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Concerning preservation of chainability upon taking a preimage under $z \mapsto z^2$

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ABSTRACT

If X is an arc in the complex plane $\mathbb{C}$ with 0 as an endpoint, then the preimage of X under $f(z) = z^2$ is also an arc, and the endpoints of $f^{-1}(X)$ are the points in the preimage of the nonzero endpoint of X. In this paper, the author explores necessary and sufficient conditions under which a chainable continuum in $\mathbb{C}$ has chainable preimage under f. The paper contains an example of a chainable continuum X (the simple three-fold Knaster continuum) embedded in the complex plane in such a way that 0 is an endpoint of X and the preimage of X under the square map is not chainable.

1. Introduction

For this paper, let $\overline{\mathbb{C}}$ denote the one-point compactification of the complex plane, and let ∞ be the point in $\overline{\mathbb{C}} \setminus \mathbb{C}$. Let 0 denote the origin in $\overline{\mathbb{C}}$. A *continuum* is a nonempty, compact, connected metric space. A continuum is a *planar continuum* if it can be embedded in $\mathbb{C}$. Let f refer to the map on the complex plane given by $f(z) = z^2$. An *arc* is a continuum which is homeomorphic to the unit interval. An ϵ*-chain* is a finite collection of open sets, called links, indexed by $\{1, 2, \ldots, n\}$, each with diameter less than ϵ, such that each link intersects only the previous and subsequent links of the chain. The *end-links* of a chain are the first and last links of the chain. A nondegenerate continuum X is *chainable* if for every $\epsilon > 0$ there exists an ϵ-chain covering X. A point p in a chainable continuum X is an *endpoint* if there exist ϵ-chains as mentioned above in which p is in an end-link of each chain.

Let C be a finite collection of n open sets such that the intersection of any three elements of C is empty. Define the *nerve of* C as a graph G in the following manner. G has n vertices, each corresponding to a unique element of C, and for any two distinct vertices of G there exists an edge connecting the vertices if and only if the intersection of the corresponding elements of C is nonempty. A *tree-chain* is a finite open cover whose nerve is a tree. If for every $\epsilon > 0$ there exists a finite open cover of a continuum X whose nerve is isomorphic to some graph P, then X is P*-like*. Note that a chainable continuum is arc-like by the above definition.

A map $g : X \to Y$ is called an ϵ*-map* if for any $x \in X$, the diameter of $g^{-1}(f(x))$ is less than ϵ. It is known that a continuum X is chainable if and only if for every $\epsilon > 0$ there exists an ϵ-map from X onto the unit interval $I = [0, 1]$ (see, for example, [4, p. 114]).

If X is an arc in $\mathbb{C}$ with endpoints 0 and p, then $f^{-1}(X)$ can be written as $X_1 \cup X_2$, where X_1 and X_2 are arcs such that $X_1 \cap X_2 = \{0\}$ and each X_i maps homeomorphically via f onto X. Then $f^{-1}(X)$ is also an arc, and the endpoints of $f^{-1}(X)$ are exactly the points in $f^{-1}(\{p\})$. This may lead us to believe that the same is true for all chainable continua

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which contain 0 as an endpoint. We will give conditions for which the above statement is true and an example for which it is false.

2. Preservation of lack of chainability

The following theorem was proven by Rosenholtz in [8].

Theorem 1 *(Rosenholtz). Let $g : X \to Y$ be an open map. If X is chainable, then Y is chainable.*

Corollary 1. *Let $f(z) = z^2$. If $X \subset \mathbb{C}$ is not chainable, then $f^{-1}(X)$ is not chainable.*

Proof. This follows directly from Theorem 1 and the fact that f restricted to the domain $f^{-1}(X)$ is an open map. □

The *span* of a continuum X is the supremum of all ϵ for which there exists $Z \subset X \times X$ with $\pi_1(Z) = \pi_2(Z)$ and for every point $(x, y) \in Z$ we have $d(x, y) > \epsilon$. Lelek showed in [3] that chainability implies span zero for a continuum. Kawamura [2] proved the following theorem.

Theorem 2 *(Kawamura). Let $g : X \to Y$ be an open map. If X is has span zero, then Y has span zero.*

Corollary 2. *Let $f(z) = z^2$. If $X \subset \mathbb{C}$ is has positive span, then $f^{-1}(X)$ has positive span.*

Proof. This follows directly from Theorem 2 and the fact that f restricted to the domain $f^{-1}(X)$ is an open map. □

Question. Let $f(z) = z^2$. If $X \subset \mathbb{C}$ is a chainable continuum such that $f^{-1}(X)$ is also chainable, must 0 be an endpoint of X?

3. Sufficient conditions for preservation of chainability

The previous section demonstrates that chainability of X is necessary for chainability of $f^{-1}(X)$. We now give some sufficient conditions for which $f^{-1}(X)$ is chainable. Let p be a point in a planar continuum X. The point p is *accessible* if there exists an arc α in $\overline{\mathbb{C}}$ which only intersects X at p. The point p is accessible from the unbounded complement of X if there exists an arc α in $\overline{\mathbb{C}}$ from p to ∞ such that $\alpha \cap X = \{p\}$.

Lemma 1. *Let X be an embedding of a planar continuum containing 0 such that 0 is accessible from the unbounded complement of X. Then $f^{-1}(X)$ is the union of two subcontinua X_1 and X_2, each homeomorphic to X via f, such that $X_1 \cap X_2 = \{0\}$.*

Proof. Let α be an arc with 0 and ∞ as endpoints such that $\alpha \cap X = \{0\}$. Then $f^{-1}(\alpha)$ is a simple closed curve separating $\overline{\mathbb{C}}$ into two components C_1 and C_2, such that f restricted to either of these components is a homeomorphism. Let $X_i = (f^{-1}(X) \cap C_i) \cup \{0\}$ for $i = 1, 2$. □

Lemma 2. *Let X be the union of two chainable continua X_1 and X_2 such that $X_1 \cap X_2 = \{p\}$ and p is an endpoint of both X_1 and X_2. Then X is chainable.*

Proof. Let f_1 be an $\epsilon/2$-map from X_1 onto $[0, 1/2]$ such that $f_1(p) = 1/2$. Let f_2 be an $\epsilon/2$-map from X_2 onto $[1/2, 1]$ such that $f_2(p) = 1/2$. Let f_3 the union of f_1 and f_2. Then f_3 is an ϵ-map from X onto $[0, 1]$. □

Theorem 3. *Let $f(z) = z^2$. Let X be an embedding of a chainable planar continuum such that 0 is an endpoint of X. If 0 is accessible, then $f^{-1}(X)$ is chainable.*

Proof. Since X is chainable, it does not separate $\mathbb{C}$, so 0 is accessible from the unbounded complement of X. The remainder of the proof follows directly from Lemmas 1 and 2. □

The converse of Theorem 3 is false. Let α be an arc with 0 as an endpoint, and let X be the closure of a ray spiraling around α with only α as the remainder. Each point in α is inaccessible. Then $f^{-1}(X)$ is the closure of two disjoint rays spiraling around $f^{-1}(\alpha)$, with $f^{-1}(\alpha)$ as the remainder. This continuum is chainable.

Definition 1. Let X be an embedding of a chainable continuum with p as an endpoint. Define p to be a *strong endpoint* of X if for every $\epsilon > 0$, there is an ϵ-chain covering X whose links are convex, such that p is in an end-link of the chain.

Theorem 4. *Let $f(z) = z^2$. Let X be a planar embedding of a chainable continuum with 0 as a strong endpoint of X. Then $f^{-1}(X)$ is chainable.*

Proof. Let C be an ϵ-chain of convex links covering X with 0 in an end-link. Let D be the chain covering $f^{-1}(X)$ whose links are exactly the components of preimages of links of C. It is easy to show that D is a chain with small mesh. Then $f^{-1}(X)$ is chainable. □

While it has been shown that any chainable continuum with one or more endpoints can be embedded in the plane as a continuum with 0 as a strong endpoint, our particular embedding determines the shape of the preimage. In addition, Bing [1] showed that there exist embeddings of chainable continua which cannot be covered by chains of connected links of arbitrarily small mesh. Therefore, Theorem 3 will not hold for all embeddings of chainable continua. We will later give an example of an embedding of a chainable continuum with 0 as an endpoint whose preimage under $z \mapsto z^2$ is not chainable.

4. Preimages of endpoints

If X is an arc with endpoints 0 and p, then $f^{-1}(X)$ is an arc whose endpoints are precisely the two points which map via f onto p. However, if X is an arbitrary chainable planar continuum with chainable inverse $f^{-1}(X)$, it is not necessarily true that all points the preimage of endpoints of X are endpoints of $f^{-1}(X)$. For example, if we transform the standard embedding of the $\sin(1/x)$ continuum by a rigid translation so that 0 is an endpoint of the limit arc, then the preimage has two endpoints, each of which maps onto the locally connected endpoint of the curve. If we translate the curve so that 0 is the locally connected endpoint, then the preimage has four endpoints, each of which maps onto an endpoint on the limit arc. The following theorem tells us something about endpoints of $f^{-1}(X)$.

Theorem 5. *Let $f(z) = z^2$, and let X be a planar embedding of a chainable continuum. If $f^{-1}(X)$ is chainable and y is an endpoint of $f^{-1}(X)$, then $f(y)$ is an endpoint of X.*

Proof. In Rosenhotz's proof [8] that X must be chainable if $f^{-1}(X)$ is chainable and f is an open map, the first link of the chain D covering X is the image of the first link of the chain C covering $f^{-1}(X)$. We can always index our links so that our endpoint of $f^{-1}(X)$ is in the first link of C, so the image of this point is always in the first link of D. □

5. An example of lack of preservation of chainability

We now give an example of a chainable continuum X with 0 as an endpoint such that $f^{-1}(X)$ is not chainable.

5.1. Construction of X

Let C and D be chains. D *refines* C if every link of D is a subset of a link of C. The *pattern* of D in C is the set of pairs (i, j) such that $D_i \subset C_j$. A *pattern* P is said to be *monotone* if either (a) for every $(i_1, j_1), (i_2, j_2) \in P$, if $i_1 > i_2$ then $j_1 \geqslant j_2$ or (b) for every $(i_1, j_1), (i_2, j_2) \in P$, if $i_1 > i_2$ then $j_1 \leqslant j_2$. Let C^* denote the union of all the links of C.

Let $\widetilde{C}_0$ be a chain of convex links such that 0 is contained in exactly one link of $\widetilde{C}_0$ and the link containing 0 is a positive distance from the end-links of $\widetilde{C}_0$. We will inductively construct each C_{n+1} and $\widetilde{C}_{n+1}$. For each $n \geqslant 0$, let α_n be a simple closed curve containing 0 and ∞ which separates each link of $\widetilde{C}_n$ into two components, and separates $\widetilde{C}_n^*$ into two components. Define C_{n+1} as follows. Let $C_{n+1,0}$ be a convex link containing 0 whose closure is contained in the link of $\widetilde{C}_n$ which contains 0 and has diameter less than $1/(2^{n+1})$. Choose each subsequent link $C_{n+1,i}$ of C_{n+1} such that

(1) $\overline{C}_{n+1,i}$ is contained in a link of $\widetilde{C}_n$,
(2) $\overline{C}_{n+1,i}$ has two components, each of which is convex, one in either component of $\mathbb{C} \setminus \alpha_n$,
(3) 0 is contained in only one link of C_{n+1},
(4) the diameter of each component of $C_{n+1,i}$ is less than $1/(2^{n+1})$,
(5) C_{n+1} contains a link in the last link of $\widetilde{C}_n$,
(6) the last link of C_{n+1} is contained in the first link of $\widetilde{C}_n$, and
(7) C_{n+1} can be written as the union of two subchains such that the pattern in $\widetilde{C}_n$ of each subchain is monotone.

Then C_{n+1} is a $1/(2^n)$-chain since $\widetilde{C}_n$ is a $1/(2^n)$-chain. Now let $\widetilde{C}_{n+1}$ be a chain such that each link of $\widetilde{C}_{n+1}$ is a component of a link of C_{n+1}, and such that $\widetilde{C}^*_{n+1} = C^*_{n+1}$. Then $\widetilde{C}_{n+1}$ is a $1/(2^{n+1})$-chain with convex links.

Let $X = \bigcap_{n=1}^{\infty} \overline{C_n^*}$, and let $Y = f^{-1}(X)$.

Visual representations of the continua X and $Y = f^{-1}(X)$ are given in Fig. 1, where the gray regions represent the nesting of open covers of Y.

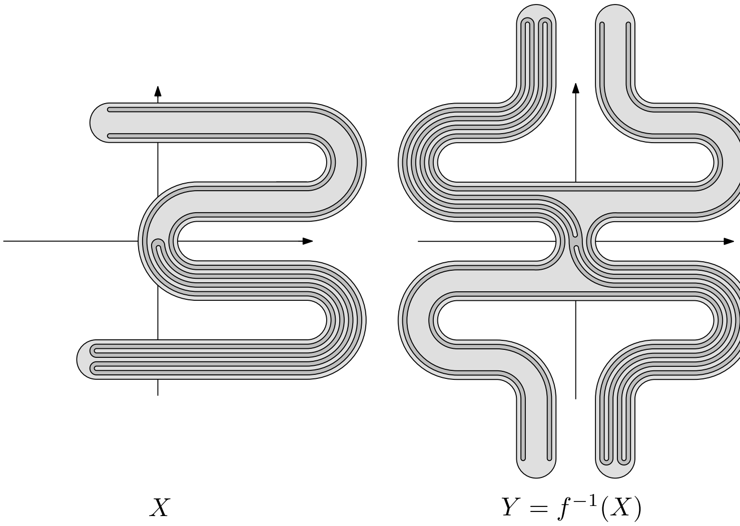

Fig. 1. Visual representations of X and Y.

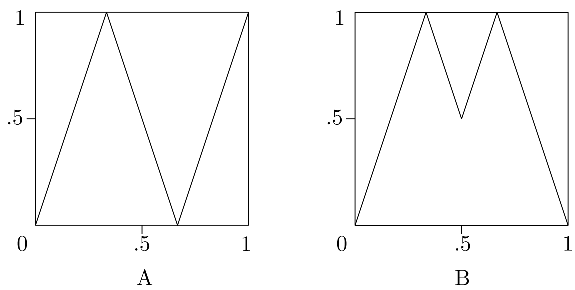

Fig. 2. Two bonding maps on the interval which each yield X.

5.2. Bonding maps

For $i = 1, 2, 3, \ldots,$ we have C_i refines $\widetilde{C}_i$ and $\widetilde{C}_i$ refines C_{i-1}. Define $\phi : I \to I$ and $\psi : I \to I$ as follows:

$$\phi(x) = \begin{cases} 1/2(3x+1), & 0 \leqslant x < 1/3, \\ 3/2(1-x), & 1/3 \leqslant x \leqslant 1, \end{cases}$$

$$\psi(x) = \begin{cases} 1-2x, & 0 \leqslant x < 1/2, \\ 2(x-1/2), & 1/2 \leqslant x \leqslant 1. \end{cases}$$

Then ϕ and ψ are maps corresponding to the pattern of C_i in $\widetilde{C}_{i-1}$ and the pattern of $\widetilde{C}_i$ in C_i, respectively.

Then the pattern from the sequence of chains $\{C_i\}_{i=1}^{\infty}$ indicates that $X \cong \varprojlim\{I, \psi \circ \phi\}$, where $\psi \circ \phi$ is given below (see Fig. 2A):

$$\psi \circ \phi(x) = \begin{cases} 3x, & 0 \leqslant x < 1/3, \\ 3(2/3 - x), & 1/3 \leqslant x < 2/3, \\ 3(x - 2/3), & 2/3 \leqslant x \leqslant 1. \end{cases}$$

In addition, the pattern from the sequence $\{\widetilde{C}_i\}_{i=1}^{\infty}$ indicates that $X \cong \varprojlim\{I, \phi \circ \psi\}$, where $\phi \circ \psi$ is given below (see Fig. 2B):

$$\phi \circ \psi(x) = \begin{cases} 3x, & 0 \leqslant x < 1/3, \\ 3(2/3 - x), & 1/3 \leqslant x < 1/2, \\ 3(x - 1/3), & 1/2 \leqslant x < 2/3, \\ 3(1 - x), & 2/3 \leqslant x \leqslant 1. \end{cases}$$

5.3. The continuum $Y = f^{-1}(X)$

Theorem 6. *If $Y = f^{-1}(X)$ as described above, then every nondegenerate proper subcontinuum of Y is an arc.*

Proof. By its construction, any nondegenerate proper subcontinuum of X is an arc. Moreover, any nondegenerate proper subcontinuum of X containing 0 is an arc with 0 as an endpoint. Any nondegenerate proper subcontinuum of Y not containing 0 maps homeomorphically onto its image, and is thus an arc. Any nondegenerate proper subcontinuum of Y containing 0 is contained in the preimage of its image; that is, in the preimage of an arc with 0 as an endpoint. Since this preimage is an arc, we have that the subcontinuum of Y is contained in an arc, and is thus an arc as well. □

Theorem 7. *Y is simple-4-od-like.*

Proof. For $i = 1, 2, 3, \ldots$, let K_i be the tree-chain whose links are exactly components of preimages under f of links of $\widetilde{C}_i$. Note that the preimage of the link containing 0 has one component, whereas the preimage of any link not containing 0 will have two components. Then each K_i is a tree-chain whose nerve is a simple-4-od, so Y is simple-4-od like. □

Theorem 8. *Let $\tilde{H}$ be a graph isomorphic to the letter H. Then Y is $\tilde{H}$-like.*

Proof. For $i = 1, 2, \ldots$, let C_{i1} be the maximal subchain of C_i containing 0 whose pattern in $\widetilde{C}_{i-1}$ is monotone. Let C_{i2} be the set of links of $\widetilde{C}_i$ which are not components of links of C_{i1}. Define $T_i = C_{i1} \cup C_{i2}$. Then T_i is a tree-chain whose nerve is a simple triod. For each link $T_{i,j}$ of T_i which is also a link in C_{i1}, let $T'_{i,j}$ denote the convex hull of $T_{i,j}$. Define M_i as follows. For the link $T_{i,0}$ of T_i containing 0, let $f^{-1}(T_{i,0})$ be a link of M_i. If $T_{i,j}$ is a link of C_{i1} but does not contain 0, then $f^{-1}(T'_{i,j})$ has two components, each of which contains a subset ξ of $f^{-1}(T_{i,j})$ such that $f(\xi) = T_{i,j}$. Define each such ξ to be a link of M_i, and for each link $T_{i,j}$ of C_{12}, define each component of $f^{-1}(T_{i,j})$ to be a link of M_i. Then M_i is a tree-chain whose nerve is homeomorphic to $\tilde{H}$, and for $i = 2, 3, 4, \ldots$ we have $M_i \subset M_{i-1}$, and the pattern of M_i in M_{i-1} can be shown by either Fig. 3 or a vertical flip of Fig. 3. The domain is given by the dashed line, the range is given by the solid line, each vertex in the domain is mapped to the nearest corresponding vertex in the range, and the map is linear on each edge of the domain. Thus Y is $\tilde{H}$-like. □

5.4. Graph-like structure

A *graph* is a one-dimensional finite simplicial complex. The vertices and edges of a graph G will be denoted $V(G)$ and $E(G)$, respectively. A *simplicial map* between two graphs is a map taking each vertex in the domain to a vertex in the range, and taking each edge of the domain to either an edge or a vertex in the range. A simplicial map is *light* if the image of each edge is nondegenerate. For two adjacent vertices v_0 and v_1, let $\langle v_0, v_1\rangle$ denote the edge connecting v_0 to v_1. A simplicial map $\phi : G_1 \to G_0$ between graphs is *ultra light* if each component of the preimage of an edge of G_0 is an edge of G_1.

A graph G' *subdivides* a graph G if $V(G) \subset V(G')$ and for every edge $e \in E(G)$ there exists an arc (e, G') in G' such that

(1) (e, G') has the same endpoints as e,
(2) $(d, G') \cap (e, G') = d \cap e$ for $d, e \in E(G)$ and $d \neq e$, and
(3) every vertex of $V(G')$ belongs to some (e, G') and every edge of $E(G')$ is an edge of some (e, G').

If G' subdivides G then for $v \in V(G), e \in E(G)$, let (v, e, G') denote the edge in (e, G') which contains v as a vertex.

If $\phi : G_1 \to G_0$ is a simplicial map and G'_0 subdivides G_0, define a simplicial map $\phi' : G'_1 \to G'_0$ to be a *subdivision of ϕ matching G'_0* if $\phi'(v) = \phi(v)$ for each $v \in V(G_1)$, and for each edge $e \in E(G_1)$,

(1) if $\phi(e)$ is degenerate then $(e, G'_1) = e$, and
(2) if $\phi(e)$ is an edge of G_0 then ϕ' is an isomorphism of (e, G'_1) onto $(\phi(e), G_0)$.

5.5. A simplicial approach

Piotr Minc defined the following terms in [7] to help determine whether certain graph-like continua are chainable.

For a given graph G, let $D(G)$ denote the graph such that

(1) the set of vertices of $D(G)$ consists of edges of G, and
(2) two vertices of $D(G)$ are adjacent if and only if they intersect as edges of G.

For a map $\phi : G_1 \to G_0$, let $K(e)$ denote the set of components of $\phi^{-1}(e)$ which are mapped by ϕ onto e. Let $K(\phi)$ denote the union of all $K(e)$. Define $D(\phi, G_1)$ as the graph such that

(1) the vertices of $D(\phi, G_1)$ are the elements of $K(\phi)$, and
(2) two vertices of $D(\phi, G_1)$ are adjacent if and only if they intersect as subgraphs of G_1.

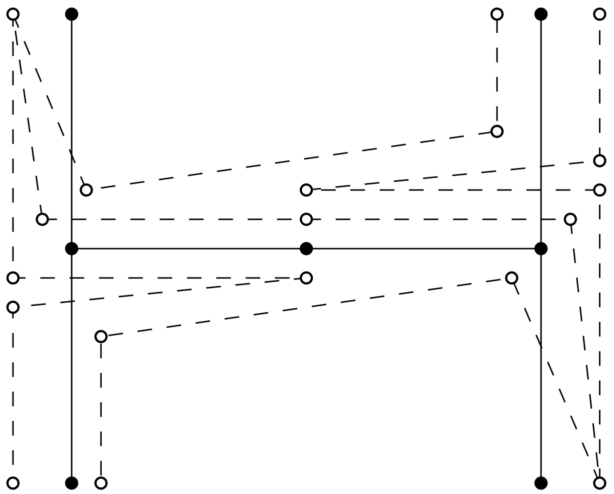

Fig. 3. The simplicial bonding map B.

Let $d[\phi] : D(\phi, G_1) \to D(G_0)$ be the map defined by $d[\phi](v) = \phi(v)$ for each vertex v in $D(\phi, G_1)$. Inductively define $d^n[\phi] = d[d^{n-1}[\phi]]$ for each $n \geqslant 1$. For a vertex v of $D(G)$, let v^* denote the corresponding edge of G.

Let $\phi : G_1 \to G_0$ be simplicial maps between graphs. Define $d[\phi, \psi] : D(\phi \circ \psi, G_2) \to D(\phi, G_1)$ to be the map such that for each vertex v of $D(\phi \circ \psi, G_2)$, we have that $d[\phi, \psi](v)$ is the vertex of $D(\phi, G_1)$ containing $\psi(v^*)$. Define $d^n[\phi, \psi] = d[d^{n-1}[\phi], d^{n-1}[\psi]]$. Note that $d^n[\phi \circ \psi] = d^n[\phi] \circ d^n[\phi, \psi]$.

The following theorems were proven by Minc.

Theorem 9 *(Minc). If $\Psi = \varprojlim\{X_i, f^i_{i-1}\}$, such that each f^i_{i-1} is a simplicial bonding map, then Ψ is chainable if and only if there exists some n such that for every $m > n$, the bonding map f^m_n can be factored through an arc.*

Theorem 10 *(Minc). Every simplicial map ϕ between two graphs can be factored through an arc if and only if $d[\phi]$ can be factored through an arc.*

For a graph G, define an *edge selection S on G* to be a function which maps every vertex v of G to a set of edges of G such that each edge in $S(v)$ contains v. If $\phi : G'_1 \to G_0$ is a simplicial map from a subdivision G'_1 of G_1 into G_0, then ϕ is *consistent on S* if there is a simplicial isomorphism λ from a subdivision H_1 of G_1 onto $D(\phi, G'_1)$ such that

(1) $(v, e, G'_1) \subset [\lambda(v)]^*$ for each $v \in V(G_1)$ and each $e \in S(v)$, and
(2) $[\lambda(v)]^* \subset (e, G_1)'$ for each $e \in E(G_1)$ and $v \in V((e, H_1)) \setminus V(G_1)$.

Let $\psi : G'_2 \to G_1$ be a simplicial map from a subdivision G'_2 of G_2 into G_1, and let S_1, S_2 be edge selections on G_1, G_2. We say that ψ *preserves* (S_1, S_2) if

(1) $\psi((v, e, G'_2)) \in S_1(\psi(v))$ for each $v \in V(G_2)$ and each $e \in S_2(v)$, and
(2) for any two different edges $e, e' \in E(G'_2)$ intersecting at a vertex v we have $\psi(e) \in S_1(\psi(v))$ or $\psi(e') \in S_1(\psi(v))$.

Let N denote the either set $\{0, 1, \ldots, n\}$ or the set of all nonnegative integers. Let N_1 denote $N \setminus \{0\}$. Let $G_0, G_1, G_2, \ldots$ be a sequence of graphs with N as the set of indices. Let Σ be a sequence of simplicial maps $\phi_1, \phi_2, \ldots$ such that for each $j \in N_1$, ϕ_j maps a graph G'_j subdividing G_j into G_{j-1}. Define a sequence of simplicial maps $\psi_1, \psi_2, \ldots$ such that $\psi_1 = \phi_1$ and for each $j \in N_1 \setminus \{1\}$, ψ_j is a subdivision of ϕ_j matching the domain of ψ_{j-1}. For each $j \in N_1$, let Σ_j denote the domain of ψ_j. Let $\Sigma_0 = G_0$. For $i, j \in N$ such that $i > j$, let Σ^i_j denote the composition $\psi_{j+1} \circ \cdots \circ \psi_i$ mapping Σ_i into Σ_j. We will say that the inverse system $\{\Sigma_j, \Sigma^i_j\}$ is *generated by the sequence Σ*.

Let S_j be an edge selection on G_j for $j \in N_1$. We say that Σ preserves the sequence $S_1, S_2, \ldots$ if ϕ_j preserves $S(j-1, S_j)$ for each $j \in N_1 \setminus \{1\}$. We say that two inverse systems $\{K_j, \kappa^i_j\}$ and $\{H_j, \nu^i_j\}$ are *isomorphic* if there is a sequence of isomorphisms $\lambda_0, \lambda_1, \ldots$ where $\lambda_j : K_j \to H_j$ such that $\lambda_j \circ \kappa^i_j = \nu^i_j \circ \lambda_i$ for $i > j \geqslant 0$.

Theorem 11 *(Minc). Let Σ be defined as above. Suppose ϕ_1 is consistent on S_1 and Σ preserves the sequence $S_1, S_2, \ldots$. Let $\lambda_1 : H_1 \to D(\phi_1, G'_1)$ be a consistency isomorphism for ϕ_1, where H_1 is a subdivision of G_1. Then the system $\{D(\Sigma^j_0, \Sigma_j), d[\Sigma^j_0, \Sigma_j]\}$ is isomorphic to the system generated by the sequence $d[\phi_1] \circ \lambda_1, \phi_2, \phi_3, \ldots$.*

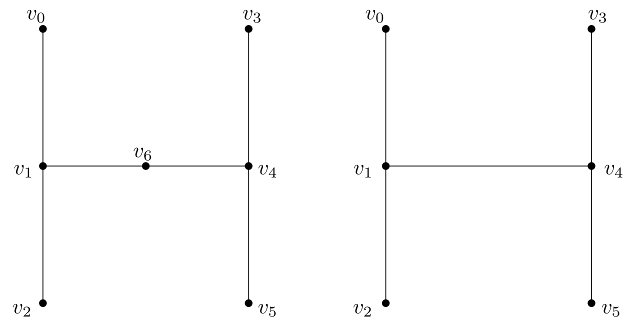

Fig. 4. A (left): A seven-vertex simplicial figure H; B (right): A six-vertex simplicial figure H.

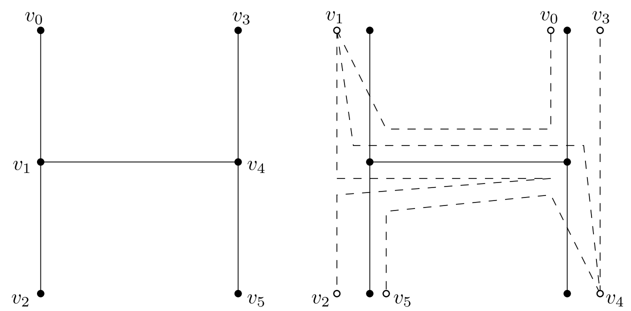

Fig. 5. The bonding map M.

Let H denote the simplicial graph with six vertices which is isomorphic to the letter H, as shown in Fig. 4B, and let H' be the subdivision of H obtained by the addition of a vertex as shown in Fig. 4A. Let $\phi : H' \to H$ be the map taking v_6 and v_4 to v_4, taking each other $v_i \in V(H')$ to the same-labeled $v_i \in V(H)$, such that ϕ is linear on each edge of H'. Then ϕ is a simplicial monotone map. Let $\sigma : H \to H$ be the symmetry satisfying $\sigma(v_0) = v_2, \sigma(v_2) = v_0, \sigma(v_3) = v_5, \sigma(v_5) = v_3$, and for all other $v_i \in V(H)$ we have $\sigma(v_i) = v_i$. Let σ' be the subdivision of σ matching H'. Note that σ and σ' can be visualized by flipping the figure-H vertically.

Let B be as defined in Fig. 3, and let M be as defined in Fig. 5. Let $\tilde{B} = \sigma' \circ B$ and $\tilde{M} = \sigma \circ M$. Then $\phi \circ B = M \circ \phi$ and $\phi \circ \tilde{B} = \tilde{M} \circ \phi$. By the proof of Theorem 8 we have that $Y \cong \varprojlim\{H_i, h_i\}$, where $H_0 = H'$ and for $i = 1, 2, \ldots,$ the map h_i is a subdivision of B or $\tilde{B}$ matching H_{i-1}. Let $P = \varprojlim\{J_i, k_i\}$, where $J_0 = H$ and for $i = 1, 2, \ldots,$ the map k_i is a subdivision of M or $\tilde{M}$ matching J_{i-1}. For each i, define $\sigma_i : H_i \to J_i$ to be the subdivision of σ matching J_i. Let σ_∞ be the induced map from $\varprojlim\{H_i, h_i\}$ to P. Then σ_∞ is monotone, and $P = \sigma_\infty(\varprojlim\{H_i, h_i\})$.

Since every proper nondegenerate subcontinuum of Y is an arc and σ_∞ is monotone, every proper nondegenerate subcontinuum of P is also an arc. So P is atriodic. Note that P is also the inverse limit of graphs with simplicial bonding maps. Then by Minc [6], since P is atriodic and is the inverse limit of graphs with simplicial bonding maps, then P is weakly chainable if and only if P is chainable.

Lemma 3. *If P is as above, then P is not chainable.*

Proof. The following proof mirrors that in Example 5.12 of [7]. Let S be the edge selection on H defined by $S(v_1) = \{\langle v_0, v_4\rangle\}$, $S(v_4) = \{\langle v_1, v_4\rangle, \langle v_4, v_5\rangle\}$, and for each other $v_i \in V(H)$, define $S(v_i)$ to be the set of all edges of H containing v_i. Then M and $\tilde{M}$ are consistent on S and preserve (S, S). Let λ and $\tilde{\lambda}$ be consistency isomorphisms for M and $\tilde{M}$ respectively. Let $M_1 = d[M] \circ \lambda$ and $\tilde{M}_1 = d[\tilde{M}] \circ \tilde{\lambda}$. Then M_1 and $\tilde{M}_1$ are consistent on S as well. Let λ' and $\tilde{\lambda}'$ be consistency isomorphisms for M_1 and $\tilde{M}_1$, respectively. Let $M_2 = d[M_1] \circ \lambda'$ and $\tilde{M}_2 = d[\tilde{M}_1] \circ \tilde{\lambda}'$. The maps M_1 and M_2 can be seen in Figs. 6A and 6B, respectively. Notice that M_2 and $\tilde{M}_2$ are ultra-light.

Let Σ be an infinite sequence of simplicial maps $\phi_1, \phi_2, \ldots,$ each of which is either M or $\tilde{M}$. By $\{\Sigma_j, \Sigma_j^i\}$ we denote the system generated by Σ. We claim that Σ_0^n cannot be factored through an arc. Clearly, this holds for $n = 1$. Now suppose the claim is true for any sequence of $n - 1$ maps each of which is either M or $\tilde{M}$. In particular, assume the claim is true for $\Sigma_2, \ldots, \Sigma_n$.

If $\Sigma_1 = M$ then set $\lambda_1 = \lambda, \psi_1 = M_1$, and $\lambda_1' = \lambda'$. If $\Sigma_1 = \tilde{M}$ then set $\lambda_1 = \tilde{\lambda}, \psi_1 = \tilde{M}_1$, and $\lambda_1' = \tilde{\lambda}'$. Using Theorem 11, we have that the system $\{D(\Sigma_0^j, \Sigma_j), d[\Sigma_0^j, \Sigma_j^i]\}_{j=0}^n$ is isomorphic to the system generated by the sequence $d[\phi_1] \circ \lambda_1, \phi_2, \ldots, \phi_n$. Using Theorem 11 again, we have that the system $\{D^2(\Sigma_0^j, \Sigma_j), d^2[\Sigma_0^j, \Sigma_j^i]\}_{j=0}^n$ is isomorphic to the

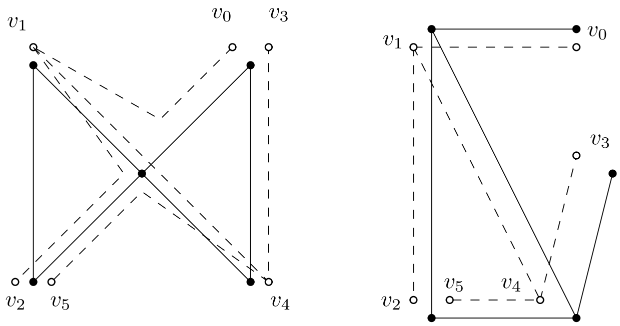

Fig. 6. A (left): The map M_1; B (right): The map M_2.

system generated by the sequence $d[\psi_1] \circ \lambda_1', \phi_2, \ldots, \phi_n$. Let Γ denote the sequence $d[\psi_1] \circ \lambda_1', \phi_2, \ldots, \phi_n$ and let $\{\Gamma_j, \Gamma_j^i\}_{j=0}^n$ denote the system generated by Γ.

Suppose Σ_0^n can be factored through an arc. Then $d^2[\Sigma_0^n]$ and thus Γ_0^n can also be factored through an arc. Since the map $\Gamma_0^1 = d[\psi_1] \circ \lambda_1'$ is either M_2 or $\tilde{M}_2$, it is ultra-light. Then Γ_1^n can be factored through an arc. But the domain of Γ_0^1 is H, so the system $\{\Gamma_j, \Gamma_j^i\}_{j=1}^n$ is generated by $\phi_2, \ldots, \phi_n$, and by our assumption Γ_1^n cannot be factored through an arc, contradicting our assumption. So the inverse limit of the system $\{\Sigma_j, \Sigma_j^i\}$ is not chainable and has positive span. $\square$

Then P is also not weakly chainable, meaning P is not the image of a chainable continuum. But P is the image of $\varprojlim\{H_i, h_i\} \cong Y$, so this implies that Y is not chainable.

Corollary 3. *If $Y = f^{-1}(X)$ as described above, then Y is an indecomposable simple-4-od like continuum which is not arc-like.*

Question. Is Y simple-triod-like? An example of a continuum which is simple-4-od-like but not simple-triod-like is given in [5].

Question. If X is a planar embedding of a chainable continuum with 0 as an endpoint, is $f^{-1}(X)$ at most simple-4-od-like?

Question. Can we characterize the collection of continua with chainable preimages under $z \mapsto z^2$?

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