Actuator and sensor placement in linear advection PDE with building system application

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ABSTRACT

We study the problem of actuator and sensor placement in a linear advection partial differential equation (PDE). The problem is motivated by its application to actuator and sensor placement in building systems for the control and detection of a scalar quantity such as temperature and contaminants. We propose a gramian based approach to the problem of actuator and sensor placement. The special structure of the advection PDE is exploited to provide an explicit formula for the controllability and observability gramian in the form of a multiplication operator. The explicit formula for the gramian, as a function of actuator and sensor location, is used to provide test criteria for the suitability of a given sensor and actuator location. Furthermore, the solution obtained using gramian based criteria is interpreted in terms of the flow of the advective vector field. In particular, the almost everywhere stability property of the advective vector field is shown to play a crucial role in deciding the location of actuators and sensors. Simulation results are performed to support the main results of this paper.

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1. Introduction

In this paper, we study the problem of actuator and sensor placement in a linear advection partial differential equation. The problem is motivated by its application to actuator and sensor location in building systems for the purpose of control of temperature and detection of contaminants. Building systems in US account for 39% of total energy consumption [1]. Design of efficient building systems not only has a significant economic benefit but also social and environmental benefits. Social benefits arise due to improved overall quality of life by enhancing occupant health, comfort and heightened aesthetic qualities. Improvement in water and air quality and reduced waste lead to environmental benefits.

The optimal placement of actuators and sensors in a building system is a difficult problem due to the complex physics that is involved. The governing equations for building system fluid flows and scalar densities are coupled nonlinear partial differential equations subjected to disturbances, various sources of uncertainties, and complicated geometry. Analysis of the building system with its full scale complexity leads to a finite element based computational approach to the actuator and sensor placement problem [2]. Such a purely computational based approach provides little insight into the obtained solution. An alternate system theoretic and dynamical systems based approach under some simplifying assumptions and physics can also be pursued [3,4]. Such an approach provides useful insights and guidelines to the complex control problems involved in building system applications. In this paper, we pursue a similar approach for the location of actuator and sensor problem in a building system.

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Under some simplifying assumptions and physics [3,4], the system equations are modeled in the form of a linear advection partial differential equation with inputs and outputs. We propose a gramian based approach to the actuator and sensor location problem. The results are an important first step towards its application to building systems. However, further research needs to be done for relaxing some of the simplifying assumptions made in this paper for the applicability of these results for the building system problem. We believe that the analytical methods developed in this paper combined with computational techniques involving detailed physics of building systems is a right approach moving forward. The main contribution of this paper is in providing explicit formula for the controllability and observability gramians as a function of actuator and sensor locations and the advection velocity field. These explicit formulas for the gramians are used to provide test criteria for deciding the location of sensors and actuators. Technical conditions for the existence of infinite time gramians are also provided. In particular, we prove that the infinite time controllability and observability gramians are well defined for almost everywhere stable and asymptotically stable advection vector fields respectively. We provide simulation results using a two dimensional fluid flow vector field for the computation of the finite time controllability and observability gramians.

An excellent review and classification of sensor and controller positioning for distributed parameter systems can be found in [5], where most of the methods involve a finite dimensional approximation of the infinite dimensional system, either before or after solving an optimization problem using the point spectrum of the infinitesimal generator. It was noted in [5] that such an approximation based method will not work for wave type systems because of the finite speed of propagation. [6] is an excellent book on sensor and actuator placement for distributed parameter systems governed by heat and diffusive type processes. [7] describes sensor and actuator placement for flexible structures. A combinatorial optimization approach for linear time invariant systems based on integer programming using the controllability and observability gramians for sensor and actuator placement can be found in [8].

The linear advection equation considered in this paper is akin to a unidirectional wave equation whose wave speed is governed by a nonlinear smooth vector field $f(x)$. Hence, actuator and sensor placement analysis based on a finite dimensional approximation as developed in earlier references will not work well for our problem. Our selection criteria for actuators and sensors uses the idea of controllability and observability gramians, but differs from what is seen in the literature slightly. The advection equation has a fundamental limitation for control as described in Theorem 5, in the sense that placing an actuator on the set B can only affect states ρ whose support is $\mathcal{R}_B^\tau = \cup_{t=0}^\tau \phi_t(B)$. This set $\mathcal{R}_B^\tau$ is precisely the support of the controllability gramian $\mathcal{C}_B^\tau$ for the advection equation (see Claim 8). This is the main reason why we consider choosing a set B for actuators that maximizes the support of $\mathcal{C}_B^\tau$. For sets B that give the same support for $\mathcal{C}_B^\tau$, we choose the one that gives lesser L^2 norm, since that will minimize the control effort (see Theorem 5, where the minimum norm control formula has the controllability gramian appearing in the denominator).

The organization of the paper is as follows. In Section 2, we describe the problem and some preliminaries from the theory of partial differential equations. In Section 3, we present the main results of the paper. In Section 4, we discuss technical conditions for the existence of the infinite time controllability and observability gramians. Simulation results are presented in Section 5 followed by conclusion in Section 6.

2. Preliminaries

We study the problem of optimal location of actuator in a linear advection partial differential equation. The motivation for this problem comes from the optimal location of actuators for the control of a scalar quantity, such as temperature or contaminants, in a room denoted by $\rho(x, t)$.

In building system applications, the evolution of $\rho(x, t)$, is governed by the velocity field $v(x, t)$ of the fluid flow. This velocity field is obtained as a solution of the following Navier–Stokes equation:

$$\frac{\partial v(x, t)}{\partial t} + v(x, t) \cdot \nabla v(x, t) = -\nabla p(x, t) + \frac{1}{\text{Re}} \Delta v(x, t)$$
$$\nabla \cdot v(x, t) = 0, \tag{1}$$

where $x \in X \subset \mathbb{R}^N$ (with $N = 2$ or 3) is the domain of the room, $v(x, t)$ is the velocity field, $p(x, t)$ is the pressure, and Re is the Reynolds number. The evolution of the scalar quantity $\rho(x, t)$ is governed by the following linear controlled partial differential equation

$$\frac{\partial \rho}{\partial t} + v(x, t) \cdot \nabla \rho(x, t) = \frac{1}{\text{Pr Re}} \Delta \rho(x, t) + \sum_{k=1}^{N} \chi_{B_k}(x) u_k(t)$$
$$y_k(x, t) = \chi_{A_k}(x) \rho(x, t), \quad k = 1, \dots, M \tag{2}$$

where Pr is the Prandtl number, $\chi_{A_k}(x)$ is the indicator function on set $A_k \subset X$, and $u_k(t) \in \mathbb{R}$ is the control input for $k = 1, \dots, N$. The form of control input $\chi_B(x)u(t)$ and output measurement $\chi_A(x)\rho(x, t)$ is motivated by the fact that the actuation and sensing can be exercised only on a small region B and A of the physical space X respectively.

Remark 1. The form of output equation in (2) is different than the one usually considered in the literature, where the sensors have access to the average state information on a set (i.e., $y_k(t) = \int_{A_k} c_k(x)\rho(x, t)$). The interpretation in our case is that the sensors have point-wise state information from the sets A_k. We choose the form in (2) because it allows us to compute the observability gramians as a explicit function of section location set A_k. Our proposed approach can also be applied to the case where the sensors have access to average state information, however, the observability gramian in that case will be a complicated function of the sensor location set A_k. Furthermore this form of output measurement is also dual to the input actuation term, in particular to Eq. (4).

The objective is to determine the optimal location of actuators and sensors, and hence the determination of indicator function $\chi_{B_k}(x)$ and $\chi_{A_k}(x)$. The terms $v(x, t) \cdot \nabla \rho(x, t)$ and $\Delta \rho(x, t)$ in (2) correspond to advection and diffusion respectively, with $D = \frac{1}{\mathrm{Re}\,\mathrm{Pr}}$ being the diffusion constant. Note that the advection–diffusion equation (2) is decoupled from the Navier–Stokes equation (1). In the case where the scalar density is temperature, this decoupling corresponds to the assumption that buoyancy forces have negligible or no effect. Furthermore, for simplicity of presentation of the main results of this paper, we now make the following assumptions.

Assumption 2. We replace the time varying velocity field $v(x, t)$ responsible for the advection of scalar density with the mean velocity field $f(x)$ i.e.,

$$f(x) := \frac{1}{T} \int_0^T v(x, t)dt.$$

Remark 3. Typically the velocity field information $v(x, t)$ is available over a finite time interval $[0, T]$ either from a simulation or from an experiment. Assumption 2 corresponds to linearizing the linear advection PDE along the mean flow field $f(x)$. It follows that if $v(x, t)$ is volume preserving i.e., $\nabla \cdot v(x, t) = 0$, then $\nabla \cdot f(x) = 0$ as well.

Assumption 4. Again for simplicity of presentation of the main results of this paper, we assume that the diffusion constant D in the advection–diffusion equation (2) is zero. As we see in the simulation section, the assumption of zero diffusion constant is justified.

We next discuss a few preliminaries on semigroup theory of partial differential equations. Consider the following ordinary differential equation (ODE):

$$\dot{x} = f(x), \qquad x(0) = x_0, \tag{3}$$

where $x \in X \subset \mathbb{R}^N$ a compact set. We denote by $\phi_t(x)$ the solution of ODE (3) starting from the initial condition x. ODE (3) is used to define two linear infinitesimal operators, $\mathcal{A}_K : L^2(X) \to L^2(X)$ and $\mathcal{A}_{\mathrm{PF}} : L^2(X) \to L^2(X)$ defined as follows:

$$\mathcal{A}_K \rho = f \cdot \nabla \rho, \qquad \mathcal{A}_{\mathrm{PF}} \rho = -\nabla \cdot (f\rho).$$

The domains of the above operators are given as follows:

$$D(\mathcal{A}_K) = \{\rho \in H^1(X) : \rho|_{\Gamma_o} = 0\},$$
$$D(\mathcal{A}_{\mathrm{PF}}) = \{\rho \in H^1(X) : \rho|_{\Gamma_i} = 0\},$$

where Γ_o and Γ_i are the outflow and inflow portions of the boundary ∂X defined as follows:

$$\Gamma_o = \{x \in \partial X : f \cdot \eta > 0\}, \qquad \Gamma_i = \{x \in \partial X : f \cdot \eta < 0\},$$

where η is the outward normal to the boundary ∂X. The semigroups corresponding to the $\mathcal{A}_K$ and $\mathcal{A}_{\mathrm{PF}}$ are called as Koopman ($\mathbb{U}_t$) and Perron–Frobenius ($\mathbb{P}_t$) operators respectively. These operators are defined as follows:

$$\mathbb{U}_t : L^2(X) \to L^2(X), \qquad (\mathbb{U}_t \rho)(x) = \rho(\phi_t(x)),$$
$$\mathbb{P}_t : L^2(X) \to L^2(X), \qquad (\mathbb{P}_t \rho)(x) = \rho(\phi_{-t}(x)) \left| \frac{\partial \phi_t(x)}{\partial x} \right|^{-1},$$

where $|\cdot|$ denotes the determinant. These semigroups can be shown to satisfy the following partial differential equations [9]:

$$\frac{\partial \rho}{\partial t} - \mathcal{A}_K \rho = 0, \quad \rho|_{\Gamma_o} = 0; \qquad \frac{\partial \rho}{\partial t} - \mathcal{A}_{\mathrm{PF}} \rho = 0, \quad \rho|_{\Gamma_i} = 0.$$

The Koopman and Perron–Frobenius semigroup operators and their infinitesimal generators are adjoint to each other i.e.,

$$\int_X (\mathbb{P}_t \rho_1)(x) \rho_2(x)dx = \int_X \rho_1(x)(\mathbb{U}_t \rho_2)(x)dx \quad \forall \rho_1, \rho_2 \in L^2(X).$$

3. Main results

The gramian based approach is one of the systematic approaches available for the optimal placement of actuators and sensors. Controllability and observability gramians measure the relative degree of controllability and observability of various states in the state space. Using the gramian based approach, actuators and sensors are placed at a location where the degree of controllability and observability of the least controllable and observable state is maximized [10,11].

3.1. Controllability gramian

For the construction of the controllability gramian, the advection–diffusion partial differential equation (2) using Assumptions 2 and 4 for a single input case can be written as follows:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (f(x)\rho) = \chi_B(x)u(x, t);$$ (4)
$$\rho|_{\Gamma_i} = 0; \qquad \rho(x, 0) = \rho_0(x).$$

In Eq. (4), we are assuming that the control input u is both a function of spatial variable x and time t. This assumption will typically not be satisfied in the building system application, however, making this assumption allows us to use existing results from linear PDE theory in the development of controllability gramian [10]. Furthermore, since $m(X) \gg m(B)$, where m is the Lebesgue measure, we expect the main conclusions of this paper to hold even when u is assumed to be only a function of time. The set B is the region of control in the state space X, and $u(x, t) \in L^2([0, \tau] : L^2(B))$ i.e., we have a control input that is square integrable in time and space, acting on the set B. The solution to (4) is given by the following:

$$\rho(x, t) = \mathbb{P}_t \rho_0(x) + \int_0^t \mathbb{P}_{t-s}(\chi_B(x)u(x, s))ds.$$

We define the controllability operator $\boldsymbol{\mathscr{B}}^\tau : L^2([0, \tau] : L^2(B)) \to L^2(X)$ as follows:

$$(\boldsymbol{\mathscr{B}}^\tau u)(x) := \int_0^t \mathbb{P}_{t-s}(\chi_B(x)u(x, s))ds.$$ (5)

The adjoint of the controllability operator $\boldsymbol{\mathscr{B}}^{\tau*} : L^2(X) \to L^2([0, \tau] : L^2(B))$ can be calculated and is given as follows:

$$(\boldsymbol{\mathscr{B}}^{\tau*}z)(x, s) = \chi_B(x)\mathbb{U}_{(\tau-s)}z(x).$$ (6)

We have the following theorem on the controllability property of the PDE (4).

Theorem 5. *Let $\mathcal{R}^\tau = \cup_{t=0}^\tau \phi_t(B)$. The PDE (4) is exactly controllable in a given time $\tau > 0$ for all initial and terminal states in the space $L^2(\mathcal{R}^\tau)$ i.e. given initial and terminal states $\rho_0(x)$ and $\rho_\tau(x)$ in $\mathcal{R}^\tau$, there exists a control $u(x, t) \in L^2([0, \tau] : L^2(B))$ such that $\rho(x, 0) = \rho_0(x)$, and $\rho(x, \tau) = \rho_\tau(x)$, where $\rho(x, t)$ is the solution of (4).*

Proof. We prove the theorem by showing the following, which is equivalent to showing that the range of the controllability operator $\boldsymbol{\mathscr{B}}^\tau$ is the same as $L^2(\mathcal{R}^\tau)$:

(1)

$$\boldsymbol{\mathscr{B}}^{\tau*}z = 0 \quad \forall (x, s) \in B \times [0, \tau] \Rightarrow z = 0 \text{ in } L^2(\mathcal{R}^\tau).$$ (7)

(2) The range of $\boldsymbol{\mathscr{B}}^\tau$ is closed.

Assume that $\boldsymbol{\mathscr{B}}^{\tau*}z = \chi_B(x)\mathbb{U}_{(\tau-s)}z(x) = 0 \ \forall (x, s) \in B \times [0, \tau]$. The assumption simply means that $z = 0$ on $\cup_{t=-\tau}^0 \phi_t(B)$. Since the set $\cup_{t=-\tau}^0 \phi_t(B)$ evolves into $\mathcal{R}^\tau = \cup_{t=0}^\tau \phi_t(B)$, we have that $z = 0$ in $L^2(\mathcal{R}^\tau)$. Next, we recall that $\boldsymbol{\mathscr{B}}^\tau : L^2([0, \tau] : L^2(B)) \to L^2(X)$ is defined by $(\boldsymbol{\mathscr{B}}^\tau u)(x) := \int_0^\tau \mathbb{P}_{-(\tau-s)}\chi_B(x)u(x, s)ds$. Let us assume that $u_n(x, t) \to u(x, t)$ is a convergent sequence in $L^2([0, \tau] : L^2(B))$. We need to show that $(\boldsymbol{\mathscr{B}}^\tau u_n)(x) \to (\boldsymbol{\mathscr{B}}^\tau u)(x)$ in $L^2(X)$. We have the following, where we have used $\|\mathbb{P}_t\|_{L^2(X)} \leq M_\omega e^{\omega t}$ from the semigroup property of $\mathbb{P}_t$:

$$\|(\boldsymbol{\mathscr{B}}^\tau u_n)(x) - (\boldsymbol{\mathscr{B}}^\tau u)(x)\|_{L^2(X)}^2 = \int_X \int_0^\tau |\mathbb{P}_{(\tau-s)}\chi_B(x)(u_n(x, s) - u(x, s))|^2 ds dx$$

$$= \int_0^\tau \|\mathbb{P}_{(\tau-s)}\chi_B(x)(u_n(x, s) - u(x, s))\|_{L^2(X)}^2$$

$$\leq \int_0^\tau \int_X M_\omega e^{\omega(\tau-s)} |\chi_B(x)(u_n(x, s) - u(x, s))|^2 dx ds$$

$$\leq C(M, \tau) \int_0^\tau \int_X |\chi_B(x)(u_n(x, s) - u(x, s))|^2 dx ds$$

$$= C(M, \tau) \|(u_n(x, s) - u(x, s))\|_{L^2([0, \tau]:L^2(B))}^2 \to 0.$$

This shows that the range of the controllability operator $\boldsymbol{\mathscr{B}}^\tau$ is closed. Hence we have exact controllability in $L^2(\mathcal{R}^\tau)$. □

The objective of this paper is to provide a solution to the optimal actuator placement problem and hence the optimal location of the set B. This motivates us to consider the following definition of controllability gramian parameterized over set B.

Definition 6. The finite time controllability gramian $\mathcal{C}_B^\tau : L^2(X) \to L^2(X)$ for the PDE (4) is given by the following:

$$\mathcal{C}_B^\tau z = \boldsymbol{\mathcal{B}}^\tau \boldsymbol{\mathcal{B}}^{\tau*} z = \int_0^\tau \mathbb{P}_{(\tau-s)}(\chi_B(x)\mathbb{U}_{(\tau-s)}z(x))ds. \tag{8}$$

Furthermore, we have the following definition for the induced two norm of the operator $\mathcal{C}_B^\tau$:

$$\|\mathcal{C}_B^\tau\|_2^2 = \max_{z \in L^2(X),\ \text{s.t.}\ \|z\|_{L^2(X)}=1} \left\langle \mathcal{C}_B^\tau z, z \right\rangle_{L^2(X)}.$$

Theorem 7. *The controllability gramian $\mathcal{C}_B^\tau : L^2(X) \to L^2(X)$ can be written as a multiplication operator as follows:*

$$(\mathcal{C}_B^\tau z)(x) = \left(\int_0^\tau \mathbb{P}_t \chi_B(x)dt \right) z(x). \tag{9}$$

Proof.

$$\begin{aligned}
\mathcal{C}_B^\tau z &= \int_0^\tau \mathbb{P}_{(\tau-s)}(\chi_B(x)\mathbb{U}_{(\tau-s)}z(x))ds \\
&= \int_0^\tau \mathbb{P}_s(\chi_B(x)\mathbb{U}_s z(x))ds = \int_0^\tau \mathbb{P}_s(\chi_B(x)z(\phi_s(x)))ds \\
&= \int_0^\tau \chi_B(\phi_{-s}(x)z(x)) \left| \frac{\partial \phi_s(x)}{\partial x} \right|^{-1} ds = \left[\int_0^\tau (\mathbb{P}_s \chi_B(x))ds \right] z(x). \quad \square
\end{aligned}$$

The explicit formula for the controllability gramian from Eq. (9) in terms of multiplication operator can be used to provide an analytical expression for the minimum energy control input.

Claim 8. $\rho_B^\tau(x) := \int_0^\tau \mathbb{P}_t \chi_B(x)dt$ *is strictly positive on $\mathcal{R}^\tau = \cup_{t=0}^\tau \phi_t(B)$ and hence $\mathcal{C}_B^\tau$ is invertible on $\mathcal{R}^\tau$ with the inverse given by*

$$(\mathcal{C}_B^\tau)^{-1}z = \frac{z}{\rho_B^\tau(x)}, \quad \forall z \in L^2(\mathcal{R}^\tau). \tag{10}$$

Proof. Since $m(B) > 0$, and B evolves into $\phi_\tau(B)$ in time τ, for every $x \in \mathcal{R}^\tau$, there exist times $0 \leq t_1(x) < t_2(x) \leq \tau$ such that $x \in \phi_t(B)\ \forall t \in [t_1(x), t_2(x)]$. Hence, by the positivity of $\mathbb{P}_t$ we have that $\mathbb{P}_t(\chi_B(x)) > 0\ \forall t \in [t_1(x), t_2(x)] \subseteq [0, \tau]$. Hence we have the following:

$$\rho_B^\tau(x) = \int_0^\tau \mathbb{P}_t \chi_B(x)dt \geq \int_{t_1(x)}^{t_2(x)} \mathbb{P}_t \chi_B(x)dt > 0 \quad \forall x \in \mathcal{R}^\tau.$$

This proves the claim. $\square$

Theorem 9. *Let $\rho_\tau(x)$ and $\rho_0(x)$ be the elements of $L^2(\mathcal{R}^\tau)$, then the minimum energy control input that is required to steer the system from initial state $\rho_0(x)$ to final state $\rho_\tau(x)$ is given by following formula*

$$u_{opt}(x, s) = \boldsymbol{\mathcal{B}}^{\tau*}(\mathcal{C}_B^\tau)^{-1}(\rho_\tau(x) - \mathbb{P}_\tau \rho_0(x)) = \chi_B(x)\mathbb{U}_{\tau-s}\left(\frac{\rho_\tau(x) - \mathbb{P}_\tau \rho_0(x)}{\rho_B^\tau(x)} \right). \tag{11}$$

The minimum energy required is given by

$$\begin{aligned}
\|u_{opt}\|^2 &= \left\langle (\rho_\tau(x) - \mathbb{P}_\tau \rho_0(x)), (\mathcal{C}_B^\tau)^{-1}(\rho_\tau(x) - \mathbb{P}_\tau \rho_0(x)) \right\rangle_{L^2(\mathcal{R}^\tau)} \\
&= \left\| \frac{(\rho_\tau(x) - \mathbb{P}_\tau \rho_0(x))}{\rho_B^\tau(x)} \right\|_{L^2(\mathcal{R}^\tau)}^2.
\end{aligned} \tag{12}$$

Proof. First, we note that controlling the initial state $\rho_0(x)$ to $\rho_\tau(x)$ is equivalent to reaching the final state $(\rho_\tau(x) - \mathbb{P}_\tau \rho_0(x))$ from the zero initial state i.e. $\rho_0(x) \equiv 0$. Hence, equivalently, we prove that $\hat{u}_{opt}(x, s) = \boldsymbol{\mathcal{B}}^{\tau*}(\mathcal{C}_B^\tau)^{-1}(\rho_\tau(x))$ is the control

input with minimum norm that reaches $\rho_\tau(x)$ in time τ. This, along with an explicit calculation of $\mathcal{B}^{\tau*}(\mathcal{C}_B^\tau)^{-1}(\rho_\tau(x))$ will prove the theorem. Next, we consider the following set of admissible control inputs:

$$\mathcal{U} = \{u(x,t) \in L^2([0,\tau]:L^2(B)) : \mathcal{B}^\tau u = \rho_\tau\}.$$

We have the following:

$$\mathcal{B}^\tau \hat{u}_{opt} = \mathcal{B}^\tau \mathcal{B}^{\tau*}(\mathcal{C}_B^\tau)^{-1}\rho_\tau = \mathcal{B}^\tau \mathcal{B}^{\tau*}(\mathcal{B}^\tau \mathcal{B}^{\tau*})^{-1}\rho_\tau = \rho_\tau.$$

Hence, we have that $\hat{u}_{opt}(x,s) = \mathcal{B}^{\tau*}(\mathcal{C}_B^\tau)^{-1}(\rho_\tau(x)) \in \mathcal{U}$. Next, we define the following operator on $L^2([0,\tau]:L^2(B))P^\tau = \mathcal{B}^{\tau*}(\mathcal{C}_B^\tau)^{-1}\mathcal{B}^\tau$. We observe the following:

$$\begin{aligned}(P^\tau)^2 &= \mathcal{B}^{\tau*}(\mathcal{C}_B^\tau)^{-1}\mathcal{B}^\tau \mathcal{B}^{\tau*}(\mathcal{C}_B^\tau)^{-1}\mathcal{B}^\tau = \mathcal{B}^{\tau*}(\mathcal{C}_B^\tau)^{-1}\mathcal{B}^\tau \\ &= P^\tau, (P^\tau)^* = (\mathcal{B}^{\tau*}(\mathcal{C}_B^\tau)^{-1}\mathcal{B}^\tau)^* = P^\tau.\end{aligned} \tag{13}$$

Hence, the operator P^τ is a projection operator on the space $L^2([0,\tau]:L^2(B))$. Then, we have the following from Bessel's inequality:

$$\|u\|^2 = \|(P^\tau)u\|^2 + \|(I - P^\tau)u\|^2 \geq \|(P^\tau)u\|^2,$$

where the norm is on the space $L^2([0,\tau]:L^2(B))$. Now, let $u \in \mathcal{U}$ be arbitrary. This means $\mathcal{B}^\tau u = \rho_\tau$. Applying $\mathcal{B}^{\tau*}(\mathcal{C}_B^\tau)^{-1}$ on both sides, we get the following:

$$P^\tau u = \mathcal{B}^{\tau*}(\mathcal{C}_B^\tau)^{-1}\mathcal{B}^\tau u = \mathcal{B}^{\tau*}(\mathcal{C}_B^\tau)^{-1}\rho_\tau = \hat{u}_{opt}.$$

Hence, Bessel's inequality above gives $\|u\|^2 \geq \|\hat{u}_{opt}\|^2$. Next, (11) and (12) can be easily shown by an explicit calculation using (6) and (10).   □

Based on the formula for the controllability gramian, we propose the following criteria for the selection of optimal actuator location and hence the set B^*.

Actuator placement criteria

(1) Maximizing the support of the controllability gramian operator i.e.,

$$B^* = \arg\max_{B \subset X} \text{supp}\left(\int_0^\tau \mathbb{P}_t \chi_B(x)dt\right). \tag{14}$$

(2) If the support of controllability gramian is maximized or if more than one choice of set A leads to the same support then the decision can be made based on maximizing the 2-norm of the support i.e.,

$$B^* = \arg\max_{B \subset X} \left\|\int_0^\tau \mathbb{P}_t \chi_B(x)dt\right\|_{L^2(X)}.$$

Using the result of Theorem 5, it follows that criterion 1 maximizes the controllability in the space X, so that the control action in a small region $B \subset X$ will have an impact over larger portion of the state space. Furthermore, it follows from the explicit formula for the minimum energy control (11) from Theorem 9 that if the actuator selection is made based on criteria 2 then the amount of control effort is minimized.

3.2. Observability gramian

For the construction of observability gramian, we consider the advection partial differential equation with a single output measurement as follows:

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (f\rho), \quad \rho|_{\Gamma_i} = 0, \qquad \rho(x,0) = \rho_0(x)$$
$$y(x,t) = \chi_A(x)\rho(x,t). \tag{15}$$

The observability operator $\mathcal{A}^\tau : L^2(X) \to L^2([0,\tau]:L^2(A))$ for (15) is defined as follows:

$$(\mathcal{A}^\tau z)(x,s) = \chi_A(x)(\mathbb{P}_s z)(x).$$

The adjoint to the observability operator $\mathcal{A}^{\tau*} : L^2([0,\tau]:L^2(A)) \to L^2(X)$ can be written as follows:

$$(\mathcal{A}^{\tau*}w)(x) = \int_0^\tau (\mathbb{U}_s \chi_A(x)w(x,s))ds.$$

Definition 10 (*Observability Gramian*). The finite time observability gramian $\mathcal{O}_A^\tau : L^2(X) \to L^2(X)$ for the PDE (15) is given by the following formula

$$(\mathcal{O}_A^\tau z)(x) = (\mathcal{A}^{\tau*}\mathcal{A}^\tau z)(x) = \int_0^\tau (\mathbb{U}_s \chi_A(x) \mathbb{P}_s z(x)) ds. \tag{16}$$

The counterpart of Theorems 5 and 9 can be proved for the observability of system (15) using a duality argument. The theorem on observability gramian similar to Theorem 7 can be stated as follows:

Theorem 11. *The observability gramian for* (15) *can be written as a multiplication operator as follows:*

$$(\mathcal{O}_A^\tau z)(x) = \left[\int_0^\tau (\mathbb{U}_s \chi_A(x)) ds \right] z(x). \tag{17}$$

Proof. The proof follows along the lines of proof of Theorem 7. □

Following criteria can be used for the optimal location of sensor:

Sensor placement criteria

The finite time observability gramian can be used to decide the criteria for the optimal location of the sensor.

(1) Maximizing the support of observability gramian operator

$$A^* = \arg\max_{A \subset X} \; \text{supp} \left(\int_0^\tau \mathbb{U}_t \chi_A(x) dt \right).$$

(2) If the support of observability gramian is maximized or if more than one choice of set B leads to the same support then the decision can be made based on maximizing the 2-norm of the support i.e.,

$$A^* = \arg\max_{A \subset X} \left\| \int_0^\tau \mathbb{U}_t \chi_A(x) dt \right\|_{L^2(X)}.$$

4. Advective vector field and gramian

In this section, we provide an interpretation for the optimal actuator and sensor location problem in terms of the flow of the advection vector field. In particular, we show that the (almost everywhere uniform) stability property of the vector field plays an important role in deciding the location of actuators and sensors.

4.1. Infinite time controllability gramian

We show that the infinite time controllability gramian can be computed for vector fields that are stable in the almost everywhere uniform sense. We now define the notion of almost everywhere uniform stability for a nonlinear system.

Definition 12 (*Almost Everywhere Uniform Stable*). Let $x_0 = 0$ be the equilibrium point of $\dot{x} = f(x)$ and B_δ be a δ neighborhood of $x_0 = 0$. The equilibrium point $x_0 = 0$ is said to be almost everywhere uniform stable if for every given $\epsilon > 0$ there exists a $T(\epsilon)$ such that

$$\int_T^\infty m(A_t) dt < \epsilon, ; \quad A_t = \{x \in X : \phi_t(x) \in A\},$$

for all measurable sets $A \subset X \setminus B_\delta$ and where m is the Lebesgue measure.

The notion of almost everywhere stability is extensively studied in [12,13]. Furthermore, a PDE based approach is also provided for the verification of almost everywhere stability in [14]. We have the following theorem regarding the infinite time controllability gramian for vector fields that are almost everywhere uniformly stable:

Theorem 13. *For vector fields that are stable in the almost everywhere uniform sense, we have*

$$(\mathcal{C}_B^\infty z)(x) = \int_0^\infty \mathbb{P}_t \chi_B(x) dt z(x) = \rho_B(x) z(x), \tag{18}$$

where $\rho_B(x)$ is the positive solution of the following PDE

$$\nabla \cdot (f(x) \rho_B(x)) = \chi_B(x); \quad \rho|_{\Gamma_i} = 0. \tag{19}$$

Proof. In [14], it was shown that $\int_0^\infty \mathbb{P}_t \chi_B(x)dt$ solves (19) if $x_0 = 0$ is stable in the almost everywhere uniform sense. This proves the theorem. $\square$

The integral $\int_X \mathcal{C}_B^\infty z(x)dx$ for the special case where $z(x) = \chi_A(x)$, the indicator function for the set A, has the interesting interpretation of residence time, which is defined as follows:

Definition 14. For an almost everywhere uniform stable vector field, consider any two measurable subsets A and B of $X \setminus B_\delta$, then the residence time of set B in set A is defined as the amount of time system trajectories starting from set B will spend in set A before entering the δ neighborhood of the equilibrium point $x = 0$. We denote this time by T_B^A.

In [15], the following was shown for a discrete time system:

$$T_B^A = \int_A \int_0^\infty \mathbb{P}_t \chi_B(x)dtdx = \int_A \rho_B(x)dx. \tag{20}$$

The proof of (20) for a continuous-time case will follow along the lines of proof in [15].

Theorem 15. *The residence time T_B^A for an almost everywhere uniformly stable vector field $f(x)$ is given by following formula*

$$T_B^A = \int_X \mathcal{C}_B^\infty \chi_A(x)dx.$$

Proof. We have the following calculation using the formula from Theorem 13 and Eq. (20):

$$(\mathcal{C}_B^\infty \chi_A(x)) = \int_0^\infty \mathbb{P}_t \chi_B(x)dt \chi_A(x) = \rho_B(x)\chi_A(x)$$

$$\Rightarrow \int_X \mathcal{C}_B^\infty \chi_A(x)dx = \int_X \rho_B(x)\chi_A(x)dx = \int_A \rho_B(x)dx = T_B^A. \quad \square$$

4.2. Infinite time observability gramian

The infinite time observability gramian is defined under the assumption that the vector field $f(x)$ is globally asymptotically stable. First, we have the following theorem that characterizes global asymptotic stability:

Theorem 16. *Let B_δ be a δ neighborhood of $x = 0$. Let $v(x) \in C^1(X \setminus \bar{B}_\delta)$ denote the solution of the following steady state transport equation:*

$$\mathcal{A}_K v = f \cdot \nabla v = -v_0(x); \ v|_{\partial \bar{B}_\delta} = 0, \tag{21}$$

where $v_0(x)$ satisfies

$$v_0(x) = 0 \quad \forall x \in \bar{B}_\delta. \tag{22}$$

Then $x = 0$ is globally asymptotically stable for (3) if and only if there exists a positive solution $v(x) \in C^1(X/\bar{B}_\delta)$ for (21) for all $v_0(x) > 0 \in C^1(X/\bar{B}_\delta)$ satisfying (22).

Proof. We prove necessity first. Let us assume that $x = 0$ is globally asymptotically stable. We construct a positive solution for (21) as follows:

$$v(x) = \int_0^\infty v_0(\phi_t(x))dt. \tag{23}$$

For a given arbitrary $x \in X/\bar{B}_\delta$, there exists a $t^+(x) \in [0, \infty)$ such that $\phi_{t^+(x)}(x) \in \partial B_\delta$. Hence, we have that $v_0(\phi_t(x)) = 0 \ \forall t \geq t^+(x)$. In particular, this means that $\int_0^\infty v_0(\phi_t(x))dt < \infty \ \forall x \in X/\bar{B}_\delta$. We also have that $0 < v(x) \in C^1(X/\bar{B}_\delta)$ by virtue of the regularity of $v_0(x)$. We show that (23) solves (21). Let $v_N(x) = \int_0^N v_0(\phi_t(x))dt$. Then, we have the following:

$$\mathcal{A}_K v_N(x) = \int_0^N \mathcal{A}_K v_0(\phi_t(x))dt$$

$$= \int_0^N \frac{d}{dt}\mathbb{U}_t v_0(x)dt = \mathbb{U}_N v_0(x) - v_0(x).$$

Global stability of $x = 0$ implies that $\lim_{N \to \infty} \mathbb{U}_N v_0(x) = \lim_{N \to \infty} v_0(\phi_N(x)) = 0$ and hence $\lim_{t \to \infty} \mathcal{A}_K v_N(x)$ exists. Also, by the Hille–Yosida semigroup generation theorem, we have that the generator $\mathcal{A}_K$ is a closed operator. Hence, we have the following:

$$f \cdot \nabla v = \mathcal{A}_K v(x) = \int_0^\infty \mathcal{A}_K \mathbb{U}_t v_0(x) = \int_0^\infty \frac{d}{dt} \mathbb{U}_t v_0(x) = -v_0(x).$$

The boundary condition $v|_{\partial \bar{B}_\delta} = 0$ is satisfied by (23) automatically. To prove sufficiency let us assume that there exists a solution $0 < v(x) \in C^1(X/\bar{B}_\delta)$ that solves (21). Then, we have the following equation along the characteristic curves which are solutions of (3):

$$\frac{d}{d\tau} v(\phi_\tau(x)) = -v_0(\phi_\tau(x)) \Rightarrow v(\phi_t(x)) - v(x) = -\int_0^t v_0(\phi_\tau(x))d\tau. \tag{24}$$

Rewriting (24), we have $v(\phi_t(x)) + \int_0^t v_0(\phi_\tau(x))d\tau = v(x)$

$$\Rightarrow \int_0^t v_0(\phi_\tau(x))d\tau \le v(x) \quad \forall x \in X/\bar{B}_\delta, \ t > 0$$

$$\Rightarrow \left\| \int_0^\infty v_0(\phi_\tau(x))d\tau \right\|_{L^\infty(X/\bar{B}_\delta)} \le \|v(x)\|_{L^\infty(X/\bar{B}_\delta)} < \infty. \tag{25}$$

To the contrary, let us assume that $x = 0$ is not globally asymptotically stable. Then, by virtue of the attractor property of $x = 0$, there exists a point $x_0 \in X/\bar{B}_\delta$ such that $\omega(x_0) \ne \{0\}$. This means that $\phi_t(x_0) \in X/\bar{B}_\delta \ \forall t > 0$, for some $\delta > 0$. Then, the set $D = \overline{\cup_{t=0}^\infty \phi_t(x_0)}$ is a compact subset of $X/\bar{B}_\delta$. Since $v_0(x) > 0 \ \forall x \in X/\bar{B}_\delta$, we have that $v_0(x) > \epsilon > 0 \ \forall x \in D$ for some positive ϵ by continuity of $v_0(x)$. Hence we have the following:

$$\int_0^\infty v_0(\phi_\tau(x_0))d\tau > \int_0^\infty \epsilon d\tau = \infty, \tag{26}$$

contradicting (25). This proves the theorem. $\quad \square$

If $x = 0$ is globally asymptotically stable, then $\Gamma_o \supseteq \partial \bar{B}_\delta$. Hence, by using a standard density argument of $C^1(X \setminus \bar{B}_\delta)$ in $L^2(X \setminus B_\delta)$, and using trace operator theory [16] for point values of H^1 functions, we can show the following theorem:

Theorem 17. *Let $v(x) \in \mathcal{D}(\mathcal{A}_K) \cap L^2(X \setminus \bar{B}_\delta)$ denote the solution of the following steady state transport equation:*

$$\mathcal{A}_K v = f \cdot \nabla v = -v_0(x); v|_{\Gamma_o} = 0. \tag{27}$$

Then $x = 0$ is globally asymptotically stable for (3) if and only if there exists a positive solution $v(x) \in \mathcal{D}(\mathcal{A}_K) \cap L^2(X/\bar{B}_\delta)$ for (21) for all $0 < v_0(x) \in \mathcal{D}(\mathcal{A}_K) \cap L^2(X/\bar{B}_\delta)$.

Theorem 18. *Let $x = 0$ be a globally stable equilibrium point for $\dot{x} = f(x)$, then the infinite time observability gramian is well defined and we have*

$$(\mathcal{O}_A^\infty z)(x) = \left[\int_0^\infty (\mathbb{U}_t \chi_A(x))dt \right] z(x) = V(x)z(x), \tag{28}$$

where $V(x)$ is the positive solution of following steady state partial differential equation:

$$\mathcal{A}_K v = f \cdot \nabla v = -\chi_A(x); v|_{\Gamma_o} = 0.$$

Proof. For a given A, if we choose $\delta > 0$ such that $A \subset X \setminus B_\delta$, then we automatically have that $\chi_A(x) = 0 \ \forall x \in \bar{B}_\delta$. Hence, global stability implies the existence of a positive solution $V(x) = \int_0^\infty (\mathbb{U}_t \chi_A(x))dt \in \mathcal{D}(\mathcal{A}_K) \cap L^2(X \setminus B_\delta)$ from Theorems 16 and 17. This shows that $V(x)$ is well defined. Finally, the formula for the infinite time observability gramian (28) is obtained by letting $\tau \to \infty$ in Theorem 11. This proves the theorem. $\quad \square$

5. Simulation

In this section, we present simulation results on the computation of finite time gramians. The purpose of the simulation section is to demonstrate the applicability of the developed theoretical results in this paper. Detailed simulation results based on the developed theoretical results will be the topic of our future publication. The vector field that we use for the purpose of simulation is the average velocity field obtained from a detailed finite element-based simulation of Navier–Stokes equation. For the purpose of simulation, we only employ a two dimensional slice of the three dimensional velocity field as
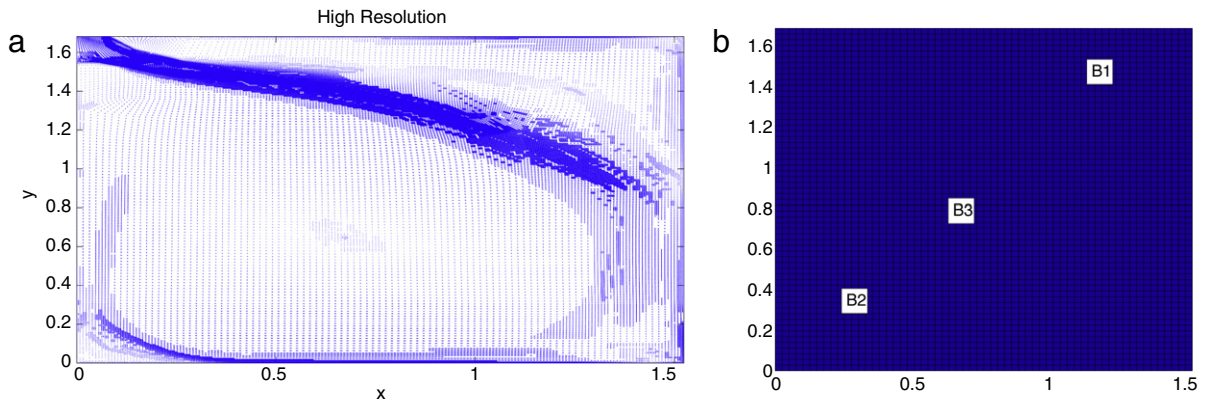
Fig. 1. (a) Velocity field; (b) Actuator locations on sets B_1, B_2 and B_3.
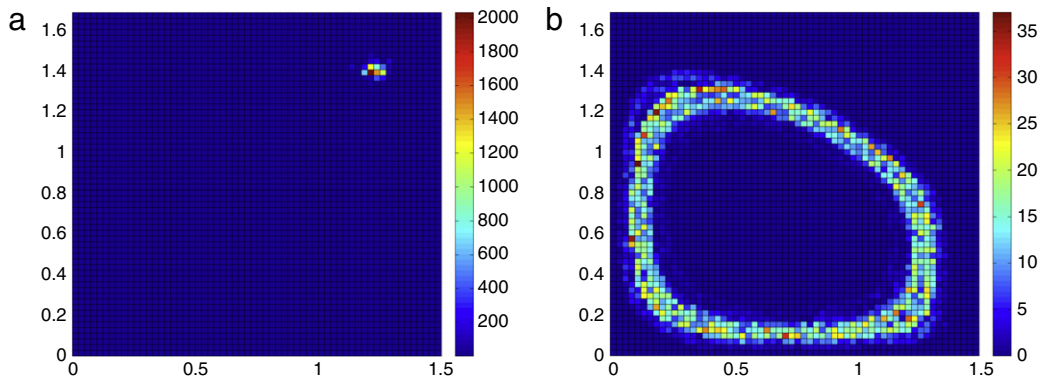

Fig. 2. Controllability gramian after 10 000 time iterations for actuator located at set (a) B_1; (b) B_2.

shown in Fig. 1(a). The dimensions of the room are as follows: $0 \leq x \leq 1.52$ m and $0 \leq y \leq 1.68$ m. The order of magnitude for the velocity field is $O(1)$. The Reynolds number of the flow is Re $= 76\,725$ and the Prandtl number Pr $= 0.729$. This makes $\frac{1}{\Pr \text{Re}} \approx O(10^{-5})$, and hence the zero diffusion constant assumption (Assumption 4) made in this paper is justified. The Reynolds number for the flow rate is in turbulent range. The $k - \epsilon$ model, which is Reynolds Average Navier–Stokes (RANS) model [17] is used to obtain the velocity field as shown in Fig. 1. A commercial CFD software Fluent was used to solve the coupled set of governing equations for pressure, temperature, turbulent kinetic energy, turbulent dissipation and velocity. No slip boundary condition was applied at all the walls.

For the purposes of computation, we employ set oriented numerical methods for the approximation of P–F semigroup $\mathbb{P}_t$ [18]. We divide the state space into finitely many square partitions denoted by $\{D_i\}_{i=1}^N$. The set D_i's are chosen such that $D_i \cap D_j = \emptyset$ for $i \neq j$ and $X = \cup_{i=1}^N D_i$. The finite dimensional matrix approximation of the P–F operator is obtained using the following formula [18]:

$$[P]_{ij} = \frac{m(\phi_{\delta t}(D_i) \cap D_j)}{m(D_i)}$$

where m is the Lebesgue measure and δt is the discretization time step and ϕ_t is the solution of vector field shown in Fig. 1(a). Using the adjoint property between the Koopman and P–F semigroup, the finite dimensional approximation of the Koopman semigroup U can be obtained as a transpose of P, namely $U = P'$.

The computation results for this section are obtained with actuators and sensors located at three different sets B_1, B_2, and B_3. The locations of these three sets are shown in Fig. 1(b).

In Figs. 2(b) and 3(a), we show the plots for the support of the controllability gramian after 10 000 time steps corresponding to two different locations of actuator sets B_2 and B_3 respectively. The support of the controllability gramian corresponding to B_2 and B_3 locations of actuator sets is approximately the same and equals 1.6. However the 2-norm of the gramian corresponding to actuator location on set B_2 is equal to 38, while for B_3 it is equal to 35. Comparing Figs. 2(a), (b), and 3(a), we see that the support of the gramian for actuator location at set B_1 is considerably smaller but it has considerably larger 2-norm compared to actuators locations at B_2 and B_3. The large value of gramian with small support in Fig. 2(a) can be very effective if one desires to perform localized control action. Comparing the support and the 2-norm of the gramian function, one can conclude that the actuator location corresponding to B_2 is optimal among B_1, B_2, and B_3.
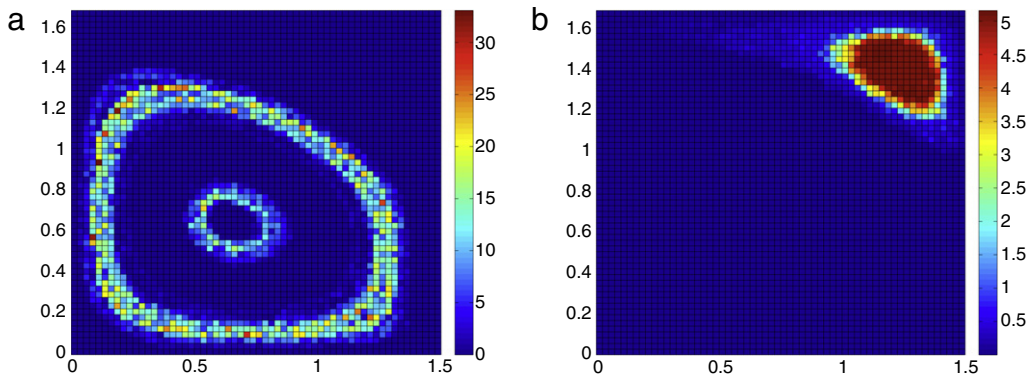
Fig. 3. (a) Controllability gramian after 10 000 time iterations for actuator located at set B_3; (b) Observability gramian after 1000 iterations with sensor location at B_1.
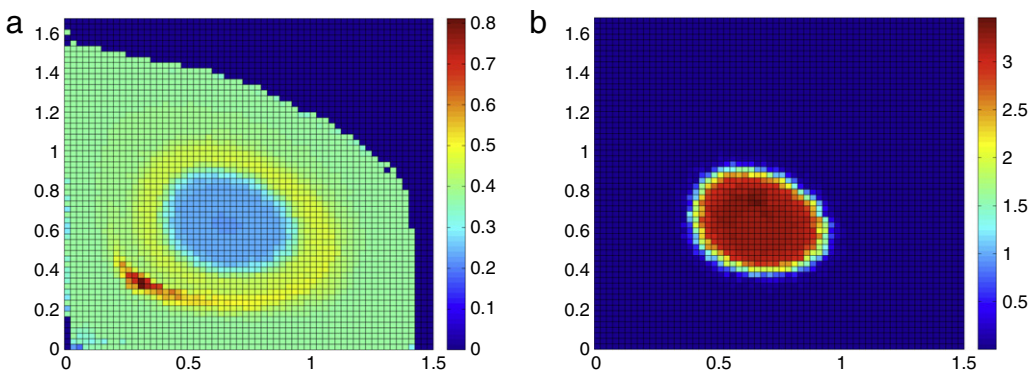

Fig. 4. Observability gramian after 1000 iterations with sensor location at (a) B_2; (b) B_3.

In Figs. 3(b), 4(a) and (b), we show the support and 2-norm of observability gramian corresponding to sensor locations on sets B_1, B_2, and B_3 respectively. The large support of the observability gramian for sensor location on set B_2 nearly outweigh the larger value and smaller support of gramian corresponding to sensors locations on set B_1 and B_3.

Remark 19. The actuator and sensor locations in the simulation example seem to be collocated. However, this is just a coincidence and in general this may not be the case. However it will be interesting question for future investigation. In particular, the combined problem of sensor and actuator placement will be the topic of our future investigation.

6. Conclusion

In this paper, controllability and observability gramian based test criteria are used to decide the suitability of given actuator and sensor locations. As compared to purely computational based methods currently existing in the literature, our proposed approach provides a systematic and insightful method for deciding the location of actuators and sensors in building systems. In particular, stability properties of the advection vector field are shown to play an important role in deciding the location of actuators and sensors. In our future research work, the explicit formula for the gramians will be exploited to provide a systematic algorithm for determining the optimal location of sensors and actuators. Furthermore some of the assumptions made in the derivation of control equations will be removed by incorporating elements of complex physics involved in building systems.

Acknowledgments

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