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Inversion of the wavelet transform using Riemannian sums ☆

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ABSTRACT

We study the approximation of the inverse wavelet transform using Riemannian sums. For a large class of wavelet functions, we show that the Riemannian sums converge to the original function as the sampling density tends to infinity. When the analysis and synthesis wavelets are the same, we also give some necessary conditions for the Riemannian sums to be convergent.

1. Introduction

The *continuous wavelet transform* of a function $f \in L^2(\mathbb{R}^d)$ with respect to $\psi \in L^2(\mathbb{R}^d)$ is defined by

$$(W_\psi f)(a,b) = \langle f, \tau(a,b)\psi\rangle = a^{-d/2}\int_{\mathbb{R}^d} f(x)\overline{\psi\left(\frac{x-b}{a}\right)}\,dx,$$

where

$$\big(\tau(a,b)\psi\big)(x) = a^{-d/2}\psi\big(a^{-1}(x-b)\big), \quad a>0,\ b\in\mathbb{R}^d.$$

The wavelet transform is a useful tool since it provides a time–frequency description for a given signal.

We call a function $\psi \in L^2(\mathbb{R}^d)$ admissible if

$$C_\psi := \int_0^{+\infty} \big|\hat{\psi}(a\omega)\big|^2 \frac{1}{a}\,da$$

is a constant for $\omega \neq 0$ and $C_\psi < +\infty$.

For the case of $d=1$, it is easy to see that the integration in the above equation only depends on $\omega = \pm 1$. For example, if $\int_0^{+\infty} |\hat{\psi}(a)|^2\,da/a = \int_0^{+\infty} |\hat{\psi}(-a)|^2\,da/a < \infty$, then C_ψ is a constant for $\omega \neq 0$ and $C_\psi = \int_0^{+\infty} |\hat{\psi}(a)|^2\,da/a < \infty$. In particular, if ψ is real-valued, then $\hat{\psi}(-a) = \hat{\psi}(a)$ and hence only $\int_0^{+\infty} |\hat{\psi}(a)|^2\,da/a < \infty$ is needed.

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Let $\psi_1, \psi_2 \in L^2(\mathbb{R}^d)$ be admissible such that

$$C_{\psi_1,\psi_2} := \int_0^{+\infty} \overline{\hat{\psi}_1(a\omega)}\hat{\psi}_2(a\omega)\frac{1}{a}\,da \tag{1.1}$$

is a non-zero constant for $\omega \neq 0$. Then we have

$$f(x) = C^{-1}_{\psi_1,\psi_2}\int_0^{+\infty} da \int_{\mathbb{R}^d} \frac{1}{a^{d+1}}(W_{\psi_1} f)(a,b)\big(\tau(a,b)\psi_2\big)(x)\,db, \tag{1.2}$$

where the convergence is in $L^2(\mathbb{R}^d)$.

For simplicity, when we write C_{ψ_1,ψ_2}, we mean that the integration in (1.1) is a constant for $\omega \neq 0$.

For the discrete wavelet transform, if $\{\tau(a_j, b_{j,k})\psi\colon\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d\}$ forms a frame for $L^2(\mathbb{R}^d)$, where $\{(a_j, b_{j,k})\colon\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d\}$ is a sequence of points of $\mathcal{G} := \{(a,b)\colon\ a > 0,\ b \in \mathbb{R}^d\}$, then we can find another frame $\{\widetilde{\psi_{j,k}}\colon\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d\}$, called the dual frame of $\{\tau(a_j, b_{j,k})\psi\colon\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d\}$, such that for any $f \in L^2(\mathbb{R}^d)$,

$$f(x) = \sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \big\langle f, \tau(a_j, b_{j,k})\psi\big\rangle \widetilde{\psi_{j,k}}(x).$$

In general, the dual frame may be not generated by dilations and translations of some function. In fact, although the theory of wavelet frames has been developed very fast in the past twenty years, e.g., see [1–4,9,12,17–19] for an overview, it is not easy to find the dual frame for a given wavelet frame [5–7,15].

Motivated by [10,11,13,14,21,22], where the convergence of Riemannian sums of the inverse windowed Fourier transform was studied, we study the approximation of f with the Riemannian sums of the integration in (1.2), i.e.,

$$S_{p,q} f(t) = S_{\psi_1,\psi_2;p,q} f(t) = \frac{q^d(p^{d/2} - p^{-d/2})}{dC_{\psi_1,\psi_2}} \sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} (W_{\psi_1} f)(a_j, b_{j,k})\big(\tau(a_j, b_{j,k})\psi_2\big)(t), \tag{1.3}$$

where $(a_j, b_{j,k}) \in E_{p,q;j,k}$,

$$E_{p,q;j,k} = \big[p^{j-1/2}, p^{j+1/2}\big) \times p^j q\left(k + \left[-\frac{1}{2}, \frac{1}{2}\right)^d\right), \quad p > 1,\ q > 0.$$

It is easy to see that $|E_{p,q;j,k}| = q^d(p^{d/2} - p^{-d/2})/d$ with respect to the left-invariant Haar measure $(1/s^{d+1})\,ds\,dt$.

We show that for a large class of wavelet functions ψ_1 and ψ_2, the operators $S_{p,q}$ are well-defined on $L^2(\mathbb{R}^d)$ and converge strongly to the identity as (p,q) tends to $(1,0)$. Moreover, the convergence is uniform with respect to sampling points $(a_j, b_{j,k})$.

The paper is organized as follows. In Section 2, we introduce some notations and preliminary results. In Section 3, we first give some necessary and sufficient conditions for the Riemannian sums of the inverse wavelet transform to be well defined. Then we give some necessary conditions for the convergence of the Riemannian sums. In Section 4, we give two convergence theorems for the Riemannian sums to be convergent as the sampling density tends to infinity.

2. Notations and preliminary results

A sequence $\{f_n\colon\ n \in \mathbb{Z}\}$ in a Hilbert space $\mathcal{H}$ is called a frame if there are constants $A, B > 0$ such that

$$A\|f\|^2 \leqslant \sum_{n\in\mathbb{Z}} \big|\langle f, f_n\rangle\big|^2 \leqslant B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

It is called a Bessel sequence if the right-hand side inequality holds.

The group action in $\mathcal{G} := \{(a,b)\colon\ a > 0,\ b \in \mathbb{R}^d\}$ is defined by

$$(a,b)(s,t) = (as, b + at).$$

The unit element is $(1,0)$ and the inverse of (a,b) is $(1/a, -b/a)$.

The Banach space $F_1(\mathbb{R}^d)$ [8] is defined by

$$F_1(\mathbb{R}^d) = \big\{f \in L^2\big(\mathbb{R}^d\big)\colon\ \|f\|_{F_1(\mathbb{R}^d)} = \|W_\varphi f\|_{L^1(\mathcal{G})} + \|W_f \varphi\|_{L^1(\mathcal{G})} < \infty\big\}$$

where

$$\varphi(x) = \big(\partial^2_{x_1} + \cdots + \partial^2_{x_d}\big)^d e^{-\pi x^2}$$

is fixed and

$$L^p(\mathcal{G}) := \left\{ F : \mathcal{G} \to \mathbb{C}\colon \left(\iint_{\mathcal{G}} \left| F(s,t) \right|^p \frac{1}{s^{d+1}} \, ds \, dt \right)^{1/p} < +\infty \right\}, \quad p \geqslant 1.$$

It was shown in [8] that the space $F_1(\mathbb{R}^d)$ contains nice wavelet atoms and is useful in wavelet analysis. To help readers to understand the space $F_1(\mathbb{R}^d)$, we introduce a result in [20] which gives a sufficient condition for a function to belong to $F_1(\mathbb{R}^d)$.

Proposition 2.1. *(See [20, Theorem 4.1].) Let $n_0 = \lfloor d/2 \rfloor + 1$. Suppose that f satisfies the following conditions,*

(i) *$\partial^\alpha f \in L^1(\mathbb{R}^d)$ for any $\alpha \in \mathbb{Z}_+^d$ with $|\alpha| \leqslant n_0$,*
(ii) *$\|x\|_2^\gamma f(x) \in L^1(\mathbb{R}^d)$ for some constant $\gamma > d/2$, and*
(iii) *$\int_{\mathbb{R}^d} x^\alpha f(x) \, dx = 0$ whenever $|\alpha| \leqslant n_0 - 1$,*

then we have $f \in F_1(\mathbb{R}^d)$. Here we use the following set of multi-indices: $\alpha = (\alpha_1, \ldots, \alpha_d)$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$, $\mathbb{Z}_+^d = \{\alpha \in \mathbb{Z}^d\colon \alpha_v \geqslant 0,\ 1 \leqslant v \leqslant d\}$, $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ and $(\partial^\alpha f)(x) = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} f(x)$.

In particular, for $d = 1$, $\hat{f}(0) = 0$, $f, f' \in L^1(\mathbb{R})$ and $|x|^\gamma f(x) \in L^1(\mathbb{R})$ with $\gamma > 1/2$ imply that $f \in F_1(\mathbb{R})$.

One of the important properties of $F_1(\mathbb{R}^d)$ is the following.

Lemma 2.2. *(See [8, Theorem 3.13].) For any $f, g \in F_1(\mathbb{R}^d)$, we have $W_g f \in L^1(\mathcal{G})$. In particular, $W_f f \in L^1(\mathcal{G})$.*

We know that the integrand in (1.2) is not necessarily integrable and the integral is not well defined pointwise. Nevertheless, we can reconstruct f from the continuous wavelet transform in the following way.

Lemma 2.3. *(See [4, Proposition 2.4.1].) Let $\psi_1, \psi_2 \in L^2(\mathbb{R}^d)$ be admissible such that $C_{\psi_1,\psi_2} \neq 0$. Then for any $f \in L^2(\mathbb{R}^d)$,*

$$f(x) = C_{\psi_1,\psi_2}^{-1} \lim_{\substack{A_1 \to 0 \\ A_2, B \to \infty}} \int_{A_1}^{A_2} da \int_{[-B,B]^d} \frac{1}{a^{d+1}} \langle f, \tau(a,b)\psi_1 \rangle \big(\tau(a,b)\psi_2\big)(x) \, db,$$

where the convergence is in $L^2(\mathbb{R}^d)$.

The following is a slightly different version of [8, Lemma 4.7], which can be proved with the same procedure.

Lemma 2.4. *(See [8, Lemma 4.7].) Let $\psi \in F_1(\mathbb{R}^d)$. Then we have:*

(i) *For any $\varepsilon > 0$, there are some $p_0 > 1$ and $q_0 > 0$ such that for any $1 < p < p_0$, $0 < q < q_0$, $(a_j, b_{j,k}) \in E_{p,q;j,k}$ and $(s,t) \in \mathcal{G}$,*

$$\sum_{\substack{j \in \mathbb{Z} \\ k \in \mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \left| (W_\psi \varphi)\left(\frac{a}{s}, \frac{b-t}{s}\right) - (W_\psi \varphi)\left(\frac{a_j}{s}, \frac{b_{j,k}-t}{s}\right) \right| \frac{da\, db}{a^{d+1}} < \varepsilon.$$

(ii) *For any $p > 1$ and $q > 0$, there exists some constant $C_{p,q} < \infty$ such that for any $(a_j, b_{j,k}) \in E_{p,q;j,k}$ and $(s,t) \in \mathcal{G}$,*

$$\sum_{\substack{j \in \mathbb{Z} \\ k \in \mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \left| (W_\psi \varphi)\left(\frac{a}{s}, \frac{b-t}{s}\right) - (W_\psi \varphi)\left(\frac{a_j}{s}, \frac{b_{j,k}-t}{s}\right) \right| \frac{da\, db}{a^{d+1}} \leqslant C_{p,q}.$$

For an admissible wavelet function g, we have

$$\begin{aligned} (W_\psi f)(a,b) &= \frac{1}{C_g} \iint_{\mathcal{G}} \frac{1}{s^{d+1}} (W_g f)(s,t) (W_\psi g)\left(\frac{a}{s}, \frac{b-t}{s}\right) ds\, dt \\ &= \frac{1}{C_g} \iint_{\mathcal{G}} \frac{1}{s^{d+1}} (W_g f)(s,t) \overline{(W_g \psi)\left(\frac{s}{a}, \frac{t-b}{a}\right)} \, ds\, dt. \end{aligned}$$

In particular, by setting $g=\varphi$, we get

$$(W_\psi f)(a,b)=\frac{1}{C_\varphi}\iint_{\mathcal{G}}\frac{1}{s^{d+1}}(W_\varphi f)(s,t)(W_\psi\varphi)\left(\frac{a}{s},\frac{b-t}{s}\right)ds\,dt$$
$$=\frac{1}{C_\varphi}\iint_{\mathcal{G}}\frac{1}{s^{d+1}}(W_\varphi f)(s,t)\overline{(W_\varphi\psi)\left(\frac{s}{a},\frac{t-b}{a}\right)}ds\,dt. \quad (2.1)$$

3. Necessary conditions for the convergence of the Riemannian sums

In this section, we study necessary conditions for the Riemannian sums $S_{p,q}f$ to converge to f as (p,q) tends to $(1,0)$. We consider only the case of $\psi_1=\psi_2=\psi$. In this case, the Riemannian sums become

$$S_{p,q}f(t)=S_{\psi,\psi;p,q}f(t)=\frac{q^d(p^{d/2}-p^{-d/2})}{dC_\psi}\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}}(W_\psi f)(a_j,b_{j,k})\big(\tau(a_j,b_{j,k})\psi\big)(t).$$

First, we give a necessary and sufficient condition for $S_{p,q}f$ to be well-defined when (p,q) is close to $(1,0)$.

Lemma 3.1. *Suppose that $\psi\in L^2(\mathbb{R}^d)$ is admissible. Then the following assertions are equivalent*:

(i) *For any $f\in L^2(\mathbb{R}^d)$, there exist constants $p_f>1$ and $q_f>0$ such that $S_{p,q}f$ is well-defined whenever $1<p<p_f$ and $0<q<q_f$.*
(ii) *$S_{p,q}f$ is well-defined for any $f\in L^2(\mathbb{R}^d)$, $p>1$ and $q>0$.*
(iii) *$\{\tau(a_j,b_{j,k})\psi\colon\ j\in\mathbb{Z},\ k\in\mathbb{Z}^d\}$ is a Bessel sequence in $L^2(\mathbb{R}^d)$ for any $p>1$, $q>0$ and $(a_j,b_{j,k})\in E_{p,q;j,k}$.*

Proof. First, we assume that (i) holds. For any $f\in L^2(\mathbb{R}^d)$, we have

$$\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}}\big|\langle f,\tau(a_j,b_{j,k})\psi\rangle\big|^2=\frac{dC_\psi}{q^d(p^{d/2}-p^{-d/2})}\langle S_{p,q}f,f\rangle<+\infty,\quad \forall 1<p<p_f,\ 0<q<q_f.$$

For any $p>1$ and $q>0$, we can find some positive integer N such that $p^{1/N}<p_f$ and $q/N<q_f$. And we can choose a sequence $\{(a'_j,b'_{j,k})\colon\ j\in\mathbb{Z},\ k\in\mathbb{Z}^d\}\subset\mathcal{G}$ such that $(a'_j,b'_{j,k})\in E_{p^{1/N},q/N;j,k}$ and it contains $\{(a_j,b_{j,k})\colon\ j\in\mathbb{Z},\ k\in\mathbb{Z}^d\}$ as a subsequence. Hence

$$\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}}\big|\langle f,\tau(a_j,b_{j,k})\psi\rangle\big|^2\leqslant\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}}\big|\langle f,\tau\big(a'_j,b'_{j,k}\big)\psi\rangle\big|^2<+\infty,\quad\forall p>1,\ q>0. \quad (3.1)$$

For $N\in\mathbb{N}$, define a bounded operator $B_N: L^2(\mathbb{R}^d)\mapsto\ell^2$ by $[B_Nf]_{j,k}:=\langle f,\tau(a_j,b_{j.k})\psi\rangle$ for $|j|\leqslant N$ and $|k|\leqslant N$; otherwise, $[B_Nf]_{j,k}=0$. Then clearly all B_N are bounded operators satisfying $\|B_Nf\|\leqslant\|B_N\|\cdot\|f\|_2$. Now it follows from (3.1) and Banach–Steinhaus theorem that there exists a positive constant B such that

$$\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}}\big|\langle f,\tau(a_j,b_{j,k})\psi\rangle\big|^2\leqslant B\|f\|_2^2,\quad\forall f\in L^2\big(\mathbb{R}^d\big).$$

Since f is arbitrary, the above inequality implies that $\{\tau(a_j,b_{j,k})\psi\colon\ j\in\mathbb{Z},\ k\in\mathbb{Z}^d\}$ is a Bessel sequence in $L^2(\mathbb{R}^d)$. Hence (iii) holds. On the other hand, it is easy to see that (iii) ⇒ (ii) and (ii) ⇒ (i) are obvious. This completes the proof. □

Next we give a necessary condition for $S_{p,q}f$ to converge to f as (p,q) tends to $(1,0)$.

Theorem 3.2. *Let $\psi\in L^2(\mathbb{R}^d)$ be admissible. Suppose that for any $f\in L^2(\mathbb{R}^d)$, there exist constants $p_f>1$ and $q_f>0$ such that $S_{p,q}f$ is well-defined whenever $1<p<p_f$, $0<q<q_f$ and the limit $\lim_{p\to1^+,q\to0^+}S_{p,q}f$ exists in the $L^2(\mathbb{R}^d)$ sense. Then we have*:

(i) *$S_{p,q}$ is well-defined on $L^2(\mathbb{R}^d)$ for any $1<p\leqslant2$, $0<q\leqslant1$ and there exists some constant $M<+\infty$ such that*

$$\|S_{p,q}\|\leqslant M,\quad\forall 1<p\leqslant2,\ 0<q\leqslant1.$$

(ii) *There is some constant $M < \infty$ such that $\{q^{d/2}(p^{d/2} - p^{-d/2})^{1/2}/(dC_\psi)^{1/2}\ \tau(a_j, b_{j,k})\psi\colon\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d\}$ is a Bessel sequence with upper frame bound M for any $1 < p \leqslant 2$ and $0 < q \leqslant 1$.*

Proof. By Lemma 3.1, $\{q^{d/2}(p^{d/2} - p^{-d/2})^{1/2}/(dC_\psi)^{1/2}\tau(a_j, b_{j,k})\psi\colon\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d\}$ is a Bessel sequence in $L^2(\mathbb{R}^d)$ for any $p > 1, q > 0$. Now we define the bounded operator $T_{p,q} : L^2(\mathbb{R}^d) \to \ell^2$ as

$$T_{p,q} f = \left\{ \frac{q^{d/2}(p^{d/2} - p^{-d/2})^{1/2}}{(dC_\psi)^{1/2}} \langle f, \tau(a_j, b_{j,k})\psi\rangle\colon\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d \right\}.$$

Suppose that $\lim_{p\to 1^+, q\to 0^+} S_{p,q} f = h$ for some $h \in L^2(\mathbb{R}^d)$. Then there are constants $p'_f > 1$ and $q'_f > 0$ such that

$$\|S_{p,q} f - h\|_2 \leqslant 1, \quad \forall 1 < p < p'_f,\ 0 < q < q'_f.$$

Hence

$$\|S_{p,q} f\|_2 \leqslant 1 + \|h\|_2, \quad \forall 1 < p < p'_f,\ 0 < q < q'_f.$$

Therefore,

$$\|T_{p,q} f\|_2^2 = \langle S_{p,q} f, f\rangle \leqslant \big(1 + \|h\|_2\big)\|f\|_2, \quad \forall 1 < p < p'_f,\ 0 < q < q'_f. \tag{3.2}$$

On the other hand, let $N = \max\{\lceil \ln 2/\ln p'_f\rceil, \lceil 1/q'_f\rceil\}$, where we use the symbol $\lceil x\rceil$ to denote the minimal integer which is greater than or equal to x. Then for any $1 < p \leqslant 2$ and $0 < q \leqslant 1$, we have $p^{1/N} \leqslant 2^{1/N} \leqslant p'_f$ and $q/N \leqslant 1/N \leqslant q'_f$. It follows that

$$\begin{aligned}
\|T_{p,q} f\|_2^2 &= \frac{q^d(p^{d/2} - p^{-d/2})}{dC_\psi} \sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \big|\langle f, \tau(a_j, b_{j,k})\psi\rangle\big|^2 \\
&\leqslant \frac{N^d(p^{d/2} - p^{-d/2})}{(p^{d/(2N)} - p^{-d/(2N)})} \cdot \frac{(q/N)^d(p^{d/(2N)} - p^{-d/(2N)})}{dC_\psi} \sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \big|\langle f, \tau\big(a'_j, b'_{j,k}\big)\psi\rangle\big|^2 \\
&\leqslant C_N N^d \langle S_{p^{1/N}, q/N} f, f\rangle \\
&\leqslant C_N N^d\big(1 + \|h\|_2\big)\|f\|_2, \quad \forall 1 < p \leqslant 2,\ 0 < q \leqslant 1.
\end{aligned}$$

Here $\{(a'_j, b'_{j,k})\colon\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d\}$ is a sequence which contains $\{(a_j, b_{j,k})\colon\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d\}$ as a subsequence and $(a'_j, b'_{j,k}) \in E_{p^{1/N}, q/N; j,k}$.

Hence

$$\sup_{\substack{1<p\leqslant 2\\ 0<q\leqslant 1}} \|T_{p,q} f\|_2^2 < +\infty, \quad \forall f \in L^2\big(\mathbb{R}^d\big).$$

By the Banach–Steinhaus theorem [16], we have

$$M := \sup_{\substack{1<p\leqslant 2\\ 0<q\leqslant 1}} \|T_{p,q}\|^2 < +\infty.$$

Hence $\{q^{d/2}(p^{d/2} - p^{-d/2})^{1/2}/(dC_\psi)^{1/2}\tau(a_j, b_{j,k})\psi\colon\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d\}$ is a Bessel sequence with upper frame bound M. Thus (ii) holds.

Since

$$\langle S_{p,q} f, f\rangle = \sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \frac{q^d(p^{d/2} - p^{-d/2})}{dC_\psi} \big|\langle f, \tau(a_j, b_{j,k})\psi\rangle\big|^2,$$

the equivalence of (i) and (ii) is obvious. This completes the proof. □

4. Sufficient conditions for the convergence of Riemannian sums

In this section, we give some sufficient conditions for the Riemannian sums with irregular sampling points to be convergent as the sampling density tends to infinity.

For $M, N \geqslant 1$, define

$$D_{M,N} = \left[2^{-M}, 2^M\right] \times \left[-2^M N, 2^M N\right]^d,$$
$$\Lambda_{M,N} = \left\{(j,k)\colon\ D_{M,N} \cap E_{p,q;j,k} \neq \emptyset\right\}.$$

First we give some conditions for the Riemannian sums to be well defined.

Lemma 4.1. *Suppose that one of the following conditions is satisfied,*

(i) $f \in L^2(\mathbb{R}^d)$, $\psi_1, \psi_2 \in F_1(\mathbb{R}^d)$ *and* $C_{\psi_1,\psi_2} \neq 0$,
(ii) $f, \psi_1 \in F_1(\mathbb{R}^d)$, $\psi_2 \in L^2(\mathbb{R}^d)$ *and* $C_{\psi_1,\psi_2} \neq 0$.

Then the series in (1.3) *converges unconditionally in* $L^2(\mathbb{R}^d)$ *for any* $p > 1$ *and* $q > 0$.

Proof. We only need to prove that for any $\varepsilon > 0$, there exist some $M, N > 0$ such that for any $M_2 > M_1 \geqslant M$ and $N_2 > N_1 \geqslant N$,

$$\left\| \frac{q^d(p^{d/2} - p^{-d/2})}{dC_{\psi_1,\psi_2}} \sum_{\substack{(j,k)\notin\Lambda_{M_1,N_1}\\(j,k)\in\Lambda_{M_2,N_2}}} \pm(W_{\psi_1} f)(a_j, b_{j,k})\tau(a_j, b_{j,k})\psi_2 \right\|_2 \leqslant \varepsilon,$$

where $\pm$'s are chosen arbitrarily.

First we assume that (i) holds. By Hölder's inequality, we have

$$\begin{aligned}
&\left\| \frac{q^d(p^{d/2} - p^{-d/2})}{dC_{\psi_1,\psi_2}} \sum_{\substack{(j,k)\notin\Lambda_{M_1,N_1}\\(j,k)\in\Lambda_{M_2,N_2}}} \pm(W_{\psi_1} f)(a_j, b_{j,k})\tau(a_j, b_{j,k})\psi_2 \right\|_2 \\
&\quad = \sup_{\|g\|_2=1} \left| \frac{q^d(p^{d/2} - p^{-d/2})}{dC_{\psi_1,\psi_2}} \sum_{\substack{(j,k)\notin\Lambda_{M_1,N_1}\\(j,k)\in\Lambda_{M_2,N_2}}} \pm(W_{\psi_1} f)(a_j, b_{j,k})\overline{(W_{\psi_2} g)(a_j, b_{j,k})} \right| \\
&\quad \leqslant \left|C^{-1}_{\psi_1,\psi_2}\right| \left(\sum_{(j,k)\notin\Lambda_{M,N}} \frac{q^d(p^{d/2} - p^{-d/2})}{d} \left|(W_{\psi_1} f)(a_j, b_{j,k})\right|^2 \right)^{1/2} \\
&\qquad \times \sup_{\|g\|_2=1} \left(\sum_{(j,k)\notin\Lambda_{M,N}} \frac{q^d(p^{d/2} - p^{-d/2})}{d} \left|(W_{\psi_2} g)(a_j, b_{j,k})\right|^2 \right)^{1/2}.
\end{aligned} \tag{4.1}$$

Observe that

$$\begin{aligned}
&\sum_{\substack{j\in\mathbb{Z}\\k\in\mathbb{Z}^d}} \frac{q^d(p^{d/2} - p^{-d/2})}{d} \left|(W_{\psi_1} f)(a_j, b_{j,k})\right|^2 \\
&\quad = \sum_{\substack{j\in\mathbb{Z}\\k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \left|(W_{\psi_1} f)(a_j, b_{j,k})\right|^2 \frac{da\,db}{a^{d+1}} \\
&\quad = \sum_{\substack{j\in\mathbb{Z}\\k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \left|(W_{\psi_1} f)(a_j, b_{j,k}) - (W_{\psi_1} f)(a, b) + (W_{\psi_1} f)(a, b)\right|^2 \frac{da\,db}{a^{d+1}} \\
&\quad \leqslant 2 \sum_{\substack{j\in\mathbb{Z}\\k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \left|(W_{\psi_1} f)(a_j, b_{j,k}) - (W_{\psi_1} f)(a, b)\right|^2 \frac{da\,db}{a^{d+1}} + 2C_{\psi_1}\|f\|_2^2.
\end{aligned} \tag{4.2}$$

By (2.1) and Hölder's inequality, we have

$$\begin{aligned}
&\left|(W_{\psi_1} f)(a, b) - (W_{\psi_1} f)(a_j, b_{j,k})\right|^2 \\
&\quad = \frac{1}{C_\varphi^2} \left| \iint_{\mathcal{G}} (W_\varphi f)(s,t) \left((W_{\psi_1}\varphi)\left(\frac{a}{s}, \frac{b-t}{s}\right) - (W_{\psi_1}\varphi)\left(\frac{a_j}{s}, \frac{b_{j,k}-t}{s}\right) \right) \frac{ds\,dt}{s^{d+1}} \right|^2
\end{aligned}$$

$$\leqslant \frac{1}{C_\varphi^2}\iint_{\mathcal{G}} \big|(W_\varphi f)(s,t)\big|^2 \left|(W_{\psi_1}\varphi)\left(\frac{a}{s},\frac{b-t}{s}\right)-(W_{\psi_1}\varphi)\left(\frac{a_j}{s},\frac{b_{j,k}-t}{s}\right)\right| \frac{ds\,dt}{s^{d+1}}$$
$$\times \iint_{\mathcal{G}} \left|(W_{\psi_1}\varphi)\left(\frac{a}{s},\frac{b-t}{s}\right)-(W_{\psi_1}\varphi)\left(\frac{a_j}{s},\frac{b_{j,k}-t}{s}\right)\right| \frac{ds\,dt}{s^{d+1}}$$
$$= \frac{1}{C_\varphi^2}\iint_{\mathcal{G}} \big|(W_\varphi f)(s,t)\big|^2 \left|(W_{\psi_1}\varphi)\left(\frac{a}{s},\frac{b-t}{s}\right)-(W_{\psi_1}\varphi)\left(\frac{a_j}{s},\frac{b_{j,k}-t}{s}\right)\right| \frac{ds\,dt}{s^{d+1}}$$
$$\times \iint_{\mathcal{G}} \left|(W_\varphi\psi_1)\left(\frac{s}{a},\frac{t-b}{a}\right)-(W_\varphi\psi_1)\left(\frac{s}{a_j},\frac{t-b_{j,k}}{a_j}\right)\right| \frac{ds\,dt}{s^{d+1}}$$
$$\leqslant \frac{2}{C_\varphi^2}\|W_\varphi\psi_1\|_{L^1(\mathcal{G})}\iint_{\mathcal{G}} \big|(W_\varphi f)(s,t)\big|^2 \left|(W_{\psi_1}\varphi)\left(\frac{a}{s},\frac{b-t}{s}\right)-(W_{\psi_1}\varphi)\left(\frac{a_j}{s},\frac{b_{j,k}-t}{s}\right)\right| \frac{ds\,dt}{s^{d+1}},$$

where we use Lemma 2.2 in the last step. Hence

$$\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \big|(W_{\psi_1} f)(a,b)-(W_{\psi_1} f)(a_j,b_{j,k})\big|^2 \frac{da\,db}{a^{d+1}}$$
$$\leqslant \frac{2}{C_\varphi^2}\|W_\varphi\psi_1\|_{L^1(\mathcal{G})} \sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \iint_{\mathcal{G}} \big|(W_\varphi f)(s,t)\big|^2$$
$$\times \left|(W_{\psi_1}\varphi)\left(\frac{a}{s},\frac{b-t}{s}\right)-(W_{\psi_1}\varphi)\left(\frac{a_j}{s},\frac{b_{j,k}-t}{s}\right)\right| \frac{ds\,dt}{s^{d+1}}\,\frac{da\,db}{a^{d+1}}$$
$$= \frac{2}{C_\varphi^2}\|W_\varphi\psi_1\|_{L^1(\mathcal{G})} \iint_{\mathcal{G}} \big|(W_\varphi f)(s,t)\big|^2 \frac{ds\,dt}{s^{d+1}}$$
$$\times \sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \left|(W_{\psi_1}\varphi)\left(\frac{a}{s},\frac{b-t}{s}\right)-(W_{\psi_1}\varphi)\left(\frac{a_j}{s},\frac{b_{j,k}-t}{s}\right)\right| \frac{da\,db}{a^{d+1}}. \tag{4.3}$$

By Lemma 2.4, for any $p>1$ and $q>0$, there exists some constant $C_1=C_1(p,q,\psi_1,\varphi)<+\infty$ such that for any $(s,t),(a_j,b_{j,k})\in\mathcal{G}$,

$$\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \left|(W_{\psi_1}\varphi)\left(\frac{a}{s},\frac{b-t}{s}\right)-(W_{\psi_1}\varphi)\left(\frac{a_j}{s},\frac{b_{j,k}-t}{s}\right)\right| \frac{da\,db}{a^{d+1}} \leqslant C_1.$$

Hence

$$\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \big|(W_{\psi_1} f)(a,b)-(W_{\psi_1} f)(a_j,b_{j,k})\big|^2 \frac{da\,db}{a^{d+1}} \leqslant \frac{2C_1}{C_\varphi}\|W_\varphi\psi_1\|_{L^1(\mathcal{G})}\|f\|_2^2.$$

By (4.2), we have

$$\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \frac{q^d(p^{d/2}-p^{-d/2})}{d}\big|(W_{\psi_1} f)(a_j,b_{j,k})\big|^2 \leqslant \frac{4C_1}{C_\varphi}\|W_\varphi\psi_1\|_{L^1(\mathcal{G})}\|f\|_2^2+2C_{\psi_1}\|f\|_2^2. \tag{4.4}$$

Similarly we can prove that there exists some constant $C_2=C_2(p,q,\psi_2,\varphi)<\infty$ such that

$$\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \frac{q^d(p^{d/2}-p^{-d/2})}{d}\big|(W_{\psi_2} g)(a_j,b_{j,k})\big|^2 \leqslant \frac{4C_2}{C_\varphi}\|W_\varphi\psi_2\|_{L^1(\mathcal{G})}\|g\|_2^2+2C_{\psi_2}\|g\|_2^2. \tag{4.5}$$

By (4.4), we can choose M and N large enough such that

$$\sum_{(j,k)\notin\Lambda_{M,N}} \frac{q^d(p^{d/2}-p^{-d/2})}{d}\big|(W_{\psi_1} f)(a_j,b_{j,k})\big|^2 \leqslant \varepsilon. \tag{4.6}$$

Now we see from (4.1), (4.5) and (4.6) that the series in (1.3) is convergent unconditionally in $L^2(\mathbb{R}^d)$.

Next we assume that (ii) holds. As in the first case, we assume that $M_2 > M_1 \geqslant M$ and $N_2 > N_1 \geqslant N$. For any $p > 1$ and $q > 0$, we have

$$\begin{aligned}
&\left\| \frac{q^d(p^{d/2} - p^{-d/2})}{dC_{\psi_1,\psi_2}} \sum_{\substack{(j,k)\notin \Lambda_{M_1,N_1} \\ (j,k)\in \Lambda_{M_2,N_2}}} \pm (W_{\psi_1} f)(a_j, b_{j,k})\tau(a_j, b_{j,k})\psi_2 \right\|_2 \\
&\quad \leqslant \sum_{(j,k)\notin \Lambda_{M,N}} \frac{q^d(p^{d/2} - p^{-d/2})}{d|C_{\psi_1,\psi_2}|} \left\| \tau(a_j, b_{j,k})\psi_2 \right\|_2 \left| (W_{\psi_1} f)(a_j, b_{j,k}) \right| \\
&\quad = \frac{\|\psi_2\|_2}{|C_{\psi_1,\psi_2}|} \sum_{(j,k)\notin \Lambda_{M,N}} \iint_{E_{p,q;j,k}} \left| (W_{\psi_1} f)(a_j, b_{j,k}) \right| \frac{da\,db}{a^{d+1}}.
\end{aligned} \tag{4.7}$$

Using the triangle inequality, we have

$$\begin{aligned}
&\sum_{\substack{j\in\mathbb{Z} \\ k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \left| (W_{\psi_1} f)(a_j, b_{j,k}) \right| \frac{da\,db}{a^{d+1}} \\
&\quad \leqslant \sum_{\substack{j\in\mathbb{Z} \\ k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \left| (W_{\psi_1} f)(a_j, b_{j,k}) - (W_{\psi_1} f)(a, b) \right| \frac{da\,db}{a^{d+1}} + \iint_{\mathcal{G}} \left| (W_{\psi_1} f)(a, b) \right| \frac{da\,db}{a^{d+1}}.
\end{aligned}$$

Since $f, \psi_1 \in F_1(\mathbb{R}^d)$, by Lemma 2.2, we have $W_{\psi_1} f \in L^1(\mathcal{G})$. Using the same technique as that for the case (i), we obtain

$$\begin{aligned}
&\sum_{\substack{j\in\mathbb{Z} \\ k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \left| (W_{\psi_1} f)(a_j, b_{j,k}) - (W_{\psi_1} f)(a, b) \right| \frac{da\,db}{a^{d+1}} \\
&\quad \leqslant \frac{1}{C_\varphi} \iint_{\mathcal{G}} \frac{1}{s^{d+1}} \left| (W_\varphi f)(s, t) \right| \sum_{\substack{j\in\mathbb{Z} \\ k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \left| (W_{\psi_1}\varphi)\left(\frac{a}{s}, \frac{b-t}{s}\right) - (W_{\psi_1}\varphi)\left(\frac{a_j}{s}, \frac{b_{j,k}-t}{s}\right) \right| \frac{da\,db}{a^{d+1}}\, ds\, dt \\
&\quad < +\infty.
\end{aligned}$$

Thus

$$\sum_{\substack{j\in\mathbb{Z} \\ k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \left| (W_{\psi_1} f)(a_j, b_{j,k}) \right| \frac{da\,db}{a^{d+1}} < +\infty.$$

Hence we can make the last term in (4.7) arbitrarily small by choosing M and N large enough. Therefore, the series in (1.3) is unconditionally convergent in $L^2(\mathbb{R}^d)$. This completes the proof. □

We are now ready to give the main result.

Theorem 4.2. *Let $\psi_1, \psi_2 \in F_1(\mathbb{R}^d)$ be admissible such that $C_{\psi_1,\psi_2} \neq 0$. Then for any $f \in L^2(\mathbb{R}^d)$, $S_{p,q} f$ in* (1.3) *is well-defined and we have*

$$\lim_{\substack{p\to 1^+ \\ q\to 0^+}} \|S_{p,q} - I\| = 0.$$

Remark 4.1. The convergence is uniform in the following sense. Given $\varepsilon > 0$, we can find some $p_0 > 1$ and $q_0 > 0$ such that for any $1 < p < p_0$, $0 < q < q_0$, and any $(a_j, b_{j,k}) \in E_{p,q;j,k}$,

$$\|S_{p,q} - I\| < \varepsilon.$$

Proof. For any $f \in L^2(\mathbb{R}^d)$, we have

$$\begin{aligned}
\|f - S_{p,q} f\|_2 &= \sup_{\|g\|_2=1} \left| \langle f - S_{p,q} f, g\rangle \right| \\
&= \left| C_{\psi_1,\psi_2}^{-1} \right| \sup_{\|g\|_2=1} \left| \sum_{\substack{j\in\mathbb{Z} \\ k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \Big((W_{\psi_1} f)(a, b)\overline{(W_{\psi_2} g)(a, b)} \right.
\end{aligned}$$

$$- (W_{\psi_1} f)(a_j, b_{j,k})\overline{(W_{\psi_2} g)(a_j, b_{j,k})}\Big) \frac{da\,db}{a^{d+1}}\bigg|$$

$$\leqslant \big|C_{\psi_1,\psi_2}^{-1}\big| \sup_{\|g\|_2=1} \bigg| \sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \big((W_{\psi_1} f)(a,b) - (W_{\psi_1} f)(a_j, b_{j,k})\big)\overline{(W_{\psi_2} g)(a,b)} \frac{da\,db}{a^{d+1}} \bigg|$$

$$+ \big|C_{\psi_1,\psi_2}^{-1}\big| \sup_{\|g\|_2=1} \bigg| \sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} (W_{\psi_1} f)(a_j, b_{j,k})\big(\overline{(W_{\psi_2} g)(a,b)} - \overline{(W_{\psi_2} g)(a_j, b_{j,k})}\big) \frac{da\,db}{a^{d+1}} \bigg|$$

$$= \big|C_{\psi_1,\psi_2}^{-1}\big|(I + II).$$

First, we estimate I. We have

$$I \leqslant \sup_{\|g\|_2=1} \sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \big|\big((W_{\psi_1} f)(a,b) - (W_{\psi_1} f)(a_j, b_{j,k})\big)\overline{(W_{\psi_2} g)(a,b)}\big| \frac{da\,db}{a^{d+1}}$$

$$\leqslant \sup_{\|g\|_2=1} \bigg(\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \big|(W_{\psi_1} f)(a,b) - (W_{\psi_1} f)(a_j, b_{j,k})\big|^2 \frac{da\,db}{a^{d+1}} \bigg)^{1/2} \bigg(\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \big|(W_{\psi_2} g)(a,b)\big|^2 \frac{da\,db}{a^{d+1}} \bigg)^{1/2}$$

$$= C_{\psi_2}^{1/2} \bigg(\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \big|(W_{\psi_1} f)(a,b) - (W_{\psi_1} f)(a_j, b_{j,k})\big|^2 \frac{da\,db}{a^{d+1}} \bigg)^{1/2}.$$

By Lemma 2.4, for any $\varepsilon > 0$, there exist some $p_1 > 1$ and $q_1 > 0$, depending only on ψ_1 and ε, such that for any $1 < p < p_1$, $0 < q < q_1$, $(s,t) \in \mathcal{G}$ and $(a_j, b_{j,k}) \in E_{p,q;j,k}$,

$$\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \left| (W_{\psi_1}\varphi)\left(\frac{a}{s}, \frac{b-t}{s}\right) - (W_{\psi_1}\varphi)\left(\frac{a_j}{s}, \frac{b_{j,k}-t}{s}\right) \right| \frac{da\,db}{a^{d+1}} < \varepsilon. \tag{4.8}$$

By (4.3), we have

$$\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \big|(W_{\psi_1} f)(a,b) - (W_{\psi_1} f)(a_j, b_{j,k})\big|^2 \frac{da\,db}{a^{d+1}} \leqslant \frac{2\varepsilon}{C_\varphi} \|f\|_2^2 \|W_\varphi \psi_1\|_{L^1(\mathcal{G})}. \tag{4.9}$$

It follows that

$$I \leqslant \frac{\varepsilon^{1/2}}{C_\varphi^{1/2}} \cdot 2^{1/2} C_{\psi_2}^{1/2} \|f\|_2 \|W_\varphi \psi_1\|_{L^1(\mathcal{G})}^{1/2}, \quad 1 < p < p_1,\ 0 < q < q_1. \tag{4.10}$$

Next we estimate II. We have

$$II \leqslant \sup_{\|g\|_2=1} \sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \big|(W_{\psi_1} f)(a_j, b_{j,k})\big((W_{\psi_2} g)(a,b) - (W_{\psi_2} g)(a_j, b_{j,k})\big)\big| \frac{da\,db}{a^{d+1}}$$

$$\leqslant \bigg(\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \big|(W_{\psi_1} f)(a_j, b_{j,k})\big|^2 \frac{da\,db}{a^{d+1}} \bigg)^{1/2}$$

$$\times \sup_{\|g\|_2=1} \bigg(\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \big|(W_{\psi_2} g)(a,b) - (W_{\psi_2} g)(a_j, b_{j,k})\big|^2 \frac{da\,db}{a^{d+1}} \bigg)^{1/2}.$$

By (4.2) and (4.9), we have

$$\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \big|(W_{\psi_1} f)(a_j, b_{j,k})\big|^2 \frac{da\,db}{a^{d+1}} \leqslant \left(\frac{4\varepsilon}{C_\varphi} \|W_\varphi \psi_1\|_{L^1(\mathcal{G})} + 2C_{\psi_1} \right) \|f\|_2^2, \quad 1 < p < p_1,\ 0 < q < q_1. \tag{4.11}$$

By a similar argument as in the proof of (4.9), we can prove that, there exist some $p_2 > 1$ and $q_2 > 0$, depending only on ψ_2 and ε, such that for any $1 < p < p_2$, $0 < q < q_2$ and $(a_j, b_{j,k}) \in E_{p,q;j,k}$,

$$\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \left|(W_{\psi_2}g)(a,b) - (W_{\psi_2}g)(a_j, b_{j,k})\right|^2 \frac{da\,db}{a^{d+1}} \leqslant \frac{2\varepsilon}{C_\varphi}\|g\|_2^2\|W_\varphi\psi_2\|_{L^1(\mathcal{G})}.$$

Hence

$$II \leqslant \frac{\varepsilon^{1/2}}{C_\varphi^{1/2}}\|W_\varphi\psi_2\|_{L^1(\mathcal{G})}^{1/2}\left(\frac{8}{C_\varphi}\|W_\varphi\psi_1\|_{L^1(\mathcal{G})} + 4C_{\psi_1}\right)^{1/2}\|f\|_2. \tag{4.12}$$

Putting (4.10) and (4.12) together, we see that for any $\varepsilon > 0$, there exist $p_0 = \min\{p_1, p_2\} > 1$ and $q_0 = \min\{q_1, q_2\} > 0$ such that for any $1 < p < p_0$, $0 < q < q_0$ and $(a_j, b_{j,k}) \in E_{p,q;j,k}$, we have

$$\|f - S_{p,q}f\|_2 \leqslant 2\varepsilon\|f\|_2, \quad \forall f \in L^2(\mathbb{R}^d).$$

This completes the proof. □

Next we show that whenever f is in $F_1(\mathbb{R}^d)$, we can choose ψ_2 more freely.

Lemma 4.3. *For any $\psi \in L^2(\mathbb{R}^d)$, $(a,b) \mapsto \tau(a,b)\psi$ is a uniformly continuous mapping from $\mathcal{G}$ to $L^2(\mathbb{R}^d)$.*

Proof. We only need to prove that for any $\varepsilon > 0$, there exist $p > 1$ and $q > 0$ such that for any $(a_1, b_1), (a_2, b_2) \in \mathcal{G}$ satisfying $p^{-1/2} < a_2/a_1 < p^{1/2}$ and $\|(b_2 - b_1)/a_1\|_\infty < q/2$,

$$\left\|\tau(a_2, b_2)\psi - \tau(a_1, b_1)\psi\right\|_2 \leqslant \varepsilon.$$

In fact, we have

$$\begin{aligned}
&\left\|\tau(a_2, b_2)\psi - \tau(a_1, b_1)\psi\right\|_2 \\
&\quad\leqslant \left\|\tau(a_2, b_2)\psi - \tau(a_1, b_2)\psi\right\|_2 + \left\|\tau(a_1, b_2)\psi - \tau(a_1, b_1)\psi\right\|_2 \\
&\quad= \left\|\psi - \tau(a_1/a_2, 0)\psi\right\|_2 + \left\|\psi - \tau\big(1, (b_1 - b_2)/a_1\big)\psi\right\|_2 \\
&\quad= \left(\int_{\mathbb{R}^d}\left|\psi(x) - \left(\frac{a_1}{a_2}\right)^{-d/2}\psi\left(\frac{a_2}{a_1}x\right)\right|^2 dx\right)^{1/2} + \left(\int_{\mathbb{R}^d}\left|\psi(x) - \psi\left(x - \frac{b_1 - b_2}{a_1}\right)\right|^2 dx\right)^{1/2} \\
&\quad\leqslant \left(\int_{\mathbb{R}^d}\left|\psi(x) - \left(\frac{a_1}{a_2}\right)^{-d/2}\psi(x)\right|^2 dx\right)^{1/2} + \left(\int_{\mathbb{R}^d}\left|\left(\frac{a_1}{a_2}\right)^{-d/2}\psi(x) - \left(\frac{a_1}{a_2}\right)^{-d/2}\psi\left(\frac{a_2}{a_1}x\right)\right|^2 dx\right)^{1/2} \\
&\qquad+ \left(\int_{\mathbb{R}^d}\left|\psi(x) - \psi\left(x - \frac{b_1 - b_2}{a_1}\right)\right|^2 dx\right)^{1/2} \\
&\quad= \left|1 - \left(\frac{a_2}{a_1}\right)^{d/2}\right|\|\psi\|_2 + \left(\frac{a_2}{a_1}\right)^{d/2}\left(\int_{\mathbb{R}^d}\left|\psi(x) - \psi\left(\frac{a_2}{a_1}x\right)\right|^2 dx\right)^{1/2} + \left(\int_{\mathbb{R}^d}\left|\psi(x) - \psi\left(x - \frac{b_1 - b_2}{a_1}\right)\right|^2 dx\right)^{1/2}.
\end{aligned}$$

We can make

$$\left(\int_{\mathbb{R}^d}\left|\psi(x) - \psi\left(\frac{a_2}{a_1}x\right)\right|^2 dx\right)^{1/2} \quad\text{and}\quad \left(\int_{\mathbb{R}^d}\left|\psi(x) - \psi\left(x - \frac{b_1 - b_2}{a_1}\right)\right|^2 dx\right)^{1/2}$$

arbitrarily small by choosing (p, q) sufficiently close to $(1, 0)$ whenever ψ is in $C_c(\mathbb{R}^d)$, the space of functions which are continuous and compactly supported. Since $C_c(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, we conclude the proof. □

Theorem 4.4. *Let $\psi_1 \in F_1(\mathbb{R}^d)$ and $\psi_2 \in L^2(\mathbb{R}^d)$ be admissible such that $C_{\psi_1,\psi_2} \neq 0$. Then for any $f \in F_1(\mathbb{R}^d)$,*

$$\lim_{\substack{p\to1^+\\ q\to0^+}} S_{p,q}f = f,$$

where the convergence is in $L^2(\mathbb{R}^d)$.

Proof. For any $f \in F_1(\mathbb{R}^d)$, similarly to the proof of Theorem 4.2, we have

$$\begin{aligned}\|f - S_{p,q} f\|_2 &\leqslant \left|C_{\psi_1,\psi_2}^{-1}\right| \sup_{\|g\|_2=1} \left| \sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \big((W_{\psi_1} f)(a,b) - (W_{\psi_1} f)(a_j, b_{j,k})\big)\overline{(W_{\psi_2} g)(a,b)}\, \frac{da\,db}{a^{d+1}} \right| \\ &\quad + \left|C_{\psi_1,\psi_2}^{-1}\right| \sup_{\|g\|_2=1} \left| \sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} (W_{\psi_1} f)(a_j, b_{j,k})\big(\overline{(W_{\psi_2} g)(a,b)} - \overline{(W_{\psi_2} g)(a_j, b_{j,k})}\big)\, \frac{da\,db}{a^{d+1}} \right| \\ &= \left|C_{\psi_1,\psi_2}^{-1}\right| (I + II). \end{aligned} \tag{4.13}$$

First, we estimate I. Fix some $\varepsilon > 0$. By Lemma 2.4, there exist some $p_1 > 1$ and $q_1 > 0$, depending only on ψ_1 and ε, such that for any $1 < p < p_1$, $0 < q < q_1$, $(s,t) \in \mathcal{G}$ and $(a_j, b_{j,k}) \in E_{p,q;j,k}$,

$$\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \left| (W_{\psi_1}\varphi)\left(\frac{a}{s}, \frac{b-t}{s}\right) - (W_{\psi_1}\varphi)\left(\frac{a_j}{s}, \frac{b_{j,k}-t}{s}\right) \right| \frac{da\,db}{a^{d+1}} \leqslant \varepsilon.$$

By (2.1), we have

$$\begin{aligned} &\sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \left|(W_{\psi_1} f)(a,b) - (W_{\psi_1} f)(a_j, b_{j,k})\right| \frac{da\,db}{a^{d+1}} \\ &\quad = \sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \frac{1}{C_\varphi} \left| \iint_{\mathcal{G}} (W_\varphi f)(s,t) \left((W_{\psi_1}\varphi)\left(\frac{a}{s}, \frac{b-t}{s}\right) - (W_{\psi_1}\varphi)\left(\frac{a_j}{s}, \frac{b_{j,k}-t}{s}\right)\right) \frac{ds\,dt}{s^{d+1}} \right| \frac{da\,db}{a^{d+1}} \\ &\quad \leqslant \frac{1}{C_\varphi} \iint_{\mathcal{G}} \left|(W_\varphi f)(s,t)\right| \frac{ds\,dt}{s^{d+1}} \sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \left| (W_{\psi_1}\varphi)\left(\frac{a}{s}, \frac{b-t}{s}\right) - (W_{\psi_1}\varphi)\left(\frac{a_j}{s}, \frac{b_{j,k}-t}{s}\right)\right| \frac{da\,db}{a^{d+1}} \\ &\quad \leqslant \frac{\varepsilon}{C_\varphi} \|W_\varphi f\|_{L^1(\mathcal{G})}. \end{aligned} \tag{4.14}$$

Since $|(W_{\psi_2} g)(a,b)| \leqslant \|\psi_2\|_2 \|g\|_2$, we see from (4.13) and (4.14) that

$$I \leqslant \frac{\varepsilon}{C_\varphi} \|\psi_2\|_2 \|W_\varphi f\|_{L^1(\mathcal{G})}. \tag{4.15}$$

Next we estimate II. Assume that $1 < p < p_1$ and $0 < q < q_1$. We have

$$\begin{aligned} II &\leqslant \sup_{\|g\|_2=1} \sum_{\substack{j\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \iint_{E_{p,q;j,k}} \left|(W_{\psi_1} f)(a_j, b_{j,k})\right| \left|(W_{\psi_2} g)(a,b) - (W_{\psi_2} g)(a_j, b_{j,k})\right| \frac{da\,db}{a^{d+1}} \\ &\leqslant \sup_{\|g\|_2=1} 2\|g\|_2 \|\psi_2\|_2 \sum_{(j,k)\notin\Lambda_{N,N}} \iint_{E_{p,q;j,k}} \left|(W_{\psi_1} f)(a_j, b_{j,k})\right| \frac{da\,db}{a^{d+1}} \\ &\quad + \sup_{\|g\|_2=1} \|f\|_2 \|\psi_1\|_2 \sum_{(j,k)\in\Lambda_{N,N}} \iint_{E_{p,q;j,k}} \left|(W_{\psi_2} g)(a,b) - (W_{\psi_2} g)(a_j, b_{j,k})\right| \frac{da\,db}{a^{d+1}}. \end{aligned} \tag{4.16}$$

Choose some N large enough such that

$$\iint_{\mathcal{G}\setminus D_{N,N}} \left|(W_{\psi_1} f)(a,b)\right| \frac{da\,db}{a^{d+1}} \leqslant \varepsilon.$$

Then we have

$$\sum_{(j,k)\notin\Lambda_{N,N}} \iint_{E_{p,q;j,k}} \left|(W_{\psi_1} f)(a_j, b_{j,k})\right| \frac{da\,db}{a^{d+1}}$$

$$\leqslant \sum_{(j,k)\notin \Lambda_{N,N}} \iint_{E_{p,q;j,k}} \left|(W_{\psi_1} f)(a_j, b_{j,k}) - (W_{\psi_1} f)(a,b)\right| \frac{da\,db}{a^{d+1}} + \sum_{(j,k)\notin \Lambda_{N,N}} \iint_{E_{p,q;j,k}} \left|(W_{\psi_1} f)(a,b)\right| \frac{da\,db}{a^{d+1}}$$

$$\leqslant \sum_{(j,k)\notin \Lambda_{N,N}} \iint_{E_{p,q;j,k}} \left|(W_{\psi_1} f)(a_j, b_{j,k}) - (W_{\psi_1} f)(a,b)\right| \frac{da\,db}{a^{d+1}} + \iint_{\mathcal{G}\setminus D_{N,N}} \left|(W_{\psi_1} f)(a,b)\right| \frac{da\,db}{a^{d+1}}$$

$$\leqslant \frac{\varepsilon}{C_\varphi} \|W_\varphi f\|_{L^1(\mathcal{G})} + \varepsilon \tag{4.17}$$

where (4.14) is used.

On the other hand, by Lemma 4.3, there exists some $\delta > 0$ such that for any $1 < p < 1+\delta$, $0 < q < \delta$, and $(a,b), (a_j, b_{j,k}) \in E_{p,q;j,k}$, we have

$$\left\|\tau(a,b)\psi_2 - \tau(a_j, b_{j,k})\psi_2\right\|_2 < \varepsilon.$$

Consequently,

$$\left|\langle g, \tau(a,b)\psi_2\rangle - \langle g, \tau(a_j, b_{j,k})\psi_2\rangle\right| \leqslant \varepsilon \|g\|_2 = \varepsilon.$$

It follows that

$$\sup_{\|g\|_2=1} \|f\|_2 \|\psi_1\|_2 \sum_{(j,k)\in \Lambda_{N,N}} \iint_{E_{p,q;j,k}} \left|(W_{\psi_2} g)(a,b) - (W_{\psi_2} g)(a_j, b_{j,k})\right| \frac{da\,db}{a^{d+1}}$$

$$\leqslant \varepsilon \|f\|_2 \|\psi_1\|_2 \cdot \frac{q^d (p^{d/2} - p^{-d/2})}{d} \cdot \#\Lambda_{N,N}$$

$$\leqslant \varepsilon \|f\|_2 \|\psi_1\|_2 \cdot \frac{q^d (p^{d/2} - p^{-d/2})}{d} \left(\frac{2N \ln 2}{\ln p} + 2\right)\left(\frac{2^{2N+1} N p^{1/2}}{q} + 2\right)^d. \tag{4.18}$$

Since

$$\lim_{(p,q)\to(1,0)} \frac{q^d (p^{d/2} - p^{-d/2})}{d} \left(\frac{2N \ln 2}{\ln p} + 2\right)\left(\frac{2^{2N+1} N p^{1/2}}{q} + 2\right)^d = 0,$$

by (4.16), (4.17) and (4.18), we can make II, and therefore $\|f - S_{p,q} f\|$, arbitrarily small by choosing $p-1$ and q small enough. This completes the proof. □

Next we consider the convergence of finite Riemannian sums. Similar to $S_{p,q} f(t)$, we define the finite Riemannian sums as

$$S_{N,p,q} f(t) = S_{\psi_1,\psi_2;N,p,q} f(t) = \frac{q^d (p^{d/2} - p^{-d/2})}{d C_{\psi_1,\psi_2}} \sum_{(j,k)\in \Lambda_{N,N}} (W_{\psi_1} f)(a_j, b_{j,k}) \big(\tau(a_j, b_{j,k})\psi_2\big)(t),$$

where $(a_j, b_{j,k}) \in E_{p,q;j,k}$.

For the case of finite Riemannian sums, we have the following result.

Theorem 4.5. *Let $\psi_1, \psi_2 \in L^2(\mathbb{R}^d)$ be admissible such that $C_{\psi_1,\psi_2} \neq 0$. Then for any $f \in L^2(\mathbb{R}^d)$,*

$$\lim_{N\to\infty} \lim_{\substack{p\to 1^+ \\ q\to 0^+}} S_{N,p,q} f = f.$$

Both limits are taken in $L^2(\mathbb{R}^d)$.

Proof. For any $f \in L^2(\mathbb{R}^d)$, we have

$$\left\| S_{N,p,q} f - C_{\psi_1,\psi_2}^{-1} \iint_{D_{N,N}} (W_{\psi_1} f)(a,b) \tau(a,b)\psi_2 \frac{da\,db}{a^{d+1}} \right\|_2$$

$$= \sup_{\|g\|_2=1} \left| \frac{q^d (p^{d/2} - p^{-d/2})}{d C_{\psi_1,\psi_2}} \sum_{(j,k)\in \Lambda_{N,N}} (W_{\psi_1} f)(a_j, b_{j,k}) \overline{(W_{\psi_2} g)(a_j, b_{j,k})} \right.$$

$$\left. - C_{\psi_1,\psi_2}^{-1} \iint_{D_{N,N}} (W_{\psi_1} f)(a,b) \overline{(W_{\psi_2} g)(a,b)} \frac{da\,db}{a^{d+1}} \right|$$

$$\begin{aligned}
&\leqslant \left|C^{-1}_{\psi_1,\psi_2}\right| \sup_{\|g\|_2=1}\Bigg| \sum_{(j,k)\in\Lambda_{N,N}} \iint\limits_{E_{p,q;j,k}} (W_{\psi_1} f)(a_j, b_{j,k})\overline{(W_{\psi_2} g)(a_j, b_{j,k})}\,\frac{da\,db}{a^{d+1}} \\
&\quad - \sum_{(j,k)\in\Lambda_{N,N}} \iint\limits_{E_{p,q;j,k}} (W_{\psi_1} f)(a, b)\overline{(W_{\psi_2} g)(a, b)}\,\frac{da\,db}{a^{d+1}}\Bigg| \\
&\quad + \left|C^{-1}_{\psi_1,\psi_2}\right| \sup_{\|g\|_2=1}\Bigg| \sum_{(j,k)\in\Lambda_{N,N}} \iint\limits_{E_{p,q;j,k}} (W_{\psi_1} f)(a, b)\overline{(W_{\psi_2} g)(a, b)}\,\frac{da\,db}{a^{d+1}} \\
&\quad - \iint\limits_{D_{N,N}} (W_{\psi_1} f)(a, b)\overline{(W_{\psi_2} g)(a, b)}\,\frac{da\,db}{a^{d+1}}\Bigg| \\
&= \left|C^{-1}_{\psi_1,\psi_2}\right|(I + II).
\end{aligned}$$

Let $\tilde{D}_{p,q;N,N} = \bigcup_{(j,k)\in\Lambda_{N,N}} E_{p,q;j,k}$. Then we have $D_{N,N} \subset \tilde{D}_{p,q;N,N}$. It is easy to see that

$$\lim_{(p,q)\to(1,0)} \iint\limits_{\tilde{D}_{p,q;N,N}\setminus D_{N,N}} \frac{da\,db}{a^{d+1}} = 0.$$

Hence

$$II \leqslant \sup_{\|g\|_2=1} \left|C^{-1}_{\psi_1,\psi_2}\right| \|\psi_1\|_2 \|\psi_2\|_2 \|f\|_2 \|g\|_2 \iint\limits_{\tilde{D}_{p,q;N,N}\setminus D_{N,N}} \frac{da\,db}{a^{d+1}} \to 0, \quad \text{as } (p,q)\to(1,0).$$

On the other hand,

$$\begin{aligned}
I &= \sup_{\|g\|_2=1}\Bigg| \sum_{(j,k)\in\Lambda_{N,N}} \iint\limits_{E_{p,q;j,k}} (W_{\psi_1} f)(a_j, b_{j,k})\overline{(W_{\psi_2} g)(a_j, b_{j,k})}\,\frac{da\,db}{a^{d+1}} \\
&\quad - \sum_{(j,k)\in\Lambda_{N,N}} \iint\limits_{E_{p,q;j,k}} (W_{\psi_1} f)(a, b)\overline{(W_{\psi_2} g)(a, b)}\,\frac{da\,db}{a^{d+1}}\Bigg| \\
&\leqslant \sup_{\|g\|_2=1} \sum_{(j,k)\in\Lambda_{N,N}} \iint\limits_{E_{p,q;j,k}} \left|(W_{\psi_1} f)(a_j, b_{j,k})\left(\overline{(W_{\psi_2} g)(a_j, b_{j,k})} - \overline{(W_{\psi_2} g)(a, b)}\right)\right| \frac{da\,db}{a^{d+1}} \\
&\quad + \sup_{\|g\|_2=1} \sum_{(j,k)\in\Lambda_{N,N}} \iint\limits_{E_{p,q;j,k}} \left|\left((W_{\psi_1} f)(a_j, b_{j,k}) - (W_{\psi_1} f)(a, b)\right)\overline{(W_{\psi_2} g)(a, b)}\right| \frac{da\,db}{a^{d+1}} \\
&\leqslant \sup_{\|g\|_2=1} \|f\|_2 \|\psi_1\|_2 \sum_{(j,k)\in\Lambda_{N,N}} \iint\limits_{E_{p,q;j,k}} \left|(W_{\psi_2} g)(a, b) - (W_{\psi_2} g)(a_j, b_{j,k})\right| \frac{da\,db}{a^{d+1}} \\
&\quad + \|\psi_2\|_2 \sum_{(j,k)\in\Lambda_{N,N}} \iint\limits_{E_{p,q;j,k}} \left|(W_{\psi_1} f)(a, b) - (W_{\psi_1} f)(a_j, b_{j,k})\right| \frac{da\,db}{a^{d+1}}.
\end{aligned}$$

Similarly to the proof of Theorem 4.4, we can prove that there exists some $\delta > 0$, depending only on ψ_1, ψ_2 and ε, such that for any $1 < p < 1 + \delta$, $0 < q < \delta$, and $(a, b), (a_j, b_{j,k}) \in E_{p,q;j,k}$,

$$\begin{aligned}
&\sup_{\|g\|_2=1} \|f\|_2 \|\psi_1\|_2 \sum_{(j,k)\in\Lambda_{N,N}} \iint\limits_{E_{p,q;j,k}} \left|(W_{\psi_2} g)(a, b) - (W_{\psi_2} g)(a_j, b_{j,k})\right| \frac{da\,db}{a^{d+1}} \\
&\quad \leqslant \varepsilon \|f\|_2 \|\psi_1\|_2 \cdot \frac{q^d (p^{d/2} - p^{-d/2})}{d} \left(\frac{2N \ln 2}{\ln p} + 2\right)\left(\frac{2^{2N+1} N p^{1/2}}{q} + 2\right)^d
\end{aligned} \tag{4.19}$$

and

$$\|\psi_2\|_2 \sum_{(j,k)\in\Lambda_{N,N}} \iint_{E_{p,q;j,k}} \left|(W_{\psi_1} f)(a_j, b_{j,k}) - (W_{\psi_1} f)(a,b)\right| \frac{da\,db}{a^{d+1}}$$
$$\leqslant \varepsilon \|f\|_2 \|\psi_2\|_2 \cdot \frac{q^d(p^{d/2} - p^{-d/2})}{d} \left(\frac{2N\ln 2}{\ln p} + 2\right)\left(\frac{2^{2N+1}Np^{1/2}}{q} + 2\right)^d. \tag{4.20}$$

Since

$$\lim_{(p,q)\to(1,0)} \frac{q^d(p^{d/2} - p^{-d/2})}{d} \left(\frac{2N\ln 2}{\ln p} + 2\right)\left(\frac{2^{2N+1}Np^{1/2}}{q} + 2\right)^d = 0,$$

we can make I arbitrarily small by choosing $p-1$ and q small enough. Hence

$$\lim_{\substack{p\to 1^+\\ q\to 0^+}} \left\| S_{N,p,q} f - C^{-1}_{\psi_1,\psi_2} \iint_{D_{N,N}} (W_{\psi_1} f)(a,b)\tau(a,b)\psi_2 \frac{da\,db}{a^{d+1}} \right\|_2 = 0.$$

By Lemma 2.3, we obtain that

$$\lim_{N\to\infty} \lim_{\substack{p\to 1^+\\ q\to 0^+}} S_{N,p,q} f = f,$$

where both limits are taken in the $L^2(\mathbb{R}^d)$ norm. This completes the proof. □

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