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Advances in Mathematics 197 (2005) 554–586
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

An endomorphism of the Khovanov invariant

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Received 14 June 2004; accepted 27 October 2004

Communicated by Tomasz Mrowka
Available online 7 December 2004

Abstract

We construct an endomorphism of the Khovanov invariant to prove H-thinness and pairing phenomena of the invariants for alternating links. As a consequence, it follows that the Khovanov invariant of an oriented nonsplit alternating link is determined by its Jones polynomial, signature, and the linking numbers of its components.

MSC: 57M27

Keywords: Khovanov invariant; H-thin; Alternating links

1. Introduction

Khovanov invariant is a cohomology theory for oriented links with values in graded abelian groups, and specializes to the Jones polynomial by taking graded Euler characteristic of those cohomology groups (Theorem 1.1). Khovanov [7] constructed the invariant in a search of connections between combinatorial invariants and differential geometric invariants of three- and four-dimensional manifolds. He interpreted his coboundary map as the image of a functor from the category of two

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0001-8708/$ - see front matter
doi:10.1016/j.aim.2004.10.015

dimensional cobordisms between one dimensional manifolds to the category of $\mathbb{Z}[c]$-modules.

The Khovanov invariant specialized by setting $c = 0$ and tensoring with $\mathbb{Q}$ (will be just called the Khovanov invariant from now on) has been computed by Bar-Natan [2,3] for the prime knots with up to 11 crossings. From Bar-Natan's data, two conjectures [2,5] on the values of Khovanov invariant for alternating knots were formulated by Bar-Natan, Garoufalidis, and Khovanov. The conjectures (Theorems 1.2 and 1.4) imply that the Khovanov invariant of an alternating knot determines and is determined by its Jones polynomial and signature.

The following is the theorem in [7] which states that the Khovanov invariant specializes to the Jones polynomial. The Khovanov invariant of a (relatively) oriented link L in rational coefficients is denoted by $\mathcal{H}(L)$ following [7], and is defined in Section 2. Its associated polynomial is denoted by $Kh(L)$ as it is in [2].

$$Kh(L)(t,q) \stackrel{\text{def}}{=} \sum t^i q^j \dim \mathcal{H}^{i,j}(L).$$

Theorem 1.1 (*Khovanov [7]*). *For an oriented link L, the graded Euler characteristic*

$$\sum_{i,j\in\mathbb{Z}} (-1)^i q^j \dim \mathcal{H}^{i,j}(L)$$

of the Khovanov invariant $\mathcal{H}(L)$ of L is equal to $(q^{-1}+q)$ times the Jones polynomial $V(L)$ of L.

$$\sum_{i,j\in\mathbb{Z}} (-1)^i q^j \dim \mathcal{H}^{i,j}(L) = (q^{-1}+q)V(L)_{\sqrt{t}=-q}.$$

In terms of the associated polynomial $Kh(L)$,

$$Kh(L)(-1,q) = (q^{-1}+q)V(L)_{\sqrt{t}=-q}.$$

The following two theorems are the conjectures in [2] proved in this paper.

Theorem 1.2 (*Bar-Natan [2, Conjecture 2] and Garoufalidis [5]*). *For any alternating knot L, the Khovanov invariants $\mathcal{H}^{i,j}(L)$ of L are supported in two lines*

$$j = 2i - \sigma(L) \pm 1.$$

In other words, the equality

$$Kh(L)(t,q) = q^{-\sigma(L)}(q^{-1} \cdot A(tq^2) + q \cdot B(tq^2))$$

holds for some polynomials A and B, where $\sigma(L)$ is the signature of L.

Theorem 1.2, in fact, holds for any (relatively) oriented nonsplit alternating link L. (See Theorem 3.12.)

Definition 1.3 (*H-thinness [8]*). A diagram/knot/link D is *H-thin* if its Khovanov invariant $\mathcal{H}(D)$ (or $\overline{\mathcal{H}}(D)$ if D is a diagram) is supported in two diagonal lines as in Theorem 1.2 up to a shift of the grading.

Theorem 1.2 implies that any nonsplit alternating link is H-thin.

Theorem 1.4 (*Bar-Natan [2, Conjecture 1] and Garoufalidis [5]*). *For an alternating knot L, its Khovanov invariants $\mathcal{H}^{i,j}(L)$ of degree difference $(1,4)$ are paired except in the 0th cohomology group.*

More precisely, in terms of the polynomial $Kh(L)$, the equality

$$Kh(L)(t,q) = q^{-s}(q^{-1} + q) + (q^{-1} + tq^2 \cdot q) \cdot C(t,q)$$

holds for some integer s and some polynomial C.

Theorem 1.4 was extended to (relatively) oriented nonsplit alternating links in Theorem 4.5.

Combining Theorems 1.2 and 1.4, we can write $Kh(L)$ as

$$Kh(L)(t,q) = q^{-\sigma(L)}\{(q^{-1} + q) + (q^{-1} + tq^2 \cdot q) \cdot Kh'(L)(tq^2)\}$$

for some polynomial $Kh'(L)$.

As it is discussed in [2,5], Theorems 1.2 and 1.4 with Theorem 1.1 imply that the Khovanov invariant, or equivalently the associated polynomial $Kh(L)$, of an alternating knot L is completely determined by the Jones polynomial and the signature of L.

Fortunately, that is not the case for nonalternating knots. A counterexample can be found in [3]: 10_{136} and 11^n_{92} both have signature -2 and the same Jones polynomial, but their Khovanov invariants do not agree.

The organization of the coming sections is as follows. Section 2 consists of a brief summary of the Khovanov invariant. Section 3 is devoted to our proof of Theorem 1.2. In Section 4, an endomorphism of Khovanov invariant is defined and used to prove 1.4.

We follow [9,4] for basic notions in knot theory and graph theory, and [7] for notations and terminologies related to the Khovanov invariant. We only need a relative orientation to define the Khovanov invariant, so an orientation and oriented can be read as a relative orientation and relatively oriented.

2. Khovanov invariant

In this section, the construction and some properties of Khovanov invariant in [7] are summarized. Khovanov's original construction is more general, but we will concentrate on a specialized case with coefficients in $\mathbb{Q}$. The interested reader should read [7].

2.1. Construction

2.1.1. Cubes of diagrams

Let L be an oriented link and D be its diagram, a regular projection of L together with the information of relative height at each double point. A double point of D can be resolved in two ways.

0-resolution 1-resolution

Let $\mathcal{I}$ be the set of double points of D. Each subset $\mathcal{J}$ of $\mathcal{I}$ corresponds to a complete resolution $D(\mathcal{J})$ of D in which points in $\mathcal{J}$ are resolved to their 1-resolutions, points not in $\mathcal{J}$ are to their 0-resolutions. Regard those subsets of $\mathcal{I}$ as vertices. For each pair of vertices $\mathcal{J}$ and $\mathcal{J}'$ satisfying $\mathcal{J} \subset \mathcal{J}'$ and $|\mathcal{J}' - \mathcal{J}| = 1$, there is a directed edge from $\mathcal{J}$ to $\mathcal{J}'$. A directed cube is constructed.

2.1.2. Cubes of modules

Let $\mathbf{A} = \mathbb{Q}\mathbf{1} \oplus \mathbb{Q}\mathbf{x}$ be a two-dimensional module over $\mathbb{Q}$ with a multiplication m, a comultiplication Δ, a unit ι, and a counit ε defined as

$$m(\mathbf{1} \otimes \mathbf{1}) = \mathbf{1},$$

$$m(\mathbf{1} \otimes \mathbf{x}) = m(\mathbf{x} \otimes \mathbf{1}) = \mathbf{x},$$

$$m(\mathbf{x} \otimes \mathbf{x}) = 0,$$

$$\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{x} + \mathbf{x} \otimes \mathbf{1},$$

$$\Delta(\mathbf{x}) = \mathbf{x} \otimes \mathbf{x},$$

$$\iota(1) = \mathbf{1},$$

$$\varepsilon(\mathbf{1}) = 0,$$

$$\varepsilon(\mathbf{x}) = 1.$$

For each vertex $\mathcal{J}$ of a cube, assign a tensor product of as many copies of $\mathbf{A}$ as the number of components of $D(\mathcal{J})$, and denote it by $\mathbf{M}_{\mathcal{J}}(D)$. There is a 1–1 correspondence between those copies of $\mathbf{A}$ and the components of $D(\mathcal{J})$.

2.1.3. Chain complexes

A chain complex can be constructed from the cube of modules. Its ith chain group is a direct sum of all the modules over vertices of i elements.

$$\overline{\mathcal{C}}^i(D) = \bigoplus_{|\mathcal{J}|=i} \mathbf{M}_{\mathcal{J}}(D).$$

To define the coboundary map d, choose an ordering of $\mathcal{I}$—the set of crossings of D, and regard $\mathcal{J} \subset \mathcal{I}$ as an ordered $|\mathcal{J}|$-tuple of its elements in the chosen order instead of just a subset of $\mathcal{I}$.

For a homogeneous element $x \in \mathbf{M}_{\mathcal{J}}(D)$, dx lies in the sum of all the modules over those vertices which are end-points of the directed edges from $\mathcal{J}$.

$$dx \in \bigoplus_{\mathcal{J} \subset \mathcal{J}', |\mathcal{J}'-\mathcal{J}|=1} \mathbf{M}_{\mathcal{J}'}(D) \qquad \text{for } x \in \mathbf{M}_{\mathcal{J}}(D).$$

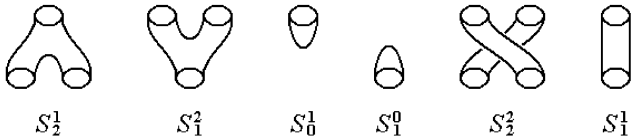
Each homogeneous component of d is defined in the way that $m : \mathbf{A} \otimes \mathbf{A} \to \mathbf{A}$ is applied to corresponding modules if two components merge into one, and $\Delta : \mathbf{A} \to \mathbf{A} \otimes \mathbf{A}$ is if one component splits to two. If the ordered $(|\mathcal{J}|+1)$-tuple $\mathcal{J}$ followed by the element in $\mathcal{J}' - \mathcal{J}$ is an odd permutation of the ordered $(|\mathcal{J}|+1)$-tuple $\mathcal{J}'$, $-m$ or $-\Delta$, instead of m or Δ, is used for the $\mathbf{M}_{\mathcal{J}}(D) \to \mathbf{M}_{\mathcal{J}'}(D)$ component of d. With this choice of signs, d satisfies $d^2 = 0$.

2.1.4. Relation to TQFT

The algebra $\mathbf{A}$ above is a Frobenius algebra and it is related to a two-dimensional topological quantum field theory. There is a functor F that maps one-dimensional manifolds consisting of n disjoint simple closed curves to $\mathbf{A}^{\otimes n}$'s, and cobordisms in

the following figure to:

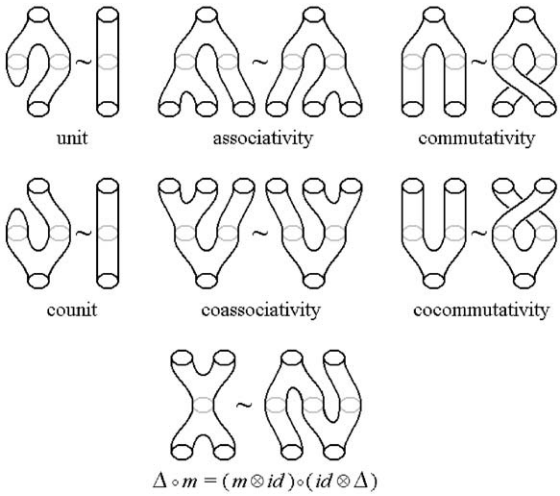
$$F(S_2^1) = m,\ F(S_1^2) = \Delta,\ F(S_0^1) = \iota,\ F(S_1^0) = \varepsilon,\ F(S_2^2) = \text{(permutation)},\ F(S_1^1) = id.$$



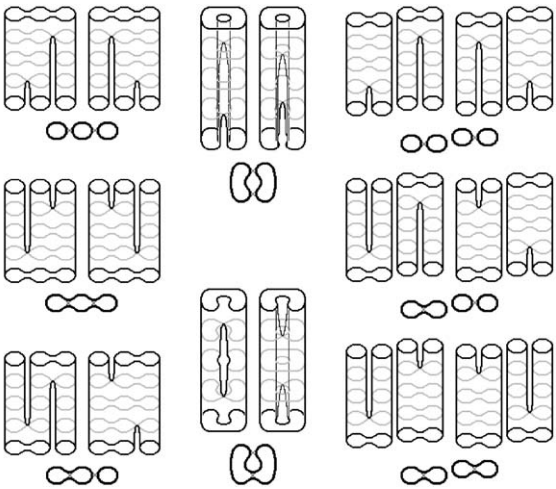
The unit, counit, (co)associativity, (co)commutativity, together with the identity

$$\Delta \circ m = (m \otimes id) \circ (id \otimes \Delta)$$

ensures well-definedness of F. (See [7].)



From the viewpoint of the previous sections, well-definedness of F implies $d^2 = 0$. The following figure tabulates all the possible relative locations of two crossings and the associated surfaces obtained by continuous change of resolutions of the two crossings one after the other.



For d^2 to be equal to 0, those in the first two columns from the left only require m and Δ to be (co)commutative, the top right one requires m to be associative, the middle right one requires Δ to be coassociative, and the bottom right one requires the identity $\Delta \circ m = (m \otimes id) \circ (id \otimes \Delta)$. The unit and counit can be dropped if we are concerned only about $d^2 = 0$. (See [1] for more discussion of Frobenius algebra associated to a two-dimensional TQFT.)

2.1.5. Grading on the chain groups

A is a graded module. **x** is of degree -1, **1** is of degree 1. $\mathbf{M}_{\mathcal{J}}(D)$ has a grading induced from that of **A**. Note that m and Δ are maps of degree -1 with respect to these gradings.

The chain group $\overline{\mathcal{C}}^i(D)$ above has a grading shifted from those of $\mathbf{M}_{\mathcal{J}}(D)$:

$$\overline{\mathcal{C}}^i(D) = \bigoplus_{|\mathcal{J}|=i} \mathbf{M}_{\mathcal{J}}(D)\{-i\}.$$

$\mathbf{M}\{k\}$ means a module identical to **M** with a shifted grading. An element of degree j in **M** is of degree $j-k$ in $\mathbf{M}\{k\}$.

Define $\overline{\mathcal{C}}^{i,j}(D)$ as the degree j component of $\overline{\mathcal{C}}^i(D)$. Due to the shifts, d is now degree preserving, so that $\overline{\mathcal{H}}^i(D)$ is also decomposed as $\bigoplus_j \overline{\mathcal{H}}^{i,j}(D)$.

A chain complex $\mathcal{C}(D)$ is defined from $\overline{\mathcal{C}}(D)$ with the orientation of D taken into account.

For each crossing of D, a sign is given as below (Note that this is opposite to the sign convention in [7].)

Let $x(D)$ and $y(D)$ be the number of negative crossings and positive crossings of D, respectively. $\mathcal{C}(D)$ is defined as

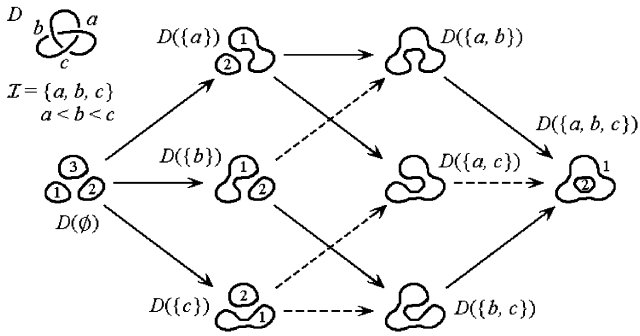
$$\mathcal{C}(D) = \overline{\mathcal{C}}(D)[x(D)]\{2x(D) - y(D)\}$$

with the same coboundary map d. Square brackets indicate a shift of the indices of chain groups, i.e.,

$$\mathcal{C}^{i,j}(D) = \overline{\mathcal{C}}^{i+x(D),\, j+2x(D)-y(D)}(D).$$

2.1.6. Example

Here is an illustration of what had happened in the previous sections to the following diagram D of the lefthanded trefoil:



Numbers for components of a resolution indicate which piece of $\mathbf{A}$ corresponds to which component, dotted edges indicate the places where $-m$ or $-\Delta$ should be used.

$$\overline{\mathcal{C}}^0(D) = \mathbf{M}_\emptyset(D) = \mathbf{A} \otimes \mathbf{A} \otimes \mathbf{A},$$

$$\overline{\mathcal{C}}^1(D) = \mathbf{M}_{\{a\}}(D) \oplus \mathbf{M}_{\{b\}}(D) \oplus \mathbf{M}_{\{c\}}(D) = (\mathbf{A} \otimes \mathbf{A}) \oplus (\mathbf{A} \otimes \mathbf{A}) \oplus (\mathbf{A} \otimes \mathbf{A}),$$

$$\overline{\mathcal{C}}^2(D) = \mathbf{M}_{\{a,b\}}(D) \oplus \mathbf{M}_{\{a,c\}}(D) \oplus \mathbf{M}_{\{b,c\}}(D) = \mathbf{A} \oplus \mathbf{A} \oplus \mathbf{A},$$

$$\overline{\mathcal{C}}^3(D) = \mathbf{M}_{\{a,b,c\}}(D) = \mathbf{A} \otimes \mathbf{A},$$

$$\mathbf{x}\otimes\mathbf{x}\otimes\mathbf{x}\in\overline{\mathcal{C}}^{0,-3}(D)\overset{d}{\longmapsto}\begin{bmatrix}0\\0\\0\end{bmatrix}\in\overline{\mathcal{C}}^{1,-3}(D),$$

$$\mathbf{1}\otimes\mathbf{x}\otimes\mathbf{1}\in\overline{\mathcal{C}}^{0,1}(D)\overset{d}{\longmapsto}\begin{bmatrix}\mathbf{x}\otimes\mathbf{1}\\\mathbf{1}\otimes\mathbf{x}\\\mathbf{x}\otimes\mathbf{1}\end{bmatrix}\in\overline{\mathcal{C}}^{1,1}(D),$$

$$\begin{bmatrix}\mathbf{1}\otimes\mathbf{x}\\\mathbf{x}\otimes\mathbf{1}\\0\end{bmatrix}\in\overline{\mathcal{C}}^{1,1}(D)\overset{d}{\longmapsto}\begin{bmatrix}0\\\mathbf{x}\\\mathbf{x}\end{bmatrix}\in\overline{\mathcal{C}}^{2,1}(D),$$

$$\begin{bmatrix}0\\\mathbf{1}\\0\end{bmatrix}\in\overline{\mathcal{C}}^{2,3}(D)\overset{d}{\longmapsto}-\mathbf{1}\otimes\mathbf{x}-\mathbf{x}\otimes\mathbf{1}\in\overline{\mathcal{C}}^{3,3}(D),$$

$$\overline{\mathcal{H}}^{0,-3}(D)=\frac{\operatorname{span}\{\mathbf{x}\otimes\mathbf{x}\otimes\mathbf{x}\}}{\{0\}},$$

$$\overline{\mathcal{H}}^{1,1}(D)=\frac{\operatorname{span}\left\{\begin{bmatrix}\mathbf{1}\otimes\mathbf{x}\\\mathbf{1}\otimes\mathbf{x}\\\mathbf{1}\otimes\mathbf{x}\end{bmatrix},\begin{bmatrix}\mathbf{1}\otimes\mathbf{x}\\\mathbf{x}\otimes\mathbf{1}\\\mathbf{x}\otimes\mathbf{1}\end{bmatrix},\begin{bmatrix}\mathbf{x}\otimes\mathbf{1}\\\mathbf{1}\otimes\mathbf{x}\\\mathbf{x}\otimes\mathbf{1}\end{bmatrix},\begin{bmatrix}\mathbf{x}\otimes\mathbf{1}\\\mathbf{x}\otimes\mathbf{1}\\\mathbf{1}\otimes\mathbf{x}\end{bmatrix}\right\}}{\operatorname{span}\left\{\begin{bmatrix}\mathbf{1}\otimes\mathbf{x}\\\mathbf{x}\otimes\mathbf{1}\\\mathbf{x}\otimes\mathbf{1}\end{bmatrix},\begin{bmatrix}\mathbf{x}\otimes\mathbf{1}\\\mathbf{1}\otimes\mathbf{x}\\\mathbf{x}\otimes\mathbf{1}\end{bmatrix},\begin{bmatrix}\mathbf{x}\otimes\mathbf{1}\\\mathbf{x}\otimes\mathbf{1}\\\mathbf{1}\otimes\mathbf{x}\end{bmatrix}\right\}},$$

$$\overline{\mathcal{H}}^{3,3}(D)=\frac{\operatorname{span}\{\mathbf{1}\otimes\mathbf{x},\mathbf{x}\otimes\mathbf{1}\}}{\operatorname{span}\{\mathbf{1}\otimes\mathbf{x}+\mathbf{x}\otimes\mathbf{1}\}},$$

$$\overline{\mathcal{H}}^{3,5}(D)=\frac{\operatorname{span}\{\mathbf{1}\otimes\mathbf{1}\}}{\{0\}},$$

$$\overline{\mathcal{H}}^{i,j}(D)=\begin{cases}\mathbb{Q} & \text{for } (i,j)=(0,-3),(1,1),(3,3),\text{ or } (3,5),\\ 0 & \text{otherwise.}\end{cases}$$

$x(D)=3$ and $y(D)=0$ for this diagram D. (A knot or a knot diagram has only one relative orientation.) Hence,

$$\mathcal{H}^{i,j}(D)=\overline{\mathcal{H}}^{i+3,j+6}(D)=\begin{cases}\mathbb{Q} & \text{for } (i,j)=(-3,-9),(-2,-5),(0,-3),\text{ or } (0,-1),\\ 0 & \text{otherwise.}\end{cases}$$

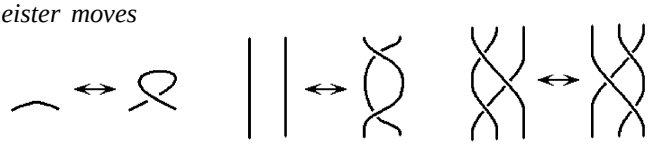
2.2. Invariance

To define $\mathcal{H}(L)$ as $\mathcal{H}(D)$, we need to see invariance of $\mathcal{H}(D)$ under a change of ordering of $\mathcal{I}$ and under the Reidemeister moves. We will just state isomorphisms. Detailed proofs can be found in [7, Chapter 5].

2.2.1. Change of ordering

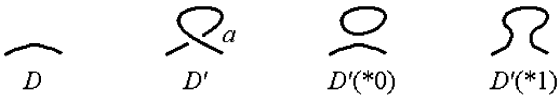
If the $|\mathcal{J}|$-tuple $\mathcal{J}$ with respect to one ordering of $\mathcal{I}$ is an even permutation of $\mathcal{J}$ with respect to the other ordering, $\mathbf{M}_{\mathcal{J}}$ component of our isomorphism between them is the identity. Otherwise, it is minus identity.

2.2.2. Reidemeister moves



Type I Type II Type III

[Type I]



Let a be the crossing which appears only in D'. The set $\mathcal{I}'$ of crossings of D' is $\mathcal{I}$, the set of crossings of D, followed by a as an ordered $|\mathcal{I}'|$-tuple. Let $D'(*0)$ and $D'(*1)$ denote D' with only its last crossing (that is a) resolved to its 0- and 1-resolutions, respectively.

As a group, $\overline{\mathcal{C}}(D')$ is a direct sum of $\overline{\mathcal{C}}(D'(*0))$ and $\overline{\mathcal{C}}(D'(*1))[-1]\{-1\}$. Denote the part of the coboundary map d' on $\overline{\mathcal{C}}(D')$ that maps from $\overline{\mathcal{C}}(D'(*0))$ to $\overline{\mathcal{C}}(D'(*1))[-1]\{-1\}$ by $d'_{0\to 1}$, and the coboundary maps on $\overline{\mathcal{C}}(D(*0))$ and $\overline{\mathcal{C}}(D(*1))$ by d'_0 and d'_1, so that

$$d'(y+z) = d'_0(y) + d'_{0\to 1}(y) - d'_1(z)$$

for $y \in \overline{\mathcal{C}}(D'(*0))$ and $z \in \overline{\mathcal{C}}(D'(*1))[-1]\{-1\}$. Similar notation should be comprehended similarly from now on.

Define

$$X_1 = \operatorname{Ker} d'_{0\to 1}$$

and

$$X_2 = \{y \otimes \mathbf{1} + z | y \in \overline{\mathcal{C}}(D), z \in \overline{\mathcal{C}}(D'(*1))[-1]\{-1\}\}.$$

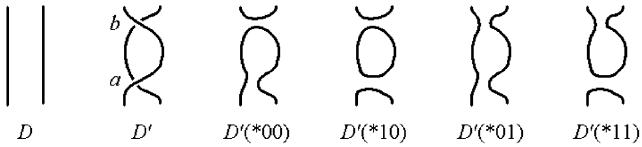
Here $\overline{\mathcal{C}}(D'(*0))$ is identified with $\overline{\mathcal{C}}(D) \otimes \mathbf{A}$.

X_1 and X_2 are subcomplexes of $\overline{\mathcal{C}}(D')$, $\overline{\mathcal{C}}(D')$ is decomposed as $X_1 \oplus X_2$ as a chain complex, X_2 is acyclic, and

$$\begin{aligned} \rho : \quad X_1 &\longrightarrow \overline{\mathcal{C}}(D)\{1\} \\ y \otimes \mathbf{1} + z \otimes \mathbf{x} &\longmapsto z \end{aligned}$$

induces an isomorphism between $\mathcal{H}(D')$ and $\mathcal{H}(D)$.

[Type II]



As before, the set $\mathcal{I}'$ of crossings of D' is $\mathcal{I}$, the set of crossings of D, followed by a, then b as an ordered $|\mathcal{I}'|$-tuple.

This time,

$$X_1 = \{z + \alpha(z) | z \in \overline{\mathcal{C}}(D'(*01))[-1]\{-1\}\},$$

$$X_2 = \{z + d'y | z, y \in \overline{\mathcal{C}}(D'(*00))\},$$

$$X_3 = \{z + y \otimes \mathbf{1} | z, y \in \overline{\mathcal{C}}(D'(*11))[-2]\{-2\}\},$$

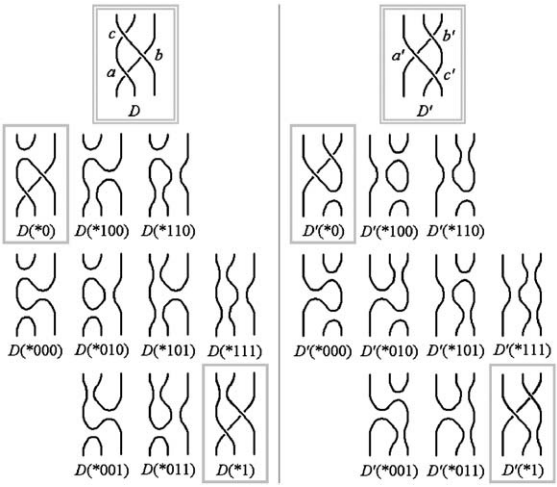
where $\alpha(z) = -d'_{01\to 11}(z) \otimes \mathbf{1} \in \overline{\mathcal{C}}(D'(*10))[-1]\{-1\} \approx \overline{\mathcal{C}}(D'(*11))[-1]\{-1\} \otimes \mathbf{A}$.

Then, $\overline{\mathcal{C}}(D')$ is a direct sum of its subcomplexes X_1, X_2, and X_3, X_2 and X_3 are acyclic, and

$$\begin{aligned} \rho : \overline{\mathcal{C}}^i(D)[-1]\{-1\} \approx \overline{\mathcal{C}}^i(D'(*01))[-1]\{-1\} &\longrightarrow X_1 \cap \overline{\mathcal{C}}^i(D') \\ z &\longmapsto (-1)^i(z + \alpha(z)) \end{aligned}$$

induces an isomorphism between $\mathcal{H}(D)$ and $\mathcal{H}(D')$.

[Type III]



Again, a, b, c and a', b', c' are the last three elements in $\mathcal{I}$ and $\mathcal{I}'$, and the others are in the same order.

Define $\alpha, \beta, \alpha', \beta'$ as

$$\begin{aligned}
\alpha:\ \overline{\mathcal{C}}(D(*110))[-2]\{-2\} &\longrightarrow \overline{\mathcal{C}}(D(*010))[-1]\{-1\} \approx \overline{\mathcal{C}}(D(*110))[-1]\{-1\} \otimes \mathbf{A},\\
z &\longmapsto z \otimes \mathbf{1}\\
\beta:\ \overline{\mathcal{C}}(D(*100))[-1]\{-1\} &\longrightarrow \overline{\mathcal{C}}(D(*010))[-1]\{-1\},\\
z &\longmapsto \alpha d_{100\to 110}(z)\\
\alpha':\ \overline{\mathcal{C}}(D'(*110))[-2]\{-2\} &\longrightarrow \overline{\mathcal{C}}(D'(*100))[-1]\{-1\} \approx \overline{\mathcal{C}}(D'(*110))[-1]\{-1\} \otimes \mathbf{A},\\
z &\longmapsto z \otimes \mathbf{1}\\
\beta':\ \overline{\mathcal{C}}(D'(*010))[-1]\{-1\} &\longrightarrow \overline{\mathcal{C}}(D'(*100))[-1]\{-1\}.\\
z &\longmapsto -\alpha' d'_{010\to 110}(z)
\end{aligned}$$

$\overline{\mathcal{C}}(D)$ and $\overline{\mathcal{C}}(D')$ can be decomposed into their subcomplexes as below.

$$\begin{aligned}
\overline{\mathcal{C}}(D) &= X_1 \oplus X_2 \oplus X_3,\\
X_1 &= \{x + \beta(x) + y \mid x \in \overline{\mathcal{C}}(D(*100))[-1]\{-1\},\ y \in \overline{\mathcal{C}}(D(*1))[-1]\{-1\}\},\\
X_2 &= \{x + dy \mid x, y \in \overline{\mathcal{C}}(D(*000))\},\\
X_3 &= \{\alpha(x) + d\alpha(y) \mid x, y \in \overline{\mathcal{C}}(D(*110))[-2]\{-2\}\};
\end{aligned}$$

$$\begin{aligned}
\overline{\mathcal{C}}(D') &= Y_1 \oplus Y_2 \oplus Y_3,\\
Y_1 &= \{x + \beta'(x) + y | x \in \overline{\mathcal{C}}(D'(*010))[-1]\{-1\}, y \in \overline{\mathcal{C}}(D'(*1))[-1]\{-1\}\},\\
Y_2 &= \{x + d'y | x, y \in \overline{\mathcal{C}}(D'(*000))\},\\
Y_3 &= \{\alpha'(x) + d'\alpha'(y) | x, y \in \overline{\mathcal{C}}(D'(*110))[-2]\{-2\}\}.
\end{aligned}$$

As before, X_2, X_3, Y_2, Y_3 are acyclic, $\overline{\mathcal{C}}(D(*100))[-1]\{-1\}$ and $\overline{\mathcal{C}}(D'(*010))[-1]\{-1\}$, $\overline{\mathcal{C}}(D(*1))[-1]\{-1\}$ and $\overline{\mathcal{C}}(D'(*1))[-1]\{-1\}$ are naturally isomorphic, and X_1 is isomorphic to Y_1 via

$$\rho : x + \beta(x) + y \longmapsto x + \beta'(x) + y.$$

2.3. Properties

The following results are proved in [7] and will be used in the coming sections.

Proposition 2.1. *For an oriented n component link diagram D,*

$$\mathcal{C}^{i,j}(D) = 0$$

unless $j \equiv n (\mathrm{mod}\, 2)$.

Corollary 2.2. *For an oriented n component link L,*

$$\mathcal{H}^{i,j}(L) = 0$$

unless $j \equiv n (\mathrm{mod}\, 2)$.

Proposition 2.3. *For a disjoint union $D \sqcup D'$ of two oriented link diagrams D and D',*

$$\mathcal{C}(D \sqcup D') = \mathcal{C}(D) \otimes \mathcal{C}(D').$$

Corollary 2.4. *For a disjoint union $L \sqcup L'$ of two oriented links L and L',*

$$\mathcal{H}(L \sqcup L') = \mathcal{H}(L) \otimes \mathcal{H}(L').$$

Proposition 2.5. *The Frobenius algebra* $(\mathbf{A}, m, \Delta, \iota, \varepsilon)$ *is isomorphic to its dual algebra* $(\mathbf{A}^*, \Delta^*, m^*, \varepsilon^*, \iota^*)$.

Proposition 2.6. *Let $D^!$ be the mirror image of an oriented link diagram D. The complex $\mathcal{C}(D^!)$ is isomorphic to the dual of $\mathcal{C}(D)$.*

Corollary 2.7. *For an oriented link L and its mirror image $L^!$,*

$$\mathcal{H}(L^!) \cong (\mathcal{H}(L))^*.$$

Theorem 2.8 (*Theorem 1.1*). *For an oriented link L, the graded Euler characteristic*

$$\sum_{i,j\in\mathbb{Z}} (-1)^i q^j \dim \mathcal{H}^{i,j}(L)$$

of the Khovanov invariant $\mathcal{H}(L)$ of L is equal to $(q^{-1}+q)$ times the Jones polynomial $V(L)$ of L.

$$\sum_{i,j\in\mathbb{Z}} (-1)^i q^j \dim \mathcal{H}^{i,j}(L) = (q^{-1}+q)V(L)_{\sqrt{t}=-q}.$$

In terms of the associated polynomial $Kh(L)$,

$$Kh(L)(-1,q) = (q^{-1}+q)V(L)_{\sqrt{t}=-q}.$$

3. H-thinness of alternating links

In this section, we prove Theorem 1.2. The proof is based on induction on the number of crossings. We will show that the support of the Khovanov invariant of a nonsplit oriented alternating link is included in the union of the supports for two such links with fewer crossings, then that the two lines of the two supports agree.

3.1. Exact sequences

Theorem 3.1. *The chain complexes $\overline{\mathcal{C}}(D)$, $\overline{\mathcal{C}}(D(*0))$, and $\overline{\mathcal{C}}(D(*1))[-1]\{-1\}$ form a short exact sequence*

$$0 \to \overline{\mathcal{C}}(D(*1))[-1]\{-1\} \to \overline{\mathcal{C}}(D) \to \overline{\mathcal{C}}(D(*0)) \to 0$$

with degree preserving maps, so that $\overline{\mathcal{H}}(D)$ is an extension of the kernel and cokernel of the connecting map δ as a bigraded $\mathbb{Q}$-module.

$$\cdots \to \overline{\mathcal{H}}^{i-1}(D(*0)) \xrightarrow{\delta} \overline{\mathcal{H}}^{i-1}(D(*1))\{-1\} \to \overline{\mathcal{H}}^{i}(D) \to \overline{\mathcal{H}}^{i}(D(*0))$$
$$\xrightarrow{\delta} \overline{\mathcal{H}}^{i}(D(*1))\{-1\} \to \cdots$$

In particular, the support of $\overline{\mathcal{H}}(D)$ is included in the union of the support of $\overline{\mathcal{H}}(D(*0))$ and $\overline{\mathcal{H}}(D(*1))[-1]\{-1\}$.

Proof. We have already seen that $\overline{\mathcal{C}}(D)$ is decomposed as $\overline{\mathcal{C}}(D(*0)) \oplus \overline{\mathcal{C}}(D(*1))[-1]\{-1\}$, and the coboundary map d can be written as

$$d(y+z) = d_0 y + d_{0\to 1} y - d_1 z.$$

Now, it is easy to see that

$$0 \to \overline{\mathcal{C}}(D(*1))[-1]\{-1\} \to \overline{\mathcal{C}}(D) \to \overline{\mathcal{C}}(D(*0)) \to 0$$

is a short exact sequence of chain complexes (after a little adjustment of sign), and that $\delta : \overline{\mathcal{H}}^i(D(*0)) \to \overline{\mathcal{H}}^{i+1}(D(*1))[-1]\{-1\}$ is induced by $d_{0\to 1}$. $\square$

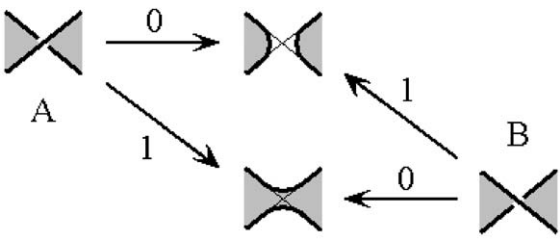
3.2. Properties of black and white coloring of alternating link diagrams

Let D be a link diagram. For brevity of the statements to follow, let us think of diagrams on S^2 rather than on $\mathbb{R}^2$. The regions of S^2 divided by D can be colored black and white in checkerboard fashion.

At each crossing, a coloring of nearby regions falls into one of the two following patterns:

As it is shown above, adjacent alternating crossings have the same coloring pattern of nearby regions. Hence, in a coloring of a nonsplit alternating diagram D, only one of the pattern A or B appears for every crossing. Reversing the coloring changes that pattern.

Resolutions of a colored diagram have induced colorings.



Definition 3.2. For a nonsplit alternating diagram D, *the coloring* of D is the coloring of D in which only pattern A appears. *The coloring* of a resolution of D is the coloring of that resolved diagram induced from the coloring of D.

For the coloring of $D(\emptyset)$ (0-resolutions of pattern A), the trace of each crossing lies in a white region. Now, our claim is:

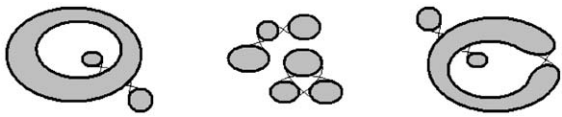
Proposition 3.3. *For a reduced nonsplit alternating diagram D, the components of $D(\emptyset)$ bound nonoverlapping black disks in the coloring of it. Each black disk corresponds to each of the black regions in the coloring of D. Furthermore, every pair of black disks are connected by a chain of black disks, which are connected by the trace of the crossings of D. Also, no trace of crossing connects a black disk to itself.*

Here is visualization of our claim for diagrams of the lefthanded trefoil and the figure 8 knot. (The unbounded black region shown below is a disk in S^2.)

Proof of Proposition 3.3. At each crossing, its 0-resolution separates incident black regions. That gives a correspondence between the black regions in the coloring of D and those in the coloring of $D(\emptyset)$. (While most white regions of D merge in the process.)

In the coloring of $D(\emptyset)$, there is no trace of crossing in black regions. That implies:

- if there is a black region which is not a disk, then D is split.
- if there is a pair of black disks which cannot be connected by any chain, then D is split.
- if there is a trace of crossing connecting a black disk to itself, that crossing is removable, so D is not reduced.



□

Definition 3.4. For a link diagram D, let $c(D)$ be the number of crossings of D, and $o(D)$ be the number of components of $D(\emptyset)$.

For D in Proposition 3.3, $o(D)$ agrees with the number of black disks in the coloring of $D(\emptyset)$.

Let $\mathcal{I}$ be an ordered set of crossings of D. Note that $D(\mathcal{I})$ agrees with $D^!(\emptyset)$, and that $o(D) + o(D^!)$ equals the total number of black and white regions in the coloring of D, which is $c(D) + 2$.

We need one further step for the inductive argument to be used in our proof of Theorem 1.2.

Proposition 3.5. *Let D be a reduced nonsplit alternating link diagram with $c(D) > 0$. Then one of the following holds*:

(A) *There is a pair of black disks in the coloring of $D(\emptyset)$ connected by exactly one crossing.*
(B) *There is a pair of black disks in the coloring of $D^!(\emptyset)$ connected by exactly one crossing.*
(C) *D is a connected sum of D' and the Hopf link, for another nonsplit alternating link diagram D' with $c(D) - 2$ crossings.*

Proof. Since $o(D) + o(D^!) = c(D) + 2$, one of the following holds:

(a) $o(D) > c(D)/2 + 1$.
(b) $o(D^!) > c(D)/2 + 1$.
(c) $o(D) = o(D^!) = c(D)/2 + 1$.

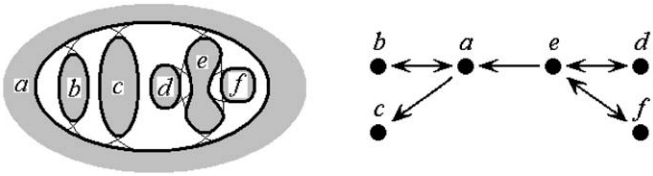
[(a) $\Rightarrow$ (A)] For $o(D)$ black disks to be connected to each other by chains of connected disks, there are at least $o(D) - 1$ different pairs that are connected by crossings. If $2(o(D) - 1) > c(D)$, then at least one of those pairs is connected by exactly one crossing.

[(b) $\Rightarrow$ (B)] Same as (a) $\Rightarrow$ (A).

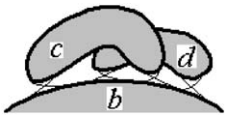
[(c) & not (A) & not (B) $\Rightarrow$ (C)] To fail (A), there are exactly $o(D) - 1$ different pairs that are connected by crossings and those pairs are connected by exactly two crossings.

Consider a graph consists of $o(D)$ vertices and $o(D)-1$ edges. Each vertex represents each black disk. For each pair of black disks connected by two crossings, there is an edge joining the corresponding pair of vertices. This graph is connected, so it is a tree.

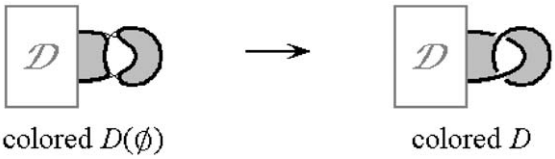
For an edge $\{a, b\}$, mark the a-end of it with arrow if the two crossings connecting the disk a and b are adjacent on the boundary of a. For example,



A vertex of a tree is called a pendent vertex if it is incident with only one edge, and an edge is called a pendent edge if it is incident with a pendent vertex. If a is a pendent vertex, the unique edge incident with a is necessarily marked at the a-end. If b is not a pendent vertex, at least two edges have marked b-end, since the two crossings connecting disks b and c and those connecting disks b and d never alternate.



If $o(D) = 2$, then there is only one edge, that is a pendent edge, and both ends of that edge is marked. For $o(D) > 2$, let p be the number of the pendent vertices. The number of the pendent edges is also p. There are at least $p + 2(o(D) - p)$ marked ends, but the number of nonpendent edges is $o(D) - 1 - p$, so there is at least one pendent edge with both ends marked. That implies (C) (up to relocation of ∞).



□

Lemma 3.6. *For a reduced nonsplit alternating diagram D, $\overline{\mathcal{H}}^{i,j}(D)$ is supported in the box $0 \leqslant i \leqslant c(D)$ and $-o(D) \leqslant j \leqslant 2c(D) - o(D) + 2$, with $\overline{\mathcal{H}}^{0,-o(D)}(D) = \overline{\mathcal{H}}^{c(D),2c(D)-o(D)+2}(D) = \mathbb{Q}$.*

Proof. First of all, it is clear from the construction of $\overline{\mathcal{C}}(D)$ that $\overline{\mathcal{C}}^{i,j}(D) = 0$ unless $0 \leqslant i \leqslant c(D)$.

When a resolution of D is changed to another resolution of D by replacing one 0-resolution by 1-resolution, the number of components either increases or decreases by one. That ensures $\overline{\mathcal{C}}^{i,j}(D)$ to be supported in $-o(D) \leqslant j \leqslant 2c(D) - o(D) + 2$.

Proposition 3.3 implies that $D(\emptyset)$ has one more component than any $D(a)$ has, because two black disks merge into one in the process. In terms of $\overline{\mathcal{C}}^{i,j}(D)$,

this means

$$\overline{\mathcal{C}}^{i,j}(D) = \begin{cases} \mathbb{Q} & \text{if } i = 0, j = -o(D), \\ 0 & \text{if } i > 0, j = -o(D), \end{cases}$$

so one half of the result follows.

For the other half, look at the other end. $D(\mathcal{I}) = D^!(\emptyset)$ also has one more component than any $D(\mathcal{I} - \{a\}) = D^!(a)$ has, so that

$$\overline{\mathcal{C}}^{i,j}(D) = \begin{cases} \mathbb{Q} & \text{if } i = c(D), j = 2c(D) - o(D) + 2, \\ 0 & \text{if } i < c(D), j = 2c(D) - o(D) + 2. \end{cases} \qquad \square$$

Let D be a diagram satisfying (A) in Proposition 3.5. Let a be a crossing of D connecting a pair of black disks that no other crossing connects. Choose an ordering of $\mathcal{I}$ in which a comes the last. Then, $D(*0)$ still has the property that $D(*0)(\emptyset)$ has one more component than any $D(*0)(b)$ has. The use of (A) is that it allows $D(*1)$ to have that property, too.

Corollary 3.7. *In the above setting, $\overline{\mathcal{H}}^{i,j}(D(*0))$ is supported in the box $0 \leqslant i \leqslant c(D(*0))$ and $-o(D(*0)) \leqslant j \leqslant 2c(D(*0)) - o(D(*0)) + 2$, with $\overline{\mathcal{H}}^{0,-o(D(*0))}(D(*0)) = \mathbb{Q}$, and $\overline{\mathcal{H}}^{i,j}(D(*1))$ is supported in the box $0 \leqslant i \leqslant c(D(*1))$ and $-o(D(*1)) \leqslant j \leqslant 2c(D(*1)) - o(D(*1)) + 2$, with $\overline{\mathcal{H}}^{0,-o(D(*1))}(D(*1)) = \overline{\mathcal{H}}^{c(D(*1)),2c(D(*1))-o(D(*1))+2}(D(*1)) = \mathbb{Q}$.*

Finally, to apply induction hypothesis to $D(*0)$ and $D(*1)$ later on, they need to be nonsplit alternating.

Proposition 3.8. *In the above setting, $D(*0)$ and $D(*1)$ are nonsplit alternating.*

Proof. Alternating property is easy to see.

To be nonsplit, their black disks in the induced coloring have to be connected. That is clear for $D(*1)$. For $D(*0)$, if the black disks of $D(\emptyset)$ are disconnected after removing a, then a was a removable crossing in D, which contradicts D being reduced. $\square$

3.3. Signature of an alternating link

This section consists of the result of [6] and an application to alternating links, to relate the shift with the signature in Theorem 1.2.

Definition 3.9 (*Goeritz matrix: following Section 1 of [6]*). Let D be an oriented link diagram. Color the regions of $\mathbb{R}^2$ (or S^2) divided by D in checkerboard fashion. Denote the white regions by $X_0, X_1, \ldots, X_n$. Assume that each crossing is incident to two distinct white regions. Assign an incidence number $\eta(a) = \pm 1$ to each crossing a as

in the figure below. For $0 \leqslant i, j \leqslant n$ define

$$g_{ij} = \begin{cases} -\sum_a \text{ incident to both } x_i \text{ and } x_j\ \eta(a) & \text{for } i \neq j, \\ -\sum_{0 \leqslant k \leqslant n, k \neq i} g_{ik} & \text{for } i = j. \end{cases}$$

The Goeritz matrix $G(D)$ of D is the $n \times n$ (not $(n+1) \times (n+1)$!) symmetric matrix $G(D) = (g_{ij})_{1 \leqslant i, j \leqslant n}$.

The signature of an oriented link can be obtained from the signature of Goeritz matrix of its diagram by adding a correction term.

Theorem 3.10 (*Gordon and Litherland [6, Theorem 6]*). *For an oriented link L,*

$$\sigma(L) = \operatorname{sign} G(D) - \mu(D)$$

for its diagram D, where $\mu(D) = \sum \eta(a)$, *summed over all crossings a of type II.* (*See the figure above.*)

Proposition 3.11. *For an oriented nonsplit alternating link L and a reduced alternating diagram D of L,* $\sigma(L) = o(D) - y(D) - 1$.

Proof. In the reversed coloring of D, $\eta(a) = 1$ for any crossing a, components of the resolution $D(\emptyset)$ bound nonoverlapping white disks, $+$ crossings are of the type II, and $-$ crossings are of the type I, so that $g_{ij} \leqslant 0$ for $i \neq j$, $g_{ii} \geqslant 0$, and $\mu(D) = y(D)$. Reducedness of D ensures that each crossing is incident to two distinct white regions.

Since

$$\sum_{1 \leqslant i, j \leqslant n} g_{ij} x_i x_j = \sum_{1 \leqslant i < j \leqslant n} |g_{ij}|(x_i - x_j)^2 + \sum_{i=1}^{n} |g_{i0}| x_i^2 \geqslant 0,$$

$G(D)$ is a positive-definite matrix, and hence,

$$\sigma(L) = \operatorname{sign} G(D) - \mu(D) = \operatorname{rank} G(D) - y(D) = o(D) - 1 - y(D). \qquad \square$$

3.4. Proof of Theorem 1.2

The proof is based on induction on the number of crossings of a link diagram. First, we prove the theorem for some number $s(L)$ instead of $\sigma(L)$, and then, show $s(L) = \sigma(L)$. For convenience of proof, we will restate Theorem 1.2 in more detailed and extended form as follows.

Theorem 3.12. *For any oriented nonsplit alternating link L, $Kh(L)(t,q)$ is supported in two lines* $\deg(q) = 2\deg(t) - \sigma(L) \pm 1$, *its nonzero coefficient of the smallest degree in t is on the line* $\deg(q) = 2\deg(t) - \sigma(L) - 1$, *its nonzero coefficient of the largest degree in t is on the line* $\deg(q) = 2\deg(t) - \sigma(L) + 1$, *and those coefficients are* 1.

In other words,

$$Kh(L)(t,q) = \sum_{i=p}^{m} (a_i t^i q^{2i-\sigma(L)-1} + b_i t^i q^{2i-\sigma(L)+1})$$

for some $p \leqslant m$ with $a_p = b_m = 1$.

The lines $\deg(q) = 2\deg(t) - s(L) - 1$ and $\deg(q) = 2\deg(t) - s(L) + 1$ will be called the upper diagonal, and the lower diagonal, respectively, and the positions of $a_p = 1$ and $b_m = 1$ will be referred to as the top at $(p, 2p - s(L) - 1)$ and the bottom at $(m, 2m - s(L) + 1)$, thinking of the table of coefficients in which the powers of t increase from left to right, and the powers of q increase from top to bottom. These terms will be applied to Khovanov's cohomology groups as well.

Theorem 3.13. *For any nonsplit alternating link diagram D, $\overline{\mathcal{H}}^{i,j}(D)$ is supported in two lines $j = 2i - s \pm 1$ for some integer s with the top and bottom on the upper diagonal and the lower diagonal, respectively.*

Proof. For the base case, the theorem holds for the unknotted diagram of unknot.

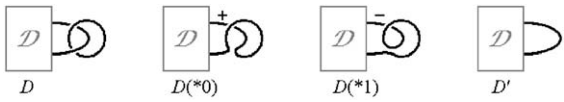
Assume that the statement is true for all such diagrams with less than c crossings. Let D be a nonsplit alternating link diagram with c crossings. If D is not reduced, then $\overline{\mathcal{H}}(D)$ is a shift of $\overline{\mathcal{H}}(D')$ for some such diagram D' with less than c crossings, so the statement is true for D as well.

Let D be reduced. By Corollary 2.7, it is enough to show the theorem for either D or $D^!$. So, we may assume that D has the property (A) or (C) in Proposition 3.5.

[Case (A)] The induction hypothesis applies to $D(*0)$ and $D(*1)$. $\overline{\mathcal{H}}(D(*0))$ is supported in two lines with the top at $(0, -o(D(*0)))$, and $\overline{\mathcal{H}}(D(*1))$ is also supported in two lines with the top at $(0, -o(D(*1)))$.

Since $o(D) = o(D(*0)) = o(D(*1)) + 1$, the upper diagonal and the lower diagonal of $\overline{\mathcal{H}}(D(*0))$ agree with those of $\overline{\mathcal{H}}(D(*1))[-1]\{-1\}$. By Theorem 3.1 and Lemma 3.6, $\overline{\mathcal{H}}(D)$ is supported in two lines with the top at $(0, -o(D))$ and the bottom at $(c, 2c - o(D) + 2)$.

[Case (C)] Our $D(*0)$ and $D(*1)$ are as below, and the induction hypothesis applies to D'.



Choose orientations for D', $D(*0)$ and $D(*1)$ accordingly. $\overline{\mathcal{H}}(D')$, $\overline{\mathcal{H}}(D(*0))$ and $\overline{\mathcal{H}}(D(*1))[-1]\{-1\}$ are shift of each other as follows.

$$\begin{aligned}\overline{\mathcal{H}}(D(*0)) &= \mathcal{H}(D(*0))[-x(D(*0))]\{-2x(D(*0)) + y(D(*0))\}\\ &= \mathcal{H}(D')[-x(D')]\{-2x(D') + y(D') + 1\}\\ &= \overline{\mathcal{H}}(D')[0]\{1\},\end{aligned}$$

$$\begin{aligned}\overline{\mathcal{H}}(D(*1))[-1]\{-1\} &= \mathcal{H}(D(*1))[-x(D(*1)) - 1]\{-2x(D(*1)) + y(D(*1)) - 1\}\\ &= \mathcal{H}(D')[-x(D') - 2]\{-2x(D') + y(D') - 3\}\\ &= \overline{\mathcal{H}}(D')[-2]\{-3\}.\end{aligned}$$

By induction hypothesis, $\overline{\mathcal{H}}(D(*0))$ is supported in two lines with the top at $(0, -o(D') - 1)$, and $\overline{\mathcal{H}}(D(*1))[-1]\{-1\}$ is also supported in two lines with the top at $(2, -o(D') + 3)$. Their upper diagonals and lower diagonals agree.

Again, by Theorem 3.1 and Lemma 3.6, $\overline{\mathcal{H}}(D)$ is supported in two lines with the top at $(0, -o(D)) = (0, -o(D') - 1)$ and the bottom at $(c, 2c - o(D) + 2)$. $\square$

Let L be an oriented nonsplit alternating link and D be a reduced alternating diagram of L. From Theorem 3.13, we can conclude that $\mathcal{H}(D) = \overline{\mathcal{H}}(D)[x(D)]\{2x(D) - y(D)\}$ has the top at $(-x(D), -2x(D) + y(D) - o(D))$. Since the top is on the upper diagonal, our $s(L)$ equals $o(D) - y(D) - 1$.

In Proposition 3.11, we saw that $\sigma(L) = o(D) - y(D) - 1$. That finishes the proof of Theorem 1.2.

4. An endomorphism of the Khovanov invariant

In this section, we prove Theorem 1.4. The strategy of our proof is as follows. We define a map Φ of degree (1,4) from the Khovanov invariant $\mathcal{H}(L)$ of any oriented link L to itself, which pairs most of $\mathcal{H}(L)$. This map Φ added to the coboundary map d gives rise to a new cohomology theory which can be computed explicitly. Then, we compare the cohomology groups of Φ on $\mathcal{H}(L)$ with the new cohomology groups of $\Phi + d$.

4.1. Definition (on chain level)

Theorem 1.4 states that there is an almost pairing of cohomology groups of degree difference (1,4), so it is natural to think of a map of degree (1,4) on the cohomology groups.

On chain level, the map Φ is defined in the same fashion as the coboundary map. Instead of

$$\begin{aligned}
\mathbf{1}\otimes\mathbf{1} &\overset{m}{\longmapsto} \mathbf{1},\\
\mathbf{1}\otimes\mathbf{x},\mathbf{x}\otimes\mathbf{1} &\overset{m}{\longmapsto} \mathbf{x},\\
\mathbf{x}\otimes\mathbf{x} &\overset{m}{\longmapsto} 0,\\
\mathbf{1} &\overset{\Delta}{\longmapsto} \mathbf{1}\otimes\mathbf{x}+\mathbf{x}\otimes\mathbf{1},\\
\mathbf{x} &\overset{\Delta}{\longmapsto} \mathbf{x}\otimes\mathbf{x}.
\end{aligned}$$

Φ's assignment is as follows:

$$\begin{aligned}
\mathbf{1}\otimes\mathbf{1},\mathbf{1}\otimes\mathbf{x},\mathbf{x}\otimes\mathbf{1} &\overset{m_\Phi}{\longmapsto} 0,\\
\mathbf{x}\otimes\mathbf{x} &\overset{m_\Phi}{\longmapsto} \mathbf{1},\\
\mathbf{1} &\overset{\Delta_\Phi}{\longmapsto} 0,\\
\mathbf{x} &\overset{\Delta_\Phi}{\longmapsto} \mathbf{1}\otimes\mathbf{1}.
\end{aligned}$$

This new multiplication m_Φ is commutative and associative.

$$m_\Phi(m_\Phi(x\otimes y)\otimes z) = m_\Phi(x\otimes m_\Phi(y\otimes z)) = 0 \text{ for any } x,y,z\in\mathbf{A}.$$

The comultiplication Δ_Φ is also cocommutative and coassociative.

$$(\Delta_\Phi\otimes id)\circ\Delta_\Phi(z) = (id\otimes\Delta_\Phi)\circ\Delta_\Phi(z) = 0 \text{ for any } z\in\mathbf{A}.$$

They also satisfy

$$\Delta_\Phi\circ m_\Phi = (m_\Phi\otimes id)\circ(id\otimes\Delta_\Phi),$$

since

$$\Delta_\Phi\circ m_\Phi(y\otimes z) = (m_\Phi\otimes id)(y\otimes\Delta_\Phi(z)) = 0 \text{ for any } y,z\in\mathbf{A}.$$

As discussed in Section 2.1.4, these properties are enough for ensuring $\Phi^2=0$.

Although m_Φ and Δ_Φ cannot have any compatible unit or counit, $(\mathbf{A},m_\Phi,\Delta_\Phi)$ is isomorphic to $(\mathbf{A}^*,\Delta_\Phi^*,m_\Phi^*)$.

4.2. Invariance of Φ

We would like to see Φ is well defined on $\mathcal{H}(L)$. For that, Φ should (anti)commute with d and be invariant under the Reidemeister moves.

4.2.1. Anticommutativity with d

From the viewpoint of Section 2.1.4, we only need to check the following identities:

(1) $m \circ (m_\Phi \otimes id) + m_\Phi \circ (m \otimes id) \;=\; m \circ (id \otimes m_\Phi) + m_\Phi \circ (id \otimes m)$,
(2) $(\Delta \otimes id) \circ \Delta_\Phi + (\Delta_\Phi \otimes id) \circ \Delta \;=\; (id \otimes \Delta) \circ \Delta_\Phi + (id \otimes \Delta_\Phi) \circ \Delta$,
(3) $\Delta \circ m_\Phi + \Delta_\Phi \circ m \;=\; (m \otimes id) \circ (id \otimes \Delta_\Phi) + (m_\Phi \otimes id) \circ (id \otimes \Delta)$.

Proof. (2) can be checked in the following table.

	$(\Delta \otimes id) \circ \Delta_\Phi$	$(\Delta_\Phi \otimes id) \circ \Delta$	$(id \otimes \Delta) \circ \Delta_\Phi$	$(id \otimes \Delta_\Phi) \circ \Delta$
$\mathbf{1} \mapsto$	0	$\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$	0	$\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$
$\mathbf{x} \mapsto$	$\mathbf{1} \otimes \mathbf{x} \otimes \mathbf{1} + \mathbf{x} \otimes \mathbf{1} \otimes \mathbf{1}$	$\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{x}$	$\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{x} + \mathbf{1} \otimes \mathbf{x} \otimes \mathbf{1}$	$\mathbf{x} \otimes \mathbf{1} \otimes \mathbf{1}$

(1) is deduced from (2) since both (A, m, Δ) and (A, m_Φ, Δ_Φ) are self dual.
A table for (3) follows.

	$\Delta \circ m_\Phi$	$\Delta_\Phi \circ m$	$(m \otimes id) \circ (id \otimes \Delta_\Phi)$	$(m_\Phi \otimes id) \circ (id \otimes \Delta)$
$\mathbf{1} \otimes \mathbf{1} \mapsto$	0	0	0	0
$\mathbf{1} \otimes \mathbf{x} \mapsto$	0	$\mathbf{1} \otimes \mathbf{1}$	$\mathbf{1} \otimes \mathbf{1}$	0
$\mathbf{x} \otimes \mathbf{1} \mapsto$	0	$\mathbf{1} \otimes \mathbf{1}$	0	$\mathbf{1} \otimes \mathbf{1}$
$\mathbf{x} \otimes \mathbf{x} \mapsto$	$\mathbf{1} \otimes \mathbf{x} + \mathbf{x} \otimes \mathbf{1}$	0	$\mathbf{x} \otimes \mathbf{1}$	$\mathbf{1} \otimes \mathbf{x}$

$\square$

4.2.2. Invariance under the Reidemeister moves

We also want Φ to commute with the isomorphisms in Section 2.2.2.
[Type I] The isomorphism was given by

$$\begin{aligned} \rho : \qquad \overline{\mathcal{C}}(D') \qquad &\longrightarrow \overline{\mathcal{C}}(D)\{1\}, \\ y \otimes \mathbf{1} + z \otimes \mathbf{x} + x \;&\longmapsto z \end{aligned}$$

for $y \otimes \mathbf{1} + z \otimes \mathbf{x} \in \overline{\mathcal{C}}(D'(*0)) \approx \overline{\mathcal{C}}(D) \otimes \mathbf{A}$ and $x \in \overline{\mathcal{C}}(D'(*1))[-1]\{-1\}$.
Then,

$$\begin{aligned} (\rho\Phi' - \Phi\rho)(y \otimes \mathbf{1} + z \otimes \mathbf{x} + x) \;&=\; \rho(\Phi'_0(y) \otimes \mathbf{1} + \Phi'_0(z) \otimes \mathbf{x} \\ &\quad + \Phi'_{0\to 1}(y \otimes \mathbf{1} + z \otimes \mathbf{x}) - \Phi'_1(x)) - \Phi(z) \\ &=\; \Phi(z) - \Phi(z) = 0. \end{aligned}$$

[Type II] Let $d(z) = 0, z \in \overline{\mathcal{C}}^i(D)[-1]\{-1\}$.

$$\begin{aligned}
(-1)^i(\rho\Phi - \Phi'\rho)(z) &= (-1)^i\rho\Phi(z) - \Phi'(z + \alpha(z)) \\
&= -(\Phi'_{01}(z) + \alpha\Phi'_{01}(z)) - (-\Phi'_{01}(z) - \Phi'_{10}\alpha(z) \\
&\quad + \Phi'_{01\to 11}(z) + \Phi'_{10\to 11}\alpha(z)) \\
&= d'_{01\to 11}\Phi'_{01}(z) \otimes \mathbf{1} - \Phi'_{11}d'_{01\to 11}(z) \otimes \mathbf{1} - \Phi'_{01\to 11}(z) \\
&= -d'(\Phi'_{01\to 11}(z) \otimes \mathbf{1}) \\
&\quad - \left(\overline{\mathcal{C}}(D'(*11))[-2]\{-2\} \text{ component of } (d'\Phi' + \Phi'd')(z)\right) \otimes \mathbf{1} \\
&= -d'(\Phi'_{01\to 11}(z) \otimes \mathbf{1}).
\end{aligned}$$

[Type III] If $d(x + \beta(x) + y) = 0$, then $\overline{\mathcal{C}}(D(*100))[-1]\{-1\}$ component of $d(x + \beta(x) + y)$, that is $-d_{100}(x)$, equals 0.

In $\overline{\mathcal{C}}(D)$,

$$\begin{aligned}
\Phi(x + \beta(x) + y) &= -\Phi_{100}(x) + \Phi_{100\to 110}(x) + \Phi_{100\to 101}(x) - \Phi_{010}(d_{100\to 110}x \otimes \mathbf{1}) \\
&\quad + \Phi_{010\to 110}(d_{100\to 110}x \otimes \mathbf{1}) + \Phi_{010\to 011}(d_{100\to 110}x \otimes \mathbf{1}) - \Phi_1(y) \\
&= -\Phi_{100}(x) + \Phi_{100\to 110}(x) + \Phi_{100\to 101}(x) \\
&\quad - \Phi_{110}d_{100\to 110}x \otimes \mathbf{1} - \Phi_1(y) \\
&\overset{(1)}{=} -\Phi_{100}(x) + \beta(-\Phi_{100}(x)) + d_{110}\Phi_{100\to 110}x \otimes \mathbf{1} + \Phi_{100\to 110}(x) \\
&\quad + \Phi_{100\to 101}(x) - \Phi_1(y) \\
&\overset{(2)}{\sim} -\Phi_{100}(x) + \beta(-\Phi_{100}(x)) + \Phi_{100\to 110\approx 011}x + \Phi_{100\to 101}(x) - \Phi_1(y).
\end{aligned}$$

(1) is from

$$\begin{aligned}
0 &= \left(\overline{\mathcal{C}}(D(*110))[-2]\{-2\} \text{ component of } (d\Phi + \Phi d)(x)\right) \\
&= d_{100\to 110}(-\Phi_{100}x) + d_{110}\Phi_{100\to 110}x + \Phi_{100\to 110}(-d_{100}x) \\
&\quad + \Phi_{110}d_{100\to 110}x,
\end{aligned}$$

and

$$\begin{aligned}
\beta(-\Phi_{100}(x)) &= d_{100\to 110}(-\Phi_{100}(x)) \otimes \mathbf{1} \\
&= -(d_{110}\Phi_{100\to 110}x + \Phi_{110}d_{100\to 110}x) \otimes \mathbf{1}.
\end{aligned}$$

(2) is from

$$\begin{aligned} d\alpha(\Phi_{100\to 110}x) &= -d_{010}(\Phi_{100\to 110}x \otimes \mathbf{1}) + d_{010\to 110}(\Phi_{100\to 110}x \otimes \mathbf{1}) \\ &\quad + d_{010\to 011}(\Phi_{100\to 110}x \otimes \mathbf{1}) \\ &= -d_{110}(\Phi_{100\to 110}x) \otimes \mathbf{1} - \Phi_{100\to 110}x + \Phi_{100\to 110\approx 011}x. \end{aligned}$$

Similarly, in $\overline{\mathcal{C}}(D')$,

$$\begin{aligned} \Phi'(x + \beta'(x) + y) &\sim -\Phi'_{010}(x) + \beta'(-\Phi'_{010}(x)) - \Phi'_{010\to 110\approx 101}(x) \\ &\quad + \Phi'_{010\to 011}(x) - \Phi'_1(y) \\ &= \rho(-\Phi_{100}(x) + \beta(-\Phi_{100}(x)) + \Phi_{100\to 101}(x) \\ &\quad + \Phi_{100\to 110\approx 011}x - \Phi_1(y)). \end{aligned}$$

4.2.3. Example

Let T be the lefthanded trefoil with the diagram D in Section 2.1.6. We have computed $\mathcal{H}(T) = \mathcal{H}(D)$ in Section 2.1.6.

Since Φ is of degree (1, 4), the only possible place Φ can be nontrivial is from $\mathcal{H}^{-3,-9}(T)$ to $\mathcal{H}^{-2,-5}(T)$. The value of Φ at a generator $[\mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x}]$ of $\mathcal{H}^{-3,-9}(T) = \mathbb{Q}$ is

$$\Phi([\mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x}]) = \left[\begin{bmatrix} \mathbf{1} \otimes \mathbf{x} \\ \mathbf{1} \otimes \mathbf{x} \\ \mathbf{1} \otimes \mathbf{x} \end{bmatrix}\right],$$

which is a generator of $\mathcal{H}^{-2,-5}(T) = \mathbb{Q}$, so Φ is nontrivial there.

$$\begin{array}{c|cccc} j \backslash i & -3 & -2 & -1 & 0 \\ \hline -9 & \mathbb{Q} & & & \\ -7 & & \cong & & \\ -5 & & \mathbb{Q} & & \\ -3 & & & & \mathbb{Q} \\ -1 & & & & \mathbb{Q} \end{array} \qquad \mathbb{Q} \xrightarrow{\Phi} \mathbb{Q}$$

4.3. $\Phi + d$ and change of variables

4.3.1. Change of variables

Let us forget the grading and make a change of variables as follows:

$$\mathbf{a} = \mathbf{x} + \mathbf{1}, \qquad \mathbf{b} = \mathbf{x} - \mathbf{1}.$$

Since $(\Phi + d)^2 = \Phi^2 + \Phi d + d\Phi + d^2 = 0$, we can regard $\Phi + d$ as a new coboundary map. Its assignment is as follows:

$$\begin{aligned}
\mathbf{a} \otimes \mathbf{a} &\overset{m_{(\Phi+d)}}{\longmapsto} 2\mathbf{a},\\
\mathbf{a} \otimes \mathbf{b}, \mathbf{b} \otimes \mathbf{a} &\overset{m_{(\Phi+d)}}{\longmapsto} 0,\\
\mathbf{b} \otimes \mathbf{b} &\overset{m_{(\Phi+d)}}{\longmapsto} -2\mathbf{b},\\
\mathbf{a} &\overset{\Delta_{(\Phi+d)}}{\longmapsto} \mathbf{a} \otimes \mathbf{a},\\
\mathbf{b} &\overset{\Delta_{(\Phi+d)}}{\longmapsto} \mathbf{b} \otimes \mathbf{b}.
\end{aligned}$$

Definition 4.1. For an oriented diagram D, $\overline{H}(D)$ is the cohomology of the chain complex $(\overline{\mathcal{C}}(D), \Phi + d)$, and $H(D)$ is that of $(\mathcal{C}(D), \Phi + d)$.

4.3.2. Invariance of $H(D)$ under the Reidemeister moves

It would not be interesting if we can define only $H(D)$, but not $H(L)$. Our proof of invariance follows [7, Chapter 5] (summarized in Section 2.2.2) with only the multiplication and comultiplication maps replaced by those of $\Phi + d$. Details are left to the reader. See Section 2.2.2 for figures.

[Type I] Define

$$\tilde{X}_1 = \mathrm{Ker}(\Phi' + d')_{0\to 1}$$

and

$$\tilde{X}_2 = \left\{ y \otimes \frac{1}{2}(\mathbf{a} - \mathbf{b}) + z \,|\, y \in \overline{\mathcal{C}}(D), z \in \overline{\mathcal{C}}(D'(*1))[-1] \right\}.$$

$\overline{\mathcal{C}}(D')$ is decomposed as $\tilde{X}_1 \oplus \tilde{X}_2$ as chain complexes, $\tilde{X}_2$ is acyclic, and

$$\begin{aligned}
\tilde{\rho} : \quad \tilde{X}_1 &\longrightarrow \overline{\mathcal{C}}(D)\\
y \otimes \mathbf{a} + z \otimes \mathbf{b} &\longmapsto y + z
\end{aligned}$$

induces an isomorphism between $H(D')$ and $H(D)$.

[Type II] This time,

$$\begin{aligned}
\tilde{X}_1 &= \{z + \tilde{\alpha}(z) | z \in \overline{\mathcal{C}}(D'(*01))[-1]\},\\
\tilde{X}_2 &= \{z + (\Phi' + d')y | z, y \in \overline{\mathcal{C}}(D'(*00))\},\\
\tilde{X}_3 &= \left\{z + y \otimes \frac{1}{2}(\mathbf{a} - \mathbf{b}) | z, y \in \overline{\mathcal{C}}(D'(*11))[-2]\right\},
\end{aligned}$$

where $\tilde{\alpha}(z) = -(\Phi' + d')_{01\to 11}(z) \otimes \frac{1}{2}(\mathbf{a}-\mathbf{b}) \in \overline{\mathcal{C}}(D'(*11))[-1] \otimes \mathbf{A} \approx \overline{\mathcal{C}}(D'(*10))[-1]$.

Then, $\overline{\mathcal{C}}(D') = \tilde{X}_1 \oplus \tilde{X}_2 \oplus \tilde{X}_3$ as chain complexes, $\tilde{X}_2$ and $\tilde{X}_3$ are acyclic, and

$$\begin{aligned}
\tilde{\rho} : \overline{\mathcal{C}}^i(D)[-1] \approx \overline{\mathcal{C}}^i(D'(*01))[-1] &\longrightarrow \tilde{X}_1 \cap \overline{\mathcal{C}}^i(D')\\
z &\longmapsto (-1)^i(z + \tilde{\alpha}(z))
\end{aligned}$$

induces an isomorphism.

[Type III] Let $\tilde{\alpha}, \tilde{\beta}, \tilde{\alpha}', \tilde{\beta}'$ be maps of complexes given by

$$\begin{aligned}
\tilde{\alpha} :\ \overline{\mathcal{C}}(D(*110))[-2] &\longrightarrow \overline{\mathcal{C}}(D(*010))[-1] \approx \overline{\mathcal{C}}(D(*110))[-1] \otimes \mathbf{A},\\
z &\longmapsto z \otimes \frac{1}{2}(\mathbf{a} - \mathbf{b})\\
\tilde{\beta} :\ \overline{\mathcal{C}}(D(*100))[-1] &\longrightarrow \overline{\mathcal{C}}(D(*010))[-1],\\
z &\longmapsto \tilde{\alpha}(\Phi + d)_{100\to 110}(z)\\
\tilde{\alpha}' :\ \overline{\mathcal{C}}(D'(*110))[-2] &\longrightarrow \overline{\mathcal{C}}(D'(*100))[-1] \approx \overline{\mathcal{C}}(D'(*110))[-1] \otimes \mathbf{A},\\
z &\longmapsto z \otimes \frac{1}{2}(\mathbf{a} - \mathbf{b})\\
\tilde{\beta}' :\ \overline{\mathcal{C}}(D'(*010))[-1] &\longrightarrow \overline{\mathcal{C}}(D'(*100))[-1].\\
z &\longmapsto -\tilde{\alpha}'(\Phi' + d')_{010\to 110}(z)
\end{aligned}$$

$\overline{\mathcal{C}}(D)$ and $\overline{\mathcal{C}}(D')$ can be decomposed as below.

$$\begin{aligned}
\overline{\mathcal{C}}(D) &= \tilde{X}_1 \oplus \tilde{X}_2 \oplus \tilde{X}_3,\\
\tilde{X}_1 &= \{x + \tilde{\beta}(x) + y | x \in \overline{\mathcal{C}}(D(*100))[-1], y \in \overline{\mathcal{C}}(D(*1))[-1]\},\\
\tilde{X}_2 &= \{x + (\Phi + d)y | x, y \in \overline{\mathcal{C}}(D(*000))\},\\
\tilde{X}_3 &= \{\tilde{\alpha}(x) + (\Phi + d)\tilde{\alpha}(y) | x, y \in \overline{\mathcal{C}}(D(*110))[-2]\};
\end{aligned}$$

$$\begin{aligned}
\overline{\mathcal{C}}(D') &= \tilde{Y}_1 \oplus \tilde{Y}_2 \oplus \tilde{Y}_3,\\
\tilde{Y}_1 &= \{x + \tilde{\beta}'(x) + y | x \in \overline{\mathcal{C}}(D'(*010))[-1], y \in \overline{\mathcal{C}}(D'(*1))[-1]\},\\
\tilde{Y}_2 &= \{x + (\Phi' + d')y | x, y \in \overline{\mathcal{C}}(D'(*000))\},\\
Y_3 &= \{\tilde{\alpha}'(x) + (\Phi' + d')\tilde{\alpha}'(y) | x, y \in \overline{\mathcal{C}}(D'(*110))[-2]\}.
\end{aligned}$$

As before, $\tilde{X}_2, \tilde{X}_3, \tilde{Y}_2, \tilde{Y}_3$ are acyclic, $\overline{\mathcal{C}}(D(*100))[-1]$ and $\overline{\mathcal{C}}(D'(*010))[-1]$, $\overline{\mathcal{C}}(D(*1))[-1]$ and $\overline{\mathcal{C}}(D'(*1))[-1]$ are naturally isomorphic, and $\tilde{X}_1$ is isomorphic to $\tilde{Y}_1$ via

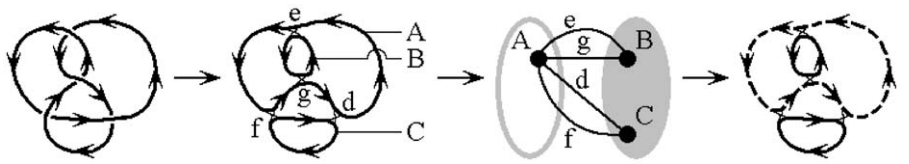
$$\tilde{\rho} : x + \tilde{\beta}(x) + y \longmapsto x + \tilde{\beta}'(x) + y.$$

4.4. Cohomology theory of $\Phi + d$

4.4.1. Resolutions of oriented links in orientation preserving way

Consider the resolution of an oriented link diagram in orientation preserving way, that is, 0-resolutions for $+$ crossings and 1-resolutions for $-$ crossings. This is the standard way to get a Seifert surface from a diagram of an oriented link.

Consider a graph whose vertices are in 1–1 correspondence with the components of this resolution and whose edges connecting a pair of vertices are in 1–1 correspondence with the crossings connecting the corresponding pair of components. Since the Seifert surface obtained by the method above is oriented, the graph has no cycle consisting of odd number of edges, so the vertices of this graph can be parted into two groups in a way that the two end-points of each edge do not belong to the same group. Accordingly, the components of this resolution can be parted into two groups in a way that each crossing connects a component in one group to another in the other group. This partition does not depend on the position of ∞.



4.4.2. Hodge theory

We can give an inner product on a chain complex so that monomials in $\mathbf{a}, \mathbf{b}$ form an orthonormal basis, then the adjoint $(\Phi + d)^*$ of $\Phi + d$ is defined as follows:

$$\mathbf{a} \otimes \mathbf{a} \overset{m_{(\Phi+d)^*}}{\longmapsto} \mathbf{a},$$

$$\mathbf{a} \otimes \mathbf{b}, \mathbf{b} \otimes \mathbf{a} \overset{m_{(\Phi+d)^*}}{\longmapsto} 0,$$

$$\mathbf{b}\otimes\mathbf{b} \overset{m_{(\Phi+d)^*}}{\longmapsto} \mathbf{b},$$

$$\mathbf{a} \overset{\Delta_{(\Phi+d)^*}}{\longmapsto} 2\mathbf{a}\otimes\mathbf{a},$$

$$\mathbf{b} \overset{\Delta_{(\Phi+d)^*}}{\longmapsto} -2\mathbf{b}\otimes\mathbf{b}.$$

By Hodge theory,

$$\overline{H}^i(D) \cong \mathrm{Ker}((\Phi+d):\overline{\mathcal{C}}^i(D)\to\overline{\mathcal{C}}^{i+1}(D))\cap\mathrm{Ker}((\Phi+d)^*:\overline{\mathcal{C}}^i(D)\to\overline{\mathcal{C}}^{i-1}(D)).$$

4.4.3. Computation of H

For an n component link, there are 2^{n-1} different (relative) orientations. Each of them gives a distinct resolution when the link is resolved in orientation preserving way. Since no crossing connects components in the same group in such a resolution, the two monomials consisting of **a** for the components in one group and **b** for those in the other clearly belong to $\mathrm{Ker}(\Phi+d)\cap\mathrm{Ker}(\Phi+d)^*$. The claim is that these are all, i.e., others are linear combinations of these.

Theorem 4.2. *The dimension of $H(L)=\oplus_{i\in\mathbb{Z}}H^i(L)$ for an oriented link L of n components equals to 2^n.*

Proof. As is in Theorem 3.1, we have a long exact sequence of cohomology groups.

$$\cdots\to\overline{H}^{i-1}(D(*0))\to\overline{H}^{i-1}(D(*1))\to\overline{H}^i(D)\to\overline{H}^i(D(*0))\to\overline{H}^i(D(*1))\to\cdots$$

So, $\dim\overline{H}(D)$ does not exceed $\dim\overline{H}(D(*0))+\dim\overline{H}(D(*1))$.

Let us prove for knots and two component links first, using induction on the number of crossings.

It clearly holds for the unknot. If we have a knot with a minimal diagram D of c crossings, then one of $D(*0)$, $D(*1)$ is a knot, the other is a two component link, and they have one fewer crossings. Suppose $D(*0)$ is a knot. One of the two relative orientations of $D(*1)$ is compatible with the relative orientation of D, and the other is compatible with that of $D(*0)$. Then in the long exact sequence, the two generators of $\overline{H}^i(D(*0))$ map to the two generators of $\overline{H}^i(D(*1))$ coming from the relative orientation compatible with that of $D(*0)$. If $D(*1)$ is a knot, then the two generators of $\overline{H}^i(D(*0))$ coming from the relative orientation compatible with that of $D(*1)$ map to the two generators of $\overline{H}^i(D(*1))$. Hence,

$$2\leqslant\dim\overline{H}(D)\leqslant\dim\overline{H}(D(*0))+\dim\overline{H}(D(*1))-4=2.$$

Let D be a minimal diagram for a two component link with $c(D) = c$. If D is a disjoint union of two knot diagrams D_1 and D_2, then $\overline{H}(D) = \overline{H}(D_1) \otimes \overline{H}(D_2)$ with $c(D_1), c(D_2) \leqslant c$, and therefore,

$$4 \leqslant \dim \overline{H}(D) \leqslant \dim \overline{H}(D_1) \cdot \dim \overline{H}(D_2) = 4.$$

If not, choose a crossing so that both $D(*0)$ and $D(*1)$ are knot diagrams. Then,

$$4 \leqslant \dim \overline{H}(D) \leqslant \dim \overline{H}(D(*0)) + \dim \overline{H}(D(*1)) = 4.$$

An n component link diagram D is either a disjoint union of link diagrams of fewer components or can be resolved to two link diagrams of $n-1$ components. The proof that $\dim \overline{H}(D) = 2^n$ goes the same way as above. $\square$

We can tell exactly to which $H^i(L)$ those generating monomials belong.

Proposition 4.3. *Let L be an oriented n component link, $S_1, \ldots, S_n$ be its components, and ℓ_{jk} be the linking number of S_j and S_k. Then,*

$$\dim H^i(L) = 2 \cdot \left| \left\{ E \subset \{2, \ldots, n\} \;\middle|\; \left(\sum_{j \in E, k \notin E} 2\ell_{jk} \right) = i \right\} \right|$$

Proof. Let O be the given (relative) orientation of L with a diagram D, and O' be another (relative) orientation obtained by reversing the orientations of S_j, $j \in E \subset \{2, \ldots, n\}$. Let $x(D), y(D)$ be the numbers of positive crossings and negative crossings with respect to O, and $x'(D), y'(D)$ be those with respect to O'.

Since a resolution in orientation preserving way is resolving $+$ crossings to its 0-resolutions and $-$ crossings to its 1-resolutions, those two monomials corresponding to O' appear in $\mathcal{H}^{x'(D)-x(D)}(D)$.

On the other hand, the number of negative crossings among the crossings between S_j and S_k does not change if none or both of the orientations of S_j and S_k are reversed, and if only one of them is reversed, the number is changed by

$$\begin{aligned} 2 \cdot \ell_{jk} &= \left(y - x \text{ among the crossings between } S_j \text{ and } S_k \right) \\ &= \left(x' - x \text{ among the crossings between } S_j \text{ and } S_k \right). \end{aligned}$$

Therefore,

$$x'(D) - x(D) = \sum_{j \in E, k \notin E} 2\ell_{jk}. \qquad \square$$

4.5. Proof of Theorem 1.4

In the previous section, we have computed H. To prove Theorem 1.4, we are going to relate H to $\mathrm{Ker}(\Phi : \mathcal{H} \to \mathcal{H})/\mathrm{Im}(\Phi : \mathcal{H} \to \mathcal{H})$.

Theorem 4.4. *For any H-thin link L,*

$$H(L) \cong \frac{\mathrm{Ker}(\Phi : \mathcal{H}(L) \to \mathcal{H}(L))}{\mathrm{Im}(\Phi : \mathcal{H}(L) \to \mathcal{H}(L))}.$$

Proof. For any link L, $(\overline{\mathcal{C}}(L), d(L), \Phi(L))$ is a double complex up to an index shift. In the spectral sequence of the double complex $(\overline{\mathcal{C}}(L), d(L), \Phi(L))$, the E_2 and E_∞ term are isomorphic to $\mathrm{Ker}(\Phi : \mathcal{H}(L) \to \mathcal{H}(L))/\mathrm{Im}(\Phi : \mathcal{H}(L) \to \mathcal{H}(L))$ and $H(L)$, respectively.

If L is H-thin, then d_2 and thereafter must be zero maps because of their degree. Hence,

$$H(L) \cong E_\infty \cong E_2 \cong \frac{\mathrm{Ker}(\Phi : \mathcal{H}(L) \to \mathcal{H}(L))}{\mathrm{Im}(\Phi : \mathcal{H}(L) \to \mathcal{H}(L))}. \qquad \square$$

4.6. Extension of Theorem 1.4 for alternating links

Theorems 1.2 and 1.4 imply that the Khovanov invariant of an alternating knot determines and is determined by its Jones polynomial and signature. This can be extended to oriented alternating links.

Let L be a link satisfying the hypothesis in Proposition 4.3 and nonsplit alternating. We already know that two monomials corresponding to an orientation O' belong to $H^{\sum_{j\in E, k\notin E} 2\ell_{jk}}(L)$.

To find out their degrees, consider $(\mathbf{a}\otimes\cdots\otimes\mathbf{a}\otimes\mathbf{b}\otimes\cdots\otimes\mathbf{b})\pm(\mathbf{b}\otimes\cdots\otimes\mathbf{b}\otimes\mathbf{a}\otimes\cdots\otimes\mathbf{a})$.

$$\begin{aligned}&(\mathbf{a}\otimes\cdots\otimes\mathbf{a}\otimes\mathbf{b}\otimes\cdots\otimes\mathbf{b}) + (\mathbf{b}\otimes\cdots\otimes\mathbf{b}\otimes\mathbf{a}\otimes\cdots\otimes\mathbf{a})\\ &\quad = 2\cdot\sum(\text{monomials in } \mathbf{1} \text{ and } \mathbf{x} \text{ with even number of } \mathbf{1}),\end{aligned}$$

$$\begin{aligned}&(\mathbf{a}\otimes\cdots\otimes\mathbf{a}\otimes\mathbf{b}\otimes\cdots\otimes\mathbf{b}) - (\mathbf{b}\otimes\cdots\otimes\mathbf{b}\otimes\mathbf{a}\otimes\cdots\otimes\mathbf{a})\\ &\quad = (-)2\cdot\sum(\text{monomials in } \mathbf{1} \text{ and } \mathbf{x} \text{ with odd number of } \mathbf{1}).\end{aligned}$$

Degrees of monomials in $\mathbf{1}$ and $\mathbf{x}$ with even numbers of $\mathbf{1}$ are the same in (mod 4), degrees of monomials in $\mathbf{1}$ and $\mathbf{x}$ with odd numbers of $\mathbf{1}$ are also the same in (mod 4), and those two differ by 2 in (mod 4). Therefore, we can conclude that one of $(\mathbf{a}\otimes\cdots\otimes\mathbf{a}\otimes\mathbf{b}\otimes\cdots\otimes\mathbf{b})\pm(\mathbf{b}\otimes\cdots\otimes\mathbf{b}\otimes\mathbf{a}\otimes\cdots\otimes\mathbf{a})$ is mapped to upper diagonal, the other to lower diagonal.

Now, Theorem 1.4 can be extended as follows.

Theorem 4.5. *For an n component oriented nonsplit alternating link L with its components $S_1, \ldots, S_n$ and linking numbers ℓ_{jk} of S_j and S_k,*

$$Kh(L)(t,q) = q^{-\sigma(L)} \left\{ (q^{-1}+q) \left(\sum_{E \subset \{2,\ldots,n\}} (tq^2)^{\sum_{j \in E, k \notin E} 2\ell_{jk}} \right) + (q^{-1} + tq^2 \cdot q) Kh'(L)(tq^2) \right\}$$

for some polynomial $Kh'(L)$.

Hence, with the linking numbers of pairs of components provided, the Khovanov invariant of an oriented nonsplit alternating link determines and is determined by its Jones polynomial and signature.

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