac{\lambda^j}{j!}L_j(x), \tag{6.12}$$

$j\in\mathbb{Z}_+^n$. If

$$\operatorname{Max}\{|\lambda\cdot e-\lambda_1x_1|,\ldots,|\lambda\cdot e-\lambda_nx_n|,\ |\lambda\cdot e|\}<1,$$

then

$$R_{-a}(d;Y)=\sum\lambda^j\frac{(a,|j|)}{j!}L_j(x), \tag{6.13}$$

$a\in\mathbb{R}$, $j\in\mathbb{Z}_+^n$, $Y=\{1-\lambda\cdot e+\lambda_1x_1,\ldots,1-\lambda\cdot e+\lambda_nx_n,\ 1-\lambda\cdot e\}$.

For the proof of (6.12) we replace b by d in (5.9) and next use (6.11), (6.9), and [3, (5.8-3)]. This gives the desired result provided that $d\in\mathbb{R}_{>}^{n+1}$. The latter restriction can be dropped because the S-function can be continued analytically in the d-parameters [3, Corollary 6.33]. The generating function (6.13) can be derived from (5.11) by the same means. Feinsilver's generating function [15] can be obtained from (6.13) by letting $a=\gamma$ and then using (4.5).

Recall that $R_0 = 1$. It is not hard to show that the R-hypergeometric function R_{-a} in (6.13) is a multiple of the Lauricella function of the fourth kind. We have

$$R_{-a}(d;\, Y) = (1 - \lambda \cdot e)^{-a} F_D(a, \beta;\, \gamma;\, z_1, \ldots, z_n),$$

where

$$z_i = \frac{-\lambda_i x_i}{1 - \lambda \cdot e},$$

$i = 1, 2, \ldots, n$. We omit further details.

Our next goal is to establish a recurrence formula obeyed by Lauricella polynomials

$$(\gamma + |k|) L_{k+d_m}(x) - \left[\gamma(1 - w_m x_m) + |k|\right] L_k(x) + \sum_{l=1}^{n} k_l \epsilon_{lm} \left[L_{k-d_l}(x) - L_{k-d_l+d_m}(x)\right] = 0, \tag{6.14}$$

$m = 1, 2, \ldots, n$. In (6.14), $\gamma \in \mathbb{R}_>$, $k = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$, d_m stands for the mth coordinate vector in $\mathbb{R}^n$, similarly d_l, $w_m = \beta_m/\gamma$, $\beta_m > 0$, $1 \leqslant m \leqslant n$,

$$\epsilon_{lm} = \begin{cases} 1, & \text{if } l \neq m, \\ 1 - x_m, & \text{if } l = m. \end{cases} \tag{6.15}$$

Here we adopt the convention that $L_k(x) = 0$ if $-k_m \in \mathbb{N}$ for some m. In order to establish the recursion (6.14), we derive first a recurrence formula for the moments of multivariate Dirichlet splines with $s = n$ and X given in (6.9). We have

$$(c + |\beta|) m_{\beta+d_m}(b;\, X) - \left[c(1 - w_m x_m) + |\beta|\right] m_\beta(b;\, X) + \sum_{l=1}^{n} \beta_l \epsilon_{lm} \left[m_{\beta-d_l}(b;\, X) - m_{\beta-d_l+d_m}(b;\, X)\right] = 0, \tag{6.16}$$

$b \in \mathbb{R}^{n+1}$, $c = b_1 + \cdots + b_{n+1}$, $\beta \in \mathbb{R}^n$, $|\beta| = \beta_1 + \cdots \beta_n$. The recursion (6.14) now follows from (6.16) by letting $b = d$, $\beta = k \in \mathbb{Z}_+^n$ and using (6.11). To complete the proof, we need to establish (6.16). To this aim we increase the indices of summation in (5.2) and (5.3) by one unit. Next we let $s = n$ and solve the resulting linear system for $m_\beta(b + e_m;\, X)$, $1 \leqslant m \leqslant n+1$. Let us note that the assumption $\mathrm{vol}_n([X]) > 0$ is equivalent to $x_m \neq 0$, $1 \leqslant m \leqslant n$. This assures uniqueness of the solution. Subtracting (5.3) from (5.2), we obtain

$$m_\beta(b + e_m;\, X) = \frac{m_\beta(b;\, X) - m_{\beta+d_m}(b;\, X)}{w_m x_m}, \tag{6.17}$$

$1 \leqslant m \leqslant n$. The remaining moment $m_\beta(b + e_{n+1};\, X)$ can be found using (5.2) and (6.17). To complete the proof of (6.16), we utilize (5.4). Replacing the index j by m and next using (6.17), we can easily obtain the assertion.

We close this section with an inequality for Lauricella polynomials. To this end, let $x_i < 1$, $1 \leqslant i \leqslant n$. It follows from (6.9) that $[X] \subset \mathbb{R}_>^n$ in the stated domain. This in conjunction with (6.11) and (5.1) provides

$$L_j(x) = \int_{[X]} y^j M(y \mid d;\, X)\, \mathrm{d}y > 0,$$

$j \in \mathbb{Z}_+^n$, $d=(\beta, \gamma - |\beta|) \in \mathbb{R}_>^{n+1}$, $y=(y_1,\ldots,y_n)$, $\mathrm{d}y=\mathrm{d}y_1\ldots\mathrm{d}y_n$. A standard argument applied to the last formula gives

$$[L_j(x)]^2 \leqslant L_{j-k}(x) L_{j+k}(x),$$

where the vector $k \in \mathbb{Z}_+^n$ is such that $j-k \in \mathbb{Z}_+^n$. In particular, if $k=e_m$, the mth coordinate vector in $\mathbb{R}^n$, then

$$[L_j(x)]^2 \leqslant L_{j-e_m}(x) L_{j+e_m}(x),$$

provided $j-e_m \in \mathbb{Z}_+^n$. Thus the function $g: \mathbb{Z}_+^n \to \mathbb{R}$, where

$$g(j_1,\ldots,j_n)=L_{j_1,\ldots,j_n},$$

is log-convex in each variable separately.

Acknowledgement

The authors wish to thank a referee for insightful comments made on the first draft of this paper, and for the suggestions for improving Proposition 5.2.

References

[1] B.C. Carlson, Appell functions and multiple averages, *SIAM J. Math. Anal.* **2** (1971) 420–430.
[2] B.C. Carlson, Appell's function F_4 as a double average, *SIAM J. Math. Anal.* 6 (1975) 960–965.
[3] B.C. Carlson, *Special Functions of Applied Mathematics* (Academic Press, New York, 1977).
[4] B.C. Carlson, B-splines, hypergeometric functions, and Dirichlet averages, *J. Approx. Theory* **67** (1991) 311–325.
[5] E. Cohen, T. Lyche and R.F. Riesenfeld, Cones and recurrence relations for simplex splines, *Constr. Approx.* **3** (1987) 131–141.
[6] H.B. Curry and I.J. Schoenberg, On Pólya frequency functions IV. The fundamental spline functions and their limits, *J. Analyse Math.* **17** (1967) 71–107.
[7] W. Dahmen, Multivariate B-splines–Recurrence relations and linear combinations of truncated powers, in: W. Schempp and K. Zeller, Eds., *Multivariate Approximation Theory* (Birkhäuser, Basel, 1979) 64–82.
[8] W. Dahmen, On multivariate B-splines, *SIAM J. Numer. Anal.* **17** (1980) 993–1012.
[9] W. Dahmen, Bernstein–Bézier representation of polynomial surfaces, in: C. de Boor, Ed., *Extension of B-spline Curve Algorithms to Surfaces*, Siggraph 86 (ACM, Dallas, TX, 1986) 1–43.
[10] W. Dahmen and C.A. Micchelli, Statistical encounters with B-splines, *Contemp. Math.* **59** (1986) 17–48.
[11] C. de Boor, Splines as linear combinations of B-splines: A survey, in: G.G. Lorentz, C.K. Chui and L.L. Schumaker, Eds., *Approximation Theory II* (Academic Press, New York, 1976) 1–47.
[12] C. de Boor, B-form basics, in: G. Farin, Ed., *Geometric Modeling* (SIAM, Philadelphia, PA, 1987) 131–148.
[13] C. de Boor and K. Höllig, Recurrence relations for multivariate B-splines, *Proc. Amer. Math. Soc.* **85** (1982) 397–400.
[14] R. Farwig, Multivariate truncated powers and B-splines with coalescent knots, *SIAM J. Numer. Anal.* **22** (1985) 592–603.
[15] P. Feinsilver, Heisenberg algebras in the theory of special functions, in: Y.S. Kim and W.W. Zachary, Eds., *The Physics of Phase Space*, Lecture Notes in Phys. **278** (Springer, Berlin, 1987) 423–425.
[16] P. Feinsilver, Orthogonal polynomials and coherent states, in: B. Gruber, L.C. Biederharn and M.D. Doehner, Eds., *Symmetries in Science V* (Plenum, New York, 1991) 159–172.

[17] H. Hakopian, Multivariate spline functions, B-spline basis, and polynomial interpolants, *SIAM J. Numer. Anal.* **19** (1982) 510–517.
[18] K. Höllig, A remark on multivariate B-splines, *J. Approx. Theory* **33** (1981) 119–125.
[19] S. Karlin, C.A. Micchelli and Y. Rinott, Multivariate splines: A probabilistic perspective, *J. Multivariate Anal.* **20** (1986) 69–90.
[20] P. Kergin, Interpolation of C^k functions, Ph.D. Thesis, Univ. Toronto, 1978.
[21] G. Lauricella, Sulle funzioni ipergeometriche a piu variabli, *Rend. Circ. Mat. Palermo* **7** (1893) 111–158.
[22] C.A. Micchelli, On a numerically efficient method for computing multivariate B-splines, in: W. Schempp and K. Zeller, Eds., *Multivariate Approximation Theory* (Basel, Birkhäuser, 1979) 211–248.
[23] C.A. Micchelli, A constructive approach to Kergin interpolation in $\mathbb{R}^k$: Multivariate B-splines and Lagrange interpolation, *Rocky Mountain J. Math.* **10** (1980) 485–497.
[24] E. Neuman, Moments and Fourier transforms of B-splines, *J. Comput. Appl. Math.* **7** (1) (1981) 51–62.
[25] E. Neuman, On complete symmetric functions, *SIAM J. Math. Anal.* **19** (1988) 736–750.
[26] E. Neuman, Inequalities involving multivariate convex functions II, *Proc. Amer. Math. Soc.* **109** (1990) 965–974.
[27] E. Neuman, Dirichlet averages and their applications to special functions, manuscript.
[28] E. Neuman and J. Pečric, Inequalities involving multivariate convex functions, *J. Math. Anal. Appl.* **137** (1989) 541–549.
[29] M. Sabin, Open questions in the application of multivariate B-splines, in: T. Lyche and L.L. Schumaker, Eds., *Mathematical Methods in Computer Aided Geometric Design* (Academic Press, Boston, 1989) 529–537.
[30] L.L. Schumaker, *Spline Functions: Basic Theory* (Wiley, New York, 1981).
[31] G.S. Watson, On the joint distribution of the circular serial correlation coefficients, *Biometrika* **4** (1956) 161–168.