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# Computing Steiner points for gradient-constrained minimum networks

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ABSTRACT

Let $T_g$ be a *gradient-constrained minimum network*, that is, a minimum length network spanning a given point set in 3-dimensional space with edges that are constrained to have gradients no more than an upper bound $m$. Such networks occur in underground mines where the slope of the declines (tunnels) cannot be too steep due to haulage constraints. Typically the gradient is less than 1/7. By defining a new metric, the gradient metric, the problem of finding $T_g$ can be approached as an unconstrained problem where embedded edges can be considered as straight but measured according to their gradients. All edges in $T_g$ are *labelled* by their gradients, being $<m$, $=m$ or $>m$, in the gradient metric space. Computing Steiner points plays a central role in constructing locally minimum networks, where the topology is fixed. A degree-3 Steiner point is *labelled minimal* if the total length of the three adjacent edges is minimized for a given labelling. In this paper we derive the formulae for computing labelled minimal Steiner points. Then we develop an algorithm for computing locally minimal Steiner points based on information from the labellings of adjacent edges. We have tested this algorithm on uniformly distributed sets of points; our results help in finding gradient-constrained minimum networks.



## 1. Introduction

In this section we review some fundamental properties of Steiner minimum trees and gradient-constrained networks and give the terminology used in this paper. In addition, the underground mine design problem, which motivated this paper, will be briefly described.

### 1.1. The Steiner tree problem

Given a point set $N$ in a metric space, the *Steiner tree problem* asks for a network $T$ spanning $N$ with minimum length. The solution $T$ is a tree with vertex set $V \supseteq N$ that spans the points in $N$, which are called *terminals*. The points in $V \setminus N$ are added to shorten the network. They are called *Steiner points*. The graph structure of a network is referred to as its *topology* and denoted by $t$. Further discussion of the Steiner tree problem can be found in the book by Hwang et al. [1].

In the classical Steiner tree problem, $N$ lies in the Euclidean $d$-space, ($d \geq 2$). In this case the following proposition holds.

**Proposition 1.1.** *Suppose $T$ is a minimum length network in Euclidean space, then*

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(1) *$T$ has a tree topology, called a* Steiner minimum tree,
(2) *all edges in $T$ are straight line segments,*
(3) *any angle at a Steiner point is no more than* 120° *(*angle condition*),*
(4) *the degree of a Steiner point is at most three, that is, $T$ has a Steiner topology and is called a* Steiner tree.

Where we wish to emphasize the dependence of $T$ on $N$, or both $N$ and $t$, the network will be denoted by $T(N)$ or $T(N, t)$, respectively. Similarly, if we want to further explicitly specify the dependence of the minimal tree on the positions of the Steiner points, the set $S = \{s_1, s_2, \ldots\}$, then we will write $T$ as $T = T(N, t, S)$ where $S$ is a third variable for the function $T$.

A Steiner tree $T = T(N, t)$ is *locally optimal* on $N$ if it is the shortest among all those trees having the same topology $t$ but where the positions of the Steiner points differ. Hence, finding a locally optimal Steiner tree is a continuous optimisation problem. However, a Steiner minimum tree $T$ is a *globally optimal* tree, that is, optimal over all Steiner topologies $t$ as well as over all possible positions for the Steiner points for each $t$. Note that the topology $t$ is a discrete variable, hence the Steiner tree problem, as a global optimisation problem, is both a continuous as well as discrete optimisation problem. The Euclidean Steiner tree problem is NP-hard [1]; the number of topologies is exponential in the size of $N$.

**Remark 1.1.** The Euclidean Steiner tree problem is even harder than other well-known combinatorial network optimisation problems such as the travelling salesman problem due to this hybrid optimisation. It is more difficult in higher-dimensional Euclidean space than in the plane, as finding the locally minimum Steiner tree, even on a set of four points in Euclidean 3-space, is not algebraically solvable in general [2].

To date there are many variants on the classical Steiner tree problem with metrics related to their different applications, *e.g.* rectilinear Steiner trees and λ-trees in VLSI design [3,4], flow-dependent networks in communications [5] and Steiner minimum trees in molecular biology [6]. In recent years, a further variant has been the problem of finding Steiner minimum trees in 3-dimensional space in which the (absolute) gradients of all edges are no more than an upper bound $m$. The application is in underground mine access design [7–10] and is described further in Section 1.2. Such networks are called gradient-constrained minimum networks and this problem has been shown to be NP-hard [11]. In fact all of the aforementioned variants have been shown to be NP-hard.

### 1.2. Gradient metric and gradient-constrained networks

Let $x_p, y_p, z_p$ denote the Cartesian coordinates of a point $p$ in Euclidean space. By the gradient $g(pq)$ of an edge $pq$ we mean the absolute value of the slope from $p = (x_p, y_p, z_p)$ to $q = (x_q, y_q, z_q)$, that is,

$$g(pq) \stackrel{\text{def}}{=} \frac{|z_q - z_p|}{\sqrt{(x_q - x_p)^2 + (y_q - y_p)^2}}.$$

Here, $|\cdot|$ denotes simply the absolute value. If $g(pq) \le m$, then $pq$ is a straight line segment joining $p$ and $q$ and is referred to as *straight*. However, if $g(pq) > m$, then $pq$ cannot be represented, or embedded in 3-dimensional Euclidean space, as a straight line segment without violating the gradient constraint. Instead it can be represented by a zig-zag line joining $p$ and $q$ with each segment having gradient $m$. Such edges are referred to as *bent*.

Suppose $o$ is the origin and $p = (x_p, y_p, z_p),\ q = (x_q, y_q, z_q)$ are points in 3-space. The *gradient metric* can be defined in terms of the Euclidean and vertical metrics, denoted by $|\cdot|_{\mathrm{e}}$ and $|\cdot|_{\mathrm{v}}$ respectively:

$$|pq|_{\mathrm{g}} = \begin{cases} |pq|_{\mathrm{e}} = \sqrt{(x_p - x_q)^2 + (y_p - y_q)^2 + (z_p - z_q)^2} & \text{if } g(pq) \le m, \\ |pq|_{\mathrm{v}} = (\sqrt{1 + m^{-2}})|z_p - z_q| & \text{if } g(pq) \ge m. \end{cases} \tag{1}$$

It is easy to see that the unit ball for the gradient metric looks like a drum, that is, like a ball whose North and South poles are cut off by horizontal planes of equal distance from the ball's centre (Fig. 1). Therefore the gradient metric, and consequently the length function of a gradient-constrained network, is convex but not strictly convex. A convex set is strictly convex if the relative open line segment between any two points on the boundary of the convex set lies strictly in the interior of the convex set.

A *gradient-constrained minimum network* $T_g = T_g(N)$ is a minimum length network spanning a given (finite) point set $N$ in 3-dimensional space with edges whose (absolute) gradients are all no more than an upper bound $m$. Such networks occur in underground mines where ore is accessed and hauled to the surface via a network of gently sloping declines (tunnels). The declines cannot be too steep (Fig. 2) as driving up steep inclines requires more fuel and there is more wear and tear on the trucks. The typical maximum gradient of the tunnels is about 1:7 (≈0.14) [8,9]. These networks may be many kilometres long. The development costs of the declines and associated haulage costs over the life of a mine are a major part of the overall mine costs. Saving 10 m in length of a decline results in savings of about 100,000 US dollars over the life of the mine. Hence minimising the length of the network is key to the viability and profitability of an underground mine.

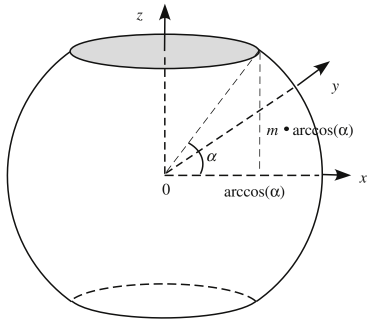


**Fig. 1.** The unit ball of the gradient metric.

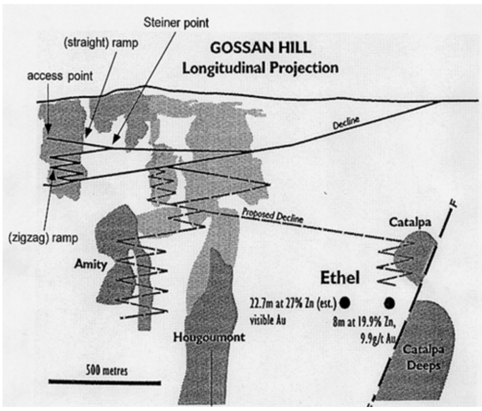


**Fig. 2.** An underground mining network.

### *1.3. Locally minimal gradient-constrained networks*

$T_g$ is a minimum length network in a special metric space, the *gradient metric space*. The gradient metric allows us to consider the gradient-constrained minimum network problem as an unconstrained problem. That is, we can assume all edges in $T_g$ are straight lines whose lengths are given by the gradient metric depending on their gradients. The gradient metric is convex (see [8], Lemma 4.2) and hence we immediately obtain the following lemma.

**Lemma 1.2.** *If a gradient-constrained network has a fixed tree topology but variable Steiner points, then its length is a convex (but not strictly convex) function of the positions of the Steiner points.*

As a globally optimal network must be locally optimal, the globally optimal solution can theoretically be found by considering all the locally optimal solutions where the topology is fixed. For an algorithm that finds the globally optimal network to be efficient, it is important to understand the fundamental properties of locally minimal networks in order that these networks can be rapidly found. General algorithms for global optimisation problems, such as a branch-and-bound algorithm, have been developed and can be used in the gradient-constrained network problem. Hence our research is focused on being able to find *locally minimal gradient-constrained networks*, or LMGCNs.

Let $S$ be the set of Steiner points and $E$ the set of edges of the locally minimal gradient-constrained network $T_g^\ell$ for a given set $N$ of terminals and a given topology $t$, where the superscript $\ell$ means 'local'. Let $|e|_m$ denote the length of an edge $e \in E$ measured by the gradient metric with maximum gradient $m$, and let $L_m(T_g^\ell) = \sum_{e\in E} |e|_m$ denote the tree length, that is, the sum of the edge lengths of $T_g^\ell$. Then the problem of finding an LMGCN $T_g^\ell = T_g^\ell(N, t)$ can be expressed as an unconstrained network optimisation problem in terms of the gradient metric as follows.

*The LMGCN problem*:

*Given*: A point set $N$ in 3-space, a topology $t$ on $N$, and a real positive number $m$
*Objective*: $\min_{s_i\in S}\left(L_{g^m}(T_g^\ell(N, t, S))\right)$.

The problem of finding an LMGCN is a continuous optimisation problem as the Steiner points can lie anywhere in a gradient metric space. However in finding the positions of the Steiner points, there is also a discrete optimisation problem to be considered. The Steiner points in LMGCNs can be classified into different types, called *labellings*, according to whether the gradients of their incident edges are less than, equal to or greater than $m$ when embedded as straight lines in 3-dimensional gradient metric space. The feasible minimal labellings have been characterized by Brazil et al. [7] and will be described in Section 2. Let $T_g(s)$ denote the subtree consisting of the edges incident to a Steiner point $s$ in $T_g$. A Steiner point $s$ with a given labelling as well as its induced tree $T_g(s)$ will be called *labelled minimal* if $T_g(s)$ cannot be shortened by label-preserving perturbations of $s$. Finding the labelled minimal trees is a necessary step in finding the locally minimal tree $T_g(s)$ which cannot be shortened by any perturbation of $s$, regardless of its labelling. Thus we can tackle the LMGCN problem by solving a finite number of labelling optimisation subproblems and then selecting the best from the solutions of the subproblems. This allows the continuous optimisation problem to be approached as a discrete problem.

Our paper is based on the practical underground mining problem where the maximum gradient is less than or equal to 1/7. Brazil et al. [7] showed that in this case the degree of Steiner points is either three or four, and the location of a degree-4 Steiner point is easily determined. Therefore, in this paper we will only study the problem of computing locally minimal Steiner points of degree three in gradient-constrained minimum networks. In a previous paper [10] we studied the properties of labelled minimal Steiner points, as well as the necessary and sufficient conditions for Steiner points to be locally minimal. In this paper we first derive the formulae for computing a labelled minimal degree-3 Steiner point $s$ incident to 3 terminals $a$, $b$ and $c$ for different labellings, then we show how the information given by the triangle $\triangle abc$ can help us to rule out the infeasible labellings, or types, of $s$. Moreover, using the variational argument [12] we can further reduce the number of feasible types of $s$.

Based on these considerations we develop an algorithm for computing locally minimal degree-3 Steiner points in LMGCNs. This algorithm has been tested on 10,000 (uniformly distributed) random sets of three points. The results show that in about 98% of cases the locally minimal Steiner points can be determined by solving linear or quadratic equations, and only in 0.43% of the cases the Steiner points, in theory, cannot be exactly determined due to the nature of the equations, and thus an approximation scheme is essentially required. The results also show that in about 63% of the cases the minimal Steiner points collapse into their adjacent points. Such Steiner points are called *degenerate*. Again, note that global minimality implies local minimality, hence this experimental result implies that even in a global minimal gradient-constrained network the probability of the number of degenerate Steiner points is as high as 63% that can greatly reduce the burden of calculation in finding global minimal gradient-constrained networks.

## 2. Computing labelled minimal Steiner points

This section is divided into two subsections. In the first subsection we briefly review some fundamental properties of gradient-constrained networks and then in the second subsection we derive the formulae for degree-3 labelled Steiner points of all five types considering the labellings, namely (b/mm), (m/mm), (m/fm), (m/ff) and (f/ff), one by one in detail.

### 2.1. Fundamental properties of gradient-constrained networks

Let $pq$ be an edge in a gradient-constrained tree. If $g(pq) < m, g(pq) = m$ or $g(pq) > m$, then $pq$ is labelled 'f' (that is, 'flat edge'), 'm' (that is, 'edge of gradient $m$') or 'b' (that is, 'bent edge'), and called an *f-edge, m-edge* or *b-edge*, respectively. Denote the horizontal plane through a Steiner point $s$ by $\mathcal{H}_s$. For simplicity, a point or an edge will be said to be above (or below) $s$ if it is above (or below) $\mathcal{H}_s$. Suppose a degree-3 (non-degenerate) Steiner point $s$ has one edge $as$ above $s$ and two edges $bs$, $cs$ below $s$. Let $g_a, g_b, g_c$ denote the respective labels of these edges. Then we say the *labelling* (or *type*) of $s$ is $(g_a/g_bg_c)$. Similarly, if $as$ and $bs$ are above $s$ and $cs$ is below $s$, then the labelling is $(g_ag_b/g_c)$. If $s$ is of degree-4 and the fourth edge $ds$ lies above $s$ as does $as$ while $bs$ and $cs$ lie below $s$, then the labelling of $s$ is denoted by $(g_ag_d/g_bg_c)$. The following are fundamental properties of gradient-constrained minimum networks.

*Properties of gradient-constrained minimum networks* (*see* [7]):

(1) *The degree of a Steiner point $s$ is either three or four if $m < 1$.*
(2) *Up to symmetry there are five (non-degenerate) feasibly optimal labellings (f/ff), (m/ff), (m/mf), (m/mm) and (b/mm) if $s$ is of degree-3 and if $m < 1$.*
(3) *If $s$ is of degree 4 and if $m < 0.38$, then there is only one feasibly optimal labelling, namely (mm/mm). Moreover, two edges, say $as$ and $ds$, lie in a vertical plane, as do the other two edges, say $bs$, $cs$.*

It follows from Property (3) that if $m < 0.38$ and $s$ is of degree four, then $s$ is locally minimal and the location of $s$ is easily determined by the equations $g(as) = g(bs) = g(cs) = g(ds) = m$. As discussed in Section 1, a typical gradient in the underground mine design problem is around 1:7 or roughly 0.14 so we can assume $m < 0.38$. We will now consider the problem where $s$ is a degree-3 Steiner point and the minimum length tree $T_g$ contains only three terminals. Clearly the five feasible labellings in Property (2) are mutually exclusive. It follows that the locally minimal Steiner point, which must be labelled minimal, has a unique labelling.

A *right circular cone* is a cone whose generating lines all have the same gradient. We denote a right circular cone with apex $p$ by $\mathcal{C}_p$. Such cones play a central role in determining labelled minimal Steiner points. The two endpoints of the long axis in a non-circular ellipse are referred to as the *vertices* of an ellipse, and this term also refers to the pair of nearest points in two branches of a hyperbola. Below are some results cited from an unpublished research note of the second author [13] in which an exhaustive study on conic sections produced by two right circular cones was made. The following results are cited from the note and a sketch of the proof is given to make the paper more self-contained.

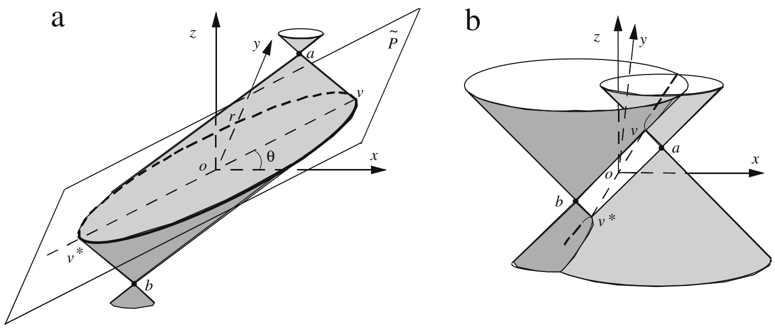


**Fig. 3.** The intersection of two right vertical cones.

**Theorem 2.1.** *Suppose $\mathcal{C}_a$, $\mathcal{C}_b$ are two right circular cones whose generating lines have the same gradient $m$ (Fig. 3). Assume $a = (u, 0, h)$, $b = (-u, 0, -h)$, $u \geq 0$, $h \geq 0$.*

(1) *The intersection $\mathcal{E}_{ab}$ of $\mathcal{C}_a$, $\mathcal{C}_b$ is an ellipse or hyperbola depending on whether $g(ab) > m$ or $g(ab) < m$, respectively.*
(2) *There are two extreme points $v = (h/m, 0, mu)$, $v^* = (-h/m, 0, -mu)$ on $\mathcal{E}_{ab}$ which are the vertices of the ellipse or hyperbola.*
(3) *Let $s$ be a point on the ellipse or hyperbola. If $\mathcal{E}_{ab}$ is an ellipse then $|sa| + |sb| = |z_b - z_a|/\sin\alpha$ is constant, where $\alpha = \arctan(m)$. If $\mathcal{E}_{ab}$ is a hyperbola then $|sa| + |sb|$ achieves a minimum when $s = v$ (or $s = v^*$).*

**Proof** (*Sketch*). Let $s = (x_s, y_s, z_s)$ be a point on the intersection of the right circular cones $\mathcal{C}_a$ and $\mathcal{C}_b$. Then the two cones are determined by

$$(x_s - u)^2 + y_s^2 = \frac{(z_s - h)^2}{m^2} \quad \text{and} \quad (x_s + u)^2 + y_s^2 = \frac{(z_s + h)^2}{m^2}. \tag{2}$$

Subtracting the first equation from the second, we obtain the following equation for the intersection $\mathcal{E}_{ab}$:

$$\frac{z_s}{x_s} = m^2 \frac{u}{h}, \quad \text{that is } z_s h = m^2 u x_s. \tag{3}$$

This equation implies that $\mathcal{E}_{ab}$ lies on a plane $\tilde{P}$ which intersects the $y$-axis. The point $a$ lies on one side of the plane and $b$ on the other (see Fig. 3). Hence, $\mathcal{E}_{ab}$ is an ellipse or a hyperbola. Using standard calculations it is not hard to prove the other statements in the theorem. □

The variational argument [12] has proved to be a powerful tool in the study of the Steiner tree problem in Euclidean space. The idea is that when one endpoint of an edge is perturbed in a certain direction, the change in the edge length can be measured by the directional derivative of the edge length. The variational argument that we use for the gradient-constrained Steiner tree problem is as follows: For a minimum tree $T_g$, the directional derivative of $|T_g|$ must be greater than or equal to zero when its Steiner points are perturbed in any direction. Note that under an arbitrarily small perturbation the only edges which can change their labellings are m-edges. Suppose $e = sa$ is an edge in $T_g$, and $s$ is a Steiner point which is perturbed to $s'$ in direction $\mathbf{u}$. Let $\dot{e}_{\mathbf{u}}$ (or simply $\dot{e}$ if $\mathbf{u}$ is known) denote the directional derivative of the length of $e$. The following theorem is easily proved (see [7] Lemma 1):

**Theorem 2.2.** (i) *If $e$ is an f-edge, then $\dot{e}_{\mathbf{u}} = -\cos(\angle ass')$.*
(ii) *If $e$ is a b-edge, then $\dot{e}_{\mathbf{u}} = -\cos(\angle zss')\sqrt{1 + m^{-2}}$ where $z$ is a point on the vertical line through $s$ such that $\angle asz \leq \pi/2$.*
(iii) *If $e$ is an m-edge, then $\dot{e}_{\mathbf{u}}$ is equal to either $-\cos(\angle ass')$ or $-\cos(\angle zss')\sqrt{1 + m^{-2}}$, depending on whether $g(s'a) \leq m$ or $g(s'a) > m$.*

In the following subsection we derive all the formulae used in our algorithm to compute labelled minimal Steiner points. In the derivation we suppose $s = (x_s, y_s, z_s)$ is a degree-3 Steiner point with one edge $as$ lying above $\mathcal{H}_s$ and two edges $bs$, $cs$ lying below $\mathcal{H}_s$. To determine the three coordinates of $s$, a system $\mathcal{S}$ of three equations is needed. If $s$ has $k$ ($0 \leq k \leq 3$) m-edges, then the $k$ gradient constraints of the m-edges constitute part of the system $\mathcal{S}$. If $k = 3$, then $\mathcal{S}$ is determined, otherwise, the other $(3 - k)$ edges are f-edges or b-edges. These $(3 - k)$ equations in $\mathcal{S}$ can be obtained by the variational argument: Any perturbation of $s$ cannot reduce the length of the 3-point tree $T_g$, that is, $\dot{L}(T_g) = \sum_{i=1}^{3} \dot{e}_i = 0$, where $e_i$, $i = 1, 2, 3$, are three edges incident to $s$, according to Theorem 2.2.

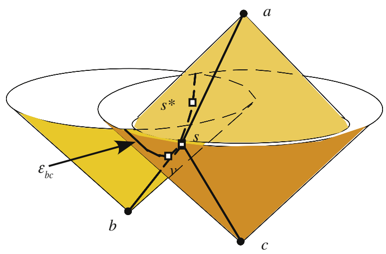


**Fig. 4.** Labelling (m/mm).

### 2.2. *Formulae for computing labelled minimal Steiner points*

By the properties of gradient-constrained minimum networks as described at the beginning of Section 2.1, if the maximum gradient is less than one, then there are five feasible labellings of degree-3 Steiner points: (b/mm), (m/mm), (m/fm), (m/ff) and (f/ff). Below we derive either an explicit formula or an approximation scheme for computing a labelled Steiner point $s$, where $s$ is one of the first four types of labellings. They are derived one by one, from the easiest to the hardest, according to the degree of the system determining $s$ (as listed in the above order). For the final type (f/ff) of labelling, the gradient-constrained Steiner point is the ordinary Steiner point in Euclidean space, hence the explicit formula is omitted.

#### 2.2.1. *Labelling* (b/mm)

In the (b/mm) case, the system $\mathcal{S}$ determining the labelled minimal Steiner point $s$ contains the two equations $g(bs) = g(cs) = m$. It can be shown, using Theorems 2.1 and 2.2, that $s$ must be the vertex of the hyperbola $\mathcal{E}$ formed by the intersection of the cones $\mathcal{C}_b$ and $\mathcal{C}_c$, as depicted for the (m/mm) case in Fig. 4, since a perturbation of $as$ cannot shorten $T$ by minimality. This gives the third equation in $\mathcal{S}$. The vertex of the hyperbola $\mathcal{E}$ lies in the vertical plane through $bc$. It follows that the projections of $b, s, c$ on the $xy$-plane are collinear, and that the system $\mathcal{S}$ is

$$\left\{ g(sb) = m, g(sc) = m, \frac{y_s - y_c}{x_s - x_c} = \frac{y_b - y_c}{x_b - x_c} \right\}.$$

$\mathcal{S}$ is a linear system, and the unique solution of $\mathcal{S}$, the labelled minimal Steiner point named $s_{\mathrm{v}}$, is

$$s_{\mathrm{v}} = \left( x_c + \frac{(x_b - x_c)}{2f_{\mathrm{v}}}(f_{\mathrm{v}} + z_b - z_c), y_c + \frac{(y_b - y_c)}{2f_{\mathrm{v}}}(f_{\mathrm{v}} + z_b - z_c), \frac{f_{\mathrm{v}} + z_b + z_c}{2} \right), \tag{4}$$

where $f_{\mathrm{v}} = m\sqrt{(x_b - x_c)^2 + (y_b - y_c)^2}$.

#### 2.2.2. *Labelling* (m/mm)

In the (m/mm) case, the labelled minimal Steiner point $s = (x_s, y_s, z_s)$ is determined by the system $\mathcal{S} : \{g(as) = g(bs) = g(cs) = m\}$. As in the (b/mm) case, the equations $g(bs) = g(cs) = m$ determine that $s$ lies on the upper branch of a hyperbola $\mathcal{E}$ formed by cones $\mathcal{C}_b$ and $\mathcal{C}_c$ (Fig. 4). Since the hyperbola $\mathcal{E}$ and the cone $\mathcal{C}_a$ are both convex, they intersect at most at two points, that is, the above system $\mathcal{S}$ has at most two real solutions, say $s, s^*$. Moreover, by Theorem 2.1, if $z_s \le z_{s^*}$, then $|as| + |bs| + |cs| \le |as^*| + |bs^*| + |cs^*|$, that is, $s$ is the labelled minimal Steiner point. Without loss of generality, after a transformation we can assume

$$b = (u, 0, h), \qquad c = (-u, 0, -h), \quad u \ge 0,\ h \ge 0,\ (h/u) \le m,$$

and we have $z_a \ge \max\{z_b, z_c\} = h$, say. By solving $g(bs) = g(cs) = m$ with respect to $x_s, z_s$, we obtain the parametric form of the hyperbola with $y_s$ as a parameter. Note that $y_s$ has the same sign as $y_a$. Define $y_s = \kappa y$ where

$$\kappa = \mathrm{sign}(y_a) = \begin{cases} -1 & \text{if } y_a < 0; \\ 0 & \text{if } y_a = 0; \\ 1 & \text{if } y_a > 0. \end{cases}$$

Then the parametric form for this labelled minimal Steiner point $s$, named $s_{\mathrm{m}}$, is

$$s_{\mathrm{m}} = \left( \frac{h}{m} f(y), \kappa y, um \cdot f(y) \right), \tag{5}$$

where $f(y) = \sqrt{1 + m^2y^2/(m^2u^2 - h^2)}$. Using this parametric expression for $s$, the equation $g(as) = m$, the third equation needed to determine $s$ becomes

$$(2m^2y_a)y + m^2(u^2 - x_a^2 - y_a^2) + z_a^2 - h^2 = 2m(hx_a - uz_a)f(y).$$

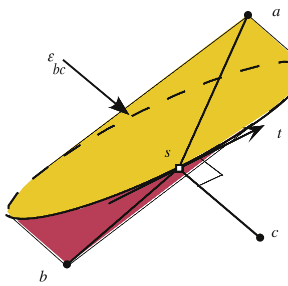


**Fig. 5.** Labelling (m/mf).

Squaring both sides and simplifying yields a quadratic equation with respect to $y$:

$$c_2y^2 + c_1y + c_0 = 0, \tag{6}$$

where

$$c_2 = 4m^4\left((h^2 - m^2u^2)y_a^2 - (hx_a - z_au)^2\right),$$
$$c_1 = 4m^2y_a(h^2 - m^2u^2)\left(m^2(u^2 - x_a^2 - y_a^2) + z_a^2 - h^2\right),$$
$$c_0 = (h^2 - m^2u^2)\left((m^2(u^2 - x_a^2 - y_a^2) + z_a^2 - h^2)^2 - 4m^2(hx_a - uz_a)^2\right).$$

Note that from the coordinates of $b$ and $c$ and by Theorem 2.1 the three points $b, c$ and $v$, the vertex of the hyperbola, lie in the vertical $xz$-plane. Because all three edges are m-edges, it can be seen by the variational argument that the smaller $s_z$, the shorter the length of $T_g$. In the two solutions of Eq. (6)

$$y_1 = \frac{-c_1 - \sqrt{c_1^2 - 4c_2c_0}}{2c_2}, \qquad y_2 = \frac{-c_1 + \sqrt{c_1^2 - 4c_2c_0}}{2c_2},$$

clearly $y_1$ is smaller than $y_2$. Because $s_z$ is proportional to $f(y)$ by Expression (5) and hence proportional to $y$ by the definition of $f(y)$, of the two solutions $y_1, y_2$ we should choose $y_1$, and then $s_m$ is given by expression (5).

### 2.2.3. *Labelling* (m/mf)

In the (m/mf) case, the system $\mathcal{S}$ determining the labelled minimal Steiner point $s = (x_s, y_s, z_s)$ contains the equations $g(as) = g(bs) = m$, and hence $s$ lies on the ellipse $\mathcal{E}$ formed by cones $\mathcal{C}_a$ and $\mathcal{C}_b$ (Fig. 5). Without loss of generality, after a transformation we suppose

$$a = (u, 0, h), \qquad b = (-u, 0, -h), \quad u \geq 0,\ h \geq 0,\ (h/u) \geq m.$$

Then $\mathcal{E}$ can be expressed in a parametric form with $x$ as a parameter, and the labelled minimal Steiner point $s$ will have the following form:

$$s = \left(x, \kappa\frac{\sqrt{(m^2u^2 - h^2)(x^2m^2 - h^2)}}{mh}, \frac{m^2u}{h}x\right), \quad \kappa = \mathrm{sign}(y_c).$$

It follows that the tangent vector at $s$ is

$$\mathbf{t} = \left(1, \kappa\frac{mx}{h}\frac{\sqrt{(m^2u^2 - h^2)}}{\sqrt{(m^2x^2 - h^2)}}, \frac{m^2u}{h}\right).$$

By Theorem 2.2, if $s$ is minimal then $cs$ is perpendicular to $\mathcal{E}$, that is, $\vec{sc} \cdot \mathbf{t} = 0$. Since $\mathcal{E}$ is convex, $cs$, as the shortest of the two normal lines from $c$ to $\mathcal{E}$, is unique. After squaring and simplifying, the equation $\vec{sc} \cdot \mathbf{t} = 0$ can be transformed into a quartic polynomial

$$f_e(x) = c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0 = 0,$$

where

$$c_4 = m^2t_1^2, \qquad c_3 = 2m^2t_1t_2, \qquad c_2 = -t_1^2 + m^2t_2^2 + y_c^2hm^3(h^2 - u^2m^2),$$
$$c_1 = -2t_1t_2, \qquad c_0 = -t_2^2,$$
$$t_1 = -hu^2m^3 - u^2m^4 - h^2 + h^3m, \qquad t_2 = x_ch + uz_cm^2.$$

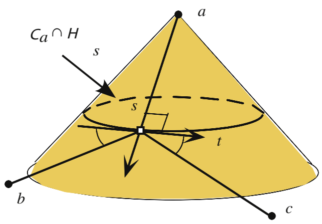


**Fig. 6.** Labelling (m/ff).

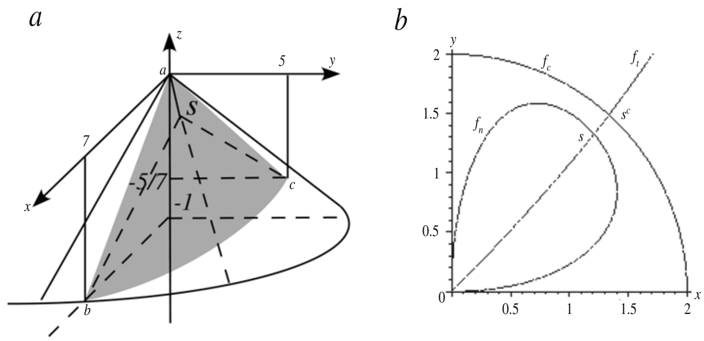

**Fig. 7.** The Steiner point for labelling (m/ff).

Let $\tilde{P}$ be the plane containing $\mathcal{E}$. Then the projection of $cs$ on $\tilde{P}$ is the shortest line perpendicular to $\mathcal{E}$ in $\tilde{P}$, and to find such a normal in a plane, a quartic equation must be solved [4]. Theoretically we could use Ferrari's standard method for solving degree-4 polynomials, but for simplicity a numerical approximation program $P_{\mathrm{slv}}(f(x))$, a *polynomial solver*, is used in our algorithm, which finds the solution of a polynomial $f(x)$ in a given domain if the solution exists and is unique. Since the domain of $x$ is $[-h/m, h/m]$, let

$$x_s = P_{\mathrm{slv}}\,(f_{\mathrm{e}}(x) \mid x \in [-h/m, h/m])$$

be the unique solution of $f_{\mathrm{e}}(x)$ in the domain $[-h/m, h/m]$. Then

$$s_{\mathrm{e}} = \left(x_s, \kappa\sqrt{\frac{(m^2u^2 - h^2)(x_s^2 m^2 - h^2)}{mh}}, \frac{m^2 u}{h} x_s\right), \tag{7}$$

is the labelled minimal Steiner point. This labelled minimal Steiner point is named $s_{\mathrm{e}}$ or, more precisely, $s_{\mathrm{e}}(c)$ since $sc$ is the f-edge.

**Remark 2.1.** Recall the definition of labelling given in Section 2: The labelling (m/mf) represents the situation that $bs$ is an m-edge and $cs$ is an f-edge. However, for a random set $\{a, b, c\}$, the labelling (m/fm) is also feasible (see Section 3). In this case, $bs$ is the f-edge and the Steiner point for the labelling (m/fm) will be named $s_{\mathrm{e}}(b)$ in our algorithm.

### *2.2.4. Labelling* (m/ff)

In the (m/ff) case, one equation in the system $\mathcal{S}$ will be given by $g(as) = m$. It follows that the labelled minimal Steiner point $s = (x_s, y_s, z_s)$ lies on cone $\mathcal{C}_a$ (Fig. 6). After a transformation, let $a$ be the origin $(0, 0, 0)$. Then $z_s = -m\sqrt{x_s^2 + y_s^2}$. Let $\mathbf{t}_c$ be the vector tangent to the circle $\mathcal{C}_a \cap \mathcal{H}_s$ at $s$. Then, by Theorem 2.2 the point $s$ is determined by

$$f_{\mathrm{t}}(x_s, y_s) = \frac{\vec{sb}\cdot\mathbf{t}_c}{|sb|} + \frac{\vec{sc}\cdot\mathbf{t}_c}{|sc|} = 0, \qquad f_{\mathrm{n}}(x_s, y_s) = \frac{\vec{sb}\cdot\vec{sa}}{|sb|\,|sa|} + \frac{\vec{sc}\cdot\vec{sa}}{|sc|\,|sa|} + 1 = 0. \tag{8}$$

Note that $f_{\mathrm{t}}$ is of degree four while $f_{\mathrm{n}}$ is of degree 8. The system may be unsolvable using radicals as shown in the following example.

*Example*: We assume $\triangle abc$ intersects $\mathcal{C}_a$ and furthermore assume both $b$ and $c$ lie on $\mathcal{C}_a$. More precisely, let

$$a = (0, 0, 0), \qquad b = (u, 0, -mu), \qquad c = (r\cos\phi, r\sin\phi, -mr),$$
$$u > 0, \qquad r > 0, \quad 0 < \phi < 180^\circ.$$

(Note that $b$ lies on cone $\mathcal{C}_a$ but $c$ lies above $\mathcal{C}_a$.) Replacing the degree 8 polynomial $f_{\mathrm{n}}$ with a simple degree 2 curve $f_{\mathrm{c}} = x^2 + y^2 - 4$, we show now that the intersection $s^c = (x, y, z)$ of $f_{\mathrm{t}}$ and $f_{\mathrm{c}}$ is unsolvable by radicals. Fig. 7(a) is an

**Table 1**
Experimental results for 10,000 sets of three random points.

| Type of $s$ | $s = b$ (degenerate) | $s_v$ | $s_p$ | $s_m$ | $s_e(c)$ | $s_e(b)$ | $s_c$ |
|---|---|---|---|---|---|---|---|
| Degree of system | – | 1 | 2 | 2 | 4 | 4 | >4 |
| Number of sets | 6271 | 3297 | 227 | 44 | 66 | 52 | 43 |

instance of $a, b, c$ in which $m = 1/7, u = 7, r = 5, \phi = 90°$. Fig. 7(b) shows the projections of $f_n, f_t, f_c$ on the horizontal plane. For this instance, $x = \sqrt{4 - y^2}$, $z = -2/7$, and $f_t = 0$ is equivalent to

$$\begin{aligned} f(y) &= (\vec{sb} \cdot \mathbf{t})^2 |sc|^2 - (\vec{sc} \cdot \mathbf{t})^2 |sb|^2 \\ &= 490y^3 - \frac{135\,620}{49} y^2 + 350\sqrt{4 - y^2} y^2 - 1400\sqrt{4 - y^2} + \frac{262\,200}{49} = 0. \end{aligned}$$

After simplification the equation becomes

$$\begin{aligned} &(7)^6(37)y^6 - (2)(7)^4(6781)y^5 + (2)^2(11)(37)(191)(239)y^4 + (2)^2(3)(5)(7)^4(19)(23)y^3 \\ &\quad - (2)^5(3)^2(5)(197\,921)y^2 + (2)^3(5)^6(1997) = 0. \end{aligned}$$

This polynomial of degree 6 is irreducible with a non-square discriminant. It is also irreducible modulo 11, 13, 17, 19, 23, etc. Its Galois group is the unsolvable transitive group $6T16$ [14]. For other values of $r$, the resulting polynomials are similar. These facts imply that the simplified problem (finding the intersection $s^c$ of $f_t$ and $f_c$) is unsolvable using radicals. Therefore, returning to the original problem of determining the intersection $s$ of $f_t$ and $f_n$ we can conclude the following:

**Theorem 2.3.** *The labelled minimal Steiner point $s$ for labelling* (m/ff) *is determined by* $\{f_t, f_n\}$ *(System (8)). The system has one real solution which is unlikely to be found in terms of radicals.*

As a result of this theorem, we use an approximation scheme in our algorithm as follows. Let $x_s = r\cos\phi$ and $y_s = r\sin\phi$. After squaring and simplifying, the system can be transformed into two polynomials in terms of $r$ and $4\phi$, $f_t^*(r, \phi), f_n^*(r, \phi)$. Let

$$r_{\max} = \max\left(\sqrt{x_b^2 + y_b^2}, \sqrt{x_c^2 + y_c^2}\right), \qquad \phi_{\min} = \min(\phi_b, \phi_c), \qquad \phi_{\max} = \max(\phi_b, \phi_c),$$

where $\phi_b, \phi_c$ in $[-\pi, \pi]$ are the angles between the positive $x$-axis and the projections of $\vec{ab}, \vec{ac}$ on the horizontal plane respectively. Then we can find $r$ and $\phi$ by iteration with an initial Steiner point $s_0$ on the cone $\mathcal{C}_a$, that is, $s_0 = (r_0\cos(\phi_0), r_0\cos(\phi_0), -mr_0)$ for any non-zero $r_0$ ($\in [0, r_{\max}]$) and $\phi_0$ ($\in [\phi_{\min}, \phi_{\max}]$):

$$\phi_{i+1} = P_{\text{slv}}\left(f_t^*(r_i, \phi) \mid \phi \in [\phi_{\min}, \phi_{\max}]\right), \tag{9}$$

$$r_{i+1} = P_{\text{slv}}\left(f_n^*(r, \phi_{i+1}) \mid r \in [0, r_{\max}]\right). \tag{10}$$

The iteration procedure moves the Steiner point on $\mathcal{C}_a$ alternately along the circle $\mathcal{C}_a \cap \mathcal{H}_s$ and the generating line $as$, and each move reduces the length of $T_g$. The aim is to find the point $s$ where the variation is zero, that is, $f_t^* = f^* - n = 0$. Because the length of $T_g$ is strictly convex, the iteration procedure converges to a unique solution $(r^*, \phi^*)$, and the labelled minimal Steiner point $s$, which we name $s_c$, is

$$s_c = (r^*\cos\phi^*, r^*\sin\phi^*, -mr^*). \tag{11}$$

*2.2.5. Labelling* (f/ff)

In the (f/ff) case, the labelled minimal Steiner point lies on the plane determined by $\triangle abc$. Hence, this unique labelled minimal Steiner point, named $s_p$, can be constructed using any method for finding planar Euclidean Steiner points, for example, by Melzak's construction [15] or by the hexagonal coordinates method [16]. In our algorithm we use a more recently developed method, called Simpson intersection method [2] because it can directly determine the Steiner point while both Melzak's construction and the hexagonal coordinates method need a pre-process: a transformation of the coordinate system. Regardless of the method, the degree of the system determining $s_p$ is two.

## 3. An algorithm for finding locally minimal Steiner points

As discussed in Section 1, a labelled minimal Steiner point is not necessarily locally minimal. However, since the locally minimal Steiner point $s$ must be labelled minimal, we can find the locally minimal Steiner point by comparing the lengths of labelled minimal trees. Note that for an actual configuration of terminals one cannot use symmetry. Therefore, from the fundamental properties of gradient-constrained minimum networks (Section 2) there are twelve different (non-degenerate) feasible labellings for a degree-3 Steiner point:

(b/mm), (m/mm), (m/mf), (m/fm), (m/ff), (f/ff), (mm/b), (mm/m), (fm/m), (mf/m), (ff/m), (ff/f).

To design an efficient algorithm (Table 1) the following can be taken into account.

(1) To make full use of the information from $\triangle abc$, first we transform $a, b, c$ so that

$$z(a) \geq z(b) \geq z(c) \quad \text{and} \quad g(ba) \geq g(cb). \tag{12}$$

Then four of the labellings (mm/b), (mm/m), (fm/m), (ff/m) in the above list can be removed. Moreover, note that (ff/f) is equivalent to (f/ff), that is, their Steiner points are the same as this is a planar problem. Similarly, (mf/m) is equivalent to (m/fm). Therefore, only the first 6 labellings remain, and the corresponding Steiner point are $s_v, s_m, s_e(c), s_e(b), s_c, s_p$.

(2) By Condition (12) if $g(cb) \geq m$, then $g(ab) \geq m$ and $s$ must collapse into $b$. Otherwise, $g(cb) < m$ and there are four cases depending on whether the gradients, $g(ba)$ and $g(ca)$, are at least $m$ or less than $m$. Note that $g(ba) < m$ implies that at least one of $g(as) < m$ and $g(bs) < m$ holds, and similarly for $g(ca) < m$. Therefore, in each of the four cases at most four labellings are feasible. For example, if $g(ba) > m$ and $g(ca) > m$, then the labelling of $s$ cannot be (f/ff). (Note that the formulae given in Section 2 also apply to degenerate labelled minimal Steiner points. Hence, in the case of $g(cb) < m$, we will not distinguish between $s$ being degenerate or non-degenerate.)

(3) We compute labelled minimal Steiner points from the easiest to the most difficult, according to the degree of the system $\mathcal{S}$ (being 1, 2, 4 or >4), that is, the order is $s_v, s_p, s_m, s_e, s_c$. We use a character string 'mmf' to denote the most difficult case: computing $s_c$.

Below is the pseudo-code of the algorithm for finding a Steiner point.

**3-Point Algorithm:**

input : vertices $a, b, c$
output: a Steiner point $s$

transform $a, b, c$ so that $z(a) \geq z(b) \geq z(c)$, $g(ba) \geq g(cb)$;
**if** $g(cb) \geq m$ **then** $s = b$;
 **elseif** $(g(ba) \leq m)$ **and** $(g(ca) \leq m)$ **then**
  compute $s_p$;
  **if** $(g(s_p a) \leq m)$ **and** $(g(bs_p) \leq m)$ **and** $(g(cs_p) \leq m)$ **then** $s = s_p$;
   **else** $s$ = 'mff';
  **end**
 **elseif** $(g(ba) > m)$ **and** $(g(ca) \leq m)$ **then**
  compute $s_p$;
  **if** $(g(s_p a) \leq m)$ **and** $(g(bs_p) \leq m)$ **and** $(g(cs_p) \leq m)$ **then** $s = s_p$;
   **else**
    compute $s_e(c)$;
    **if** $g(cs_e(c)) \leq m$ **then** $s = s_e(c)$; **else** $s$ = 'mff'; **end**
  **end**
 **elseif** $(g(ba) \leq m)$ **and** $(g(ca) > m)$ **then**
  compute $s_p$;
  **if** $(g(s_p a) \leq m)$ **and** $(g(bs_p) \leq m)$ **and** $(g(cs_p) \leq m)$ **then** $s = s_p$;
   **else**
    compute $s_e(b)$;
    **if** $g(bs_e(b)) \leq m$ **then** $s = s_e(b)$; **else** $s$ = 'mff'; **end**
  **end**
 **elseif** $(g(ba) > m)$ **and** $(g(ca) > m)$ **then**
  compute $s_v$;
  **if** $g(s_v a) \geq m$ **then** $s = s_v$;
   **else**
   compute $s_m$; **if** $1 + \cos(\angle as_m b) + \cos(\angle as_m c) < 0$ **then** $s$ = 'mff'; **end**
   compute $s_e(c)$; **if** $g(cs_e(c)) \leq m$ **then** $s = s_e(c)$; **end**
   compute $s_e(b)$; **if** $g(bs_e(b)) \leq m$ **then** $s = s_e(b)$; **else** $s = s_m$; **end**
  **end**
**end**
**if** $s$ = 'mff' **then** $s = s_c$; **end**
transform $s$ back into the original coordinate system;

## 4. Computational results and discussions

10,000 sets of random points $\{a, b, c\}$, uniformly distributed in a unit cube, were tested for $m = 1/7$. The results are listed below:

The results show that in about 98% of cases the Steiner point $s$ can be determined by solving linear or quadratic equations, that is, can be found in one step, and only in 0.43% of the cases ($s = s_c$) the labelled minimal Steiner point $s$ (in theory) cannot be exactly determined by solving the equations, and thus an approximation scheme is essentially required.

As stated in Section 1, in the application to mining, the typical maximum gradient is less than or equal to 1/7. Obviously, the smaller the maximum gradient is, the greater the number of m-edges or b-edges. Consequently more Steiner points become degenerate or have labelling b/mm. This is why we have chosen $m = 1/7$ in our experiment.

The fact that the Steiner point $s$ is degenerate (collapsing into adjacent terminals) in about 63% cases is important in implementing a heuristic for gradient-constrained minimum networks. First, we briefly review a heuristic for constructing Euclidean Steiner minimum trees as follows. We construct a minimal spanning tree as an initial tree and then construct an approximation of the Steiner minimum tree by inserting Steiner points. That is, suppose two edges adjacent to a point $b$ in the tree, say $ab$ and $bc$, meet at an angle less than 120°, then the angle condition in Proposition 1.1 is not satisfied and a Steiner point $s$ can be inserted so that $|sa| + |sb| + |sc| < |ab| + |bc|$. Consequently, the length of $T$ is reduced when the two edges $ab$, $bc$ are replaced with three edges $as$, $bs$ and $cs$, see the book by Hwang et al. [1].

The heuristic described above for the Euclidean Steiner tree problem can be applied to the gradient-constrained minimum network problem. First, because an algorithm for constructing Euclidean minimal spanning trees, either the Kruskal or the Prim algorithm, is based solely on distances between points, it can also be used to find gradient-constrained minimal spanning trees. Once the gradient-constrained minimal spanning tree on the given point set $N$ is constructed, the procedure of inserting Steiner points can be executed. That is, for each pair of adjacent edges in the gradient-constrained minimal spanning tree, we run the 3-point algorithm on the three endpoints of these edges. Our experiments indicate that in about 63% of the cases, a Steiner point will not need to be inserted as it is likely to be degenerate. Even where a Steiner point needs to be introduced, in most cases it can be found in one step by solving a linear or quadratic equation.

In summary our results indicate that we can rapidly find the locally minimal gradient-constrained Steiner trees which greatly reduces the burden of calculation in finding global minimal gradient-constrained networks.

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