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Generalized series of Bessel functions

A. Al-Jarrah[a], K.M. Dempsey[b,*], M.L. Glasser[c]

[a] *Department of Mathematics, Yarmouk University, Irbid, Jordan*
[b] *Department of Mathematics and Computer Science, Clarkson University, Potsdam, NY 13699-5815, USA*
[c] *Department of Physics and Center for Quantum Device Technology, Clarkson University, Potsdam, NY 13699-5820, USA*

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Abstract

Known series of Bessel functions, currently available in handbooks, and many of Neumann type, are generalized to arbitrary order. The underlying result is a Poisson formula due to Titchmarsh. This formula gives rise to a Neumann series involving modified Bessel functions of integral order. The latter is the basis of many of the generalized series that follow. Included are examples of generalized trigonometric identities. The paper concludes by indicating the wide range of results that can be obtained. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Infinite series involving Bessel functions occur quite frequently in both mathematical and physical analysis. In his 1922 treatise, Watson [5] refers to a *Neumann series* as being any series of the type

$$\sum_{n=0}^{\infty} a_n J_{v+n}(z), \tag{1}$$

where $J_{v+n}(z)$ is the Bessel function of the first kind of order $v+n$ and complex argument z. In the mid-1860s, Neumann had studied such series for integer values of v. The extension to complex v was investigated by Gegenbauer. For numerous references to the early works of Neumann and Gegenbauer, see the extensive bibliography in [5]. In modern usage, a Neumann series may involve any type of Bessel function.

* Corresponding author.
E-mail addresses: alia@yu.edu.jo (A. Al-Jarrah), dempsey@clarkson.edu (K.M. Dempsey), laryg@clarkson.edu (M.L. Glasser).

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2. A generalized Poisson formula

In the following theorem, a function $f(x)$ and its *Fourier cosine transform* $F_c(x)$ are related by

$$F_c(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos(xt)\,dt, \qquad f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty F_c(t)\cos(xt)\,dt. \tag{2}$$

Titchmarsh theorem [4, Theorem 45, p. 61]. *Let* $\alpha\beta=2\pi$, $\alpha>0$, *and let* $f(x)$ *be of bounded variation on* $(0,\infty)$, *and tend to* 0 *as* $x\to\infty$. *Then*

$$\sqrt{\beta}\sum_{n=1}^\infty F_c(n\beta) = \sqrt{\alpha}\lim_{N\to\infty}\left[\frac{1}{2}f(0+)+\sum_{n=1}^N\frac{f(n\alpha-)+f(n\alpha+)}{2}-\frac{1}{\alpha}\int_0^{(N+1/2)\alpha}f(t)\,dt\right]. \tag{3}$$

If also $\int_0^\infty f(t)\,dt$ *exists as an improper Riemann integral, then*

$$\sqrt{\beta}\left\{\frac{1}{2}F_c(0)+\sum_{n=1}^\infty F_c(n\beta)\right\}=\sqrt{\alpha}\left[\frac{1}{2}f(0+)+\sum_{n=1}^\infty\frac{f(n\alpha-)+f(n\alpha+)}{2}\right]. \tag{4}$$

If also $f(x)$ *is continuous, then* (4) *reduces to Poisson's formula*

$$\sqrt{\beta}\left\{\frac{1}{2}F_c(0)+\sum_{n=1}^\infty F_c(n\beta)\right\}=\sqrt{\alpha}\left\{\frac{1}{2}f(0)+\sum_{n=1}^\infty f(n\alpha)\right\}. \tag{5}$$

3. A generalized Neumann series

Eqs. (9.1.47), (9.6.38), (9.6.37) and (9.6.39) in [1], and Eqs. (57.1.18), (58.1.4), (58.1.2) and (58.1.12) in [3], list the Neumann series

$$\sum_{k=0}^\infty(-1)^k J_{2k}(z)=\frac{1}{2}J_0(z)+\frac{1}{2}\cos z, \qquad \sum_{k=0}^\infty(-1)^k I_k(z)=\frac{1}{2}I_0(z)+\frac{1}{2}e^{-z}, \tag{6}$$

$$\sum_{k=0}^\infty I_k(z)=\frac{1}{2}I_0(z)+\frac{1}{2}e^{z}, \qquad \sum_{k=0}^\infty I_{2k}(z)=\frac{1}{2}I_0(z)+\frac{1}{2}\cosh(z), \tag{7}$$

where $J_n(z)$ is the Bessel function of the first kind and $I_n(z)$ is the modified Bessel function of the first kind. For integer order n and complex argument z, they are the entire complex functions (see (9.1.21) and (9.6.19) in [1])

$$J_n(z)=\frac{i^{-n}}{\pi}\int_0^\pi e^{iz\cos\theta}\cos(n\theta)\,d\theta, \qquad I_n(z)=\frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(n\theta)\,d\theta \tag{8}$$

and have the series representations (see (9.1.10) and (9.6.10) in [1])

$$J_n(z) = \left(\frac{1}{2}z\right)^n \sum_{k=0}^{\infty} \frac{(-\frac{1}{4}z^2)^k}{k!(n+k)!}, \qquad I_n(z) = \left(\frac{1}{2}z\right)^n \sum_{k=0}^{\infty} \frac{(\frac{1}{4}z^2)^k}{k!(n+k)!}. \tag{9}$$

Inspection of these series shows that

$$I_n(\pm \mathrm{i}z) = (\pm \mathrm{i})^n J_n(z), \qquad I_n(-z) = (-1)^n I_n(z). \tag{10}$$

Theorem 1. *The Neumann series in* (7) *generalize to*

$$\sum_{k=0}^{\infty} I_{nk}(z) = \frac{1}{2} I_0(z) + \frac{1}{2n} \sum_{k=0}^{n-1} \mathrm{e}^{z\cos(2\pi k/n)}, \quad n = 1, 2, \ldots . \tag{11}$$

Proof. For ξ real, let

$$f(x) = \begin{cases} \mathrm{e}^{\xi \cos(2\pi x)}, & 0 \leqslant x \leqslant 1. \\ 0, & x > 1. \end{cases} \tag{12}$$

This function is continuous on $[0,1]$, zero elsewhere, and satisfies the conditions of Titchmarsh's theorem above. Choosing $\alpha = 1/n$ and $\beta = 2\pi n$, n a positive integer, and noticing that $f(0+) = f(1-)$, renders (4) as

$$n\sqrt{2\pi}\left\{\frac{1}{2}F_{\mathrm{c}}(0) + \sum_{k=1}^{\infty} F_{\mathrm{c}}(2\pi kn)\right\} = f(0+) + \sum_{k=1}^{n-1} f\left(\frac{k}{n}\right). \tag{13}$$

If m is a nonnegative integer, it follows from (2), (8) and (12) that

$$\begin{aligned} F_{\mathrm{c}}(2\pi m) &= \sqrt{\frac{2}{\pi}} \int_0^1 \mathrm{e}^{\xi\cos(2\pi t)} \cos(2\pi m t)\,\mathrm{d}t \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{\xi \cos\theta} \cos(m\theta)\,\mathrm{d}\theta = \sqrt{\frac{2}{\pi}} I_m(\xi). \end{aligned} \tag{14}$$

Substituting for f and F_{c} in (13) gives

$$\sum_{k=0}^{\infty} I_{nk}(\xi) = \frac{1}{2} I_0(\xi) + \frac{1}{2n} \sum_{k=0}^{n-1} \mathrm{e}^{\xi\cos(2\pi k/n)} \tag{15}$$

and the general result in (11) follows by analytic continuation; the series in (7) correspond to (11) with $n = 1$ and 2, respectively. □

Corollary 2. *The Neumann series in* (6) *generalize to*

$$\sum_{k=0}^{\infty} (\pm\mathrm{i})^{nk} J_{nk}(z) = \frac{1}{2} J_0(z) + \frac{1}{2n} \sum_{k=0}^{n-1} \mathrm{e}^{\pm \mathrm{i}z\cos(2\pi k/n)}, \quad n = 1, 2, \ldots, \tag{16}$$

$$\sum_{k=0}^{\infty}(-1)^{nk}I_{nk}(z)=\frac{1}{2}I_0(z)+\frac{1}{2n}\sum_{k=0}^{n-1}\mathrm{e}^{-z\cos(2\pi k/n)},\quad n=1,2,\ldots,\tag{17}$$

where

$$(-1)^{nk}=\begin{cases}+1, & n\ \textit{even}\\ (-1)^k, & n\ \textit{odd}.\end{cases}\tag{18}$$

Proof. These results follow immediately from (10) and (11); (6a) corresponds to (16) with $n=2$, while (6b) corresponds to (17) with $n=1$. □

Corollary 3.

$$\sum_{k=0}^{\infty}J_{nk}(z)\cos(nk\pi/2)=\frac{1}{2}J_0(z)+\frac{1}{2n}\sum_{k=0}^{n-1}\cos(z\cos(2\pi k/n)),\quad n=1,2,\ldots,\tag{19}$$

$$\sum_{k=0}^{\infty}J_{nk}(z)\sin(nk\pi/2)=\frac{1}{2n}\sum_{k=0}^{n-1}\sin(z\cos(2\pi k/n)),\quad n=1,3,\ldots\ .\tag{20}$$

Proof. Writing $(\pm \mathrm{i})^{nk}$ as $\mathrm{e}^{\pm \mathrm{i}nk\pi/2}$ then adding and subtracting the two series in (16) gives (19) and (20). For even n, both sides of (20) are zero. □

Putting $n=1$ or 2 in (19) gives (6a); putting $n=1$ and 3 in (20) gives the two series:

$$\sum_{k=0}^{\infty}(-1)^kJ_{2k+1}(z)=\frac{1}{2}\sin z,\tag{21}$$

$$\sum_{k=0}^{\infty}(-1)^{k+1}J_{6k+3}(z)=\frac{1}{6}\sin z-\frac{1}{3}\sin(z/2),\tag{22}$$

respectively, the first of which agrees with (9.1.48) in [1] and (57.1.20) in [3].

Corollary 4.

$$\sum_{k=0}^{\infty}(-1)^kJ_{2mk}(z)=\frac{1}{2}J_0(z)+\frac{1}{2m}\cos z+\frac{1}{m}\sum_{k=1}^{(m-1)/2}\cos(z\cos(\pi k/m)),\quad m=1,3,\ldots,\tag{23}$$

$$\sum_{k=0}^{\infty}J_{4mk}(z)=\frac{1}{2}J_0(z)+\frac{1}{4m}(1+\cos z)+\frac{1}{2m}\sum_{k=0}^{m-1}\cos(z\cos(\pi k/2m)),\quad m=1,2,\ldots\ .\tag{24}$$

There is no summation on the right-hand side of either identity for $m=1$.

Proof. Putting $n=2m$ and $4m$ in (19) gives

$$\sum_{k=0}^{\infty}(-1)^{mk}J_{2mk}(z)=\frac{1}{2}J_0(z)+\frac{1}{4m}\sum_{k=0}^{2m-1}\cos(z\cos(\pi k/m)),\quad m=1,3,\ldots,\tag{25}$$

$$\sum_{k=0}^{\infty} J_{4mk}(z) = \frac{1}{2}J_0(z) + \frac{1}{8m}\sum_{k=0}^{4m-1}\cos(z\cos(\pi k/2m)), \quad m = 1,2,\ldots . \tag{26}$$

The results follow from (18) and symmetry. □

Setting $m = 1$ in (23) and (24) gives (6a) and

$$\sum_{k=0}^{\infty} J_{4k}(z) = \frac{1}{2}J_0(z) + \frac{1}{4}(1 + \cos z). \tag{27}$$

Corollary 5.

$$\sum_{k=0}^{\infty} (-1)^k J_{mk}^2(z) = \frac{1}{2}J_0^2(z) + \frac{1}{2m}J_0(2z) + \frac{1}{m}\sum_{k=1}^{(m-1)/2} J_0(2z\cos(\pi k/m)), \quad m = 1,3,\ldots, \tag{28}$$

$$\sum_{k=0}^{\infty} J_{2mk}^2(z) = \frac{1}{2}J_0^2(z) + \frac{1}{4m}(1 + J_0(2z)) + \frac{1}{2m}\sum_{k=1}^{m-1} J_0(2z\cos(\pi k/2m)), \quad m = 1,2,\ldots . \tag{29}$$

Proof. Replacing the argument z in (23) and (24) by $2z\sin\theta$, and then integrating the resulting expressions from 0 to $\pi/2$ with the aid of the three definite integrals

$$\int_0^{\pi/2} [\cos(z\sin\theta), J_{2n}(2z\sin\theta), J_0(2z\sin\theta)\cos(2n\theta)]\,\mathrm{d}\theta = \frac{\pi}{2}[J_0(z), J_n^2(z), J_n^2(z)], \tag{30}$$

from (9.1.18), (11.4.7) and (11.4.8) in [1], gives (28) and (29). The interchange in the order of integration and summation is justified by uniform convergence. □

Setting $m = 1$ in (28) and (29) gives the series

$$\sum_{k=0}^{\infty} (-1)^k J_k^2(z) = \frac{1}{2}J_0^2(z) + \frac{1}{2}J_0(2z), \tag{31}$$

$$\sum_{k=0}^{\infty} J_{2k}^2(z) = \frac{1}{2}J_0^2(z) + \frac{1}{4}J_0(2z) + \frac{1}{4}, \tag{32}$$

the first of which is (57.17.14) in [3].

Corollary 6.

$$\sum_{k,\ell=0}^{\infty} (-1)^\ell I_{nk+2\ell+1}(z) = \frac{1}{2}\sum_{\ell=0}^{\infty}(-1)^\ell I_{2\ell+1}(z) + \frac{1}{4n}\left\{ \sum_{\substack{k=0 \\ k\neq n/4,\,3n/4}}^{n-1} \frac{\mathrm{e}^{z\cos(2\pi k/n)} - 1}{\cos(2\pi k/n)} + Nz \right\}, \tag{33}$$

where $N = 2$ *if* n *is divisible by* 4, *otherwise* $N = 0$.

Proof. For integer order $n\in\mathbb{Z}$, (6.511 11.) of Gradshteyn and Ryzhik [2] gives

$$\int_0^z I_n(z)\,\mathrm{d}z = 2\sum_{\ell=0}^{\infty}(-1)^{\ell} I_{n+2\ell+1}(z). \tag{34}$$

Since, $I_{-n}(z)=I_n(z)$, there is no restriction on n. The result in (33) follows directly from (34) and (11). □

Corollary 7. *The generalized Neumann series* (11) *leads in turn to generalized trignometric identities, namely,*

$$\cot\frac{nz}{2} = \frac{1}{n}\sum_{k=0}^{n-1}\frac{\sin z}{\cos(2\pi k/n)-\cos z}, \tag{35}$$

$$2^{1-n}(\cos nz - 1) = \prod_{k=0}^{n-1}[\cos z - \cos(2\pi k/n)]. \tag{36}$$

Identity (36), *for* z *real, is the* $x=0$ *case of* (91.2.7) *in* [3].

Proof. The Laplace transform of $I_\nu(at)$, is given by (see Table 17.13 in [2])

$$\int_0^{\infty} \mathrm{e}^{-pt} I_\nu(at)\,\mathrm{d}t = a^{-\nu}\frac{(p-\sqrt{p^2-a^2})^{\nu}}{\sqrt{p^2-a^2}} \quad (\Re p > |\Re a|,\ \Re \nu > -1). \tag{37}$$

Let $p > a > 0$. Then setting $z=at$ in (11), multiplying by e^{-pt}, and integrating from $t=0$ to $t=\infty$, gives

$$\frac{1}{\sqrt{p^2-a^2}}\sum_{k=0}^{\infty}\left[\left(\frac{p-\sqrt{p^2-a^2}}{a}\right)^{n}\right]^{k} = \frac{1}{2}\frac{1}{\sqrt{p^2-a^2}} + \frac{1}{2n}\sum_{k=0}^{n-1}\frac{1}{p-a\cos(2\pi k/n)}, \tag{38}$$

wherein $(p-\sqrt{p^2-a^2})/a = a/(p+\sqrt{p^2-a^2}) < 1$. Summing the geometric series gives

$$\frac{1}{1-((p-\sqrt{p^2-a^2})/a)^n} = \frac{1}{2} + \frac{\sqrt{p^2-a^2}}{2n}\sum_{k=0}^{n-1}\frac{1}{p-a\cos(2\pi k/n)}. \tag{39}$$

If $p=a\cosh x$, $x>0$, $\sqrt{p^2-a^2}=a\sinh x$, $(p-\sqrt{p^2-a^2})/a=\cosh x-\sinh x=\mathrm{e}^{-x}$, and (39) becomes

$$\frac{1}{1-\mathrm{e}^{-nx}} = \frac{1}{2} + \frac{\sinh x}{2n}\sum_{k=0}^{n-1}\frac{1}{\cosh x-\cos(2\pi k/n)} \tag{40}$$

which simplifies to

$$\frac{1+\mathrm{e}^{-nx}}{1-\mathrm{e}^{-nx}} = \coth\frac{nx}{2} = \frac{1}{n}\sum_{k=0}^{n-1}\frac{\sinh x}{\cosh x-\cos(2\pi k/n)}. \tag{41}$$

Arguing by analytic continuation,

$$\coth\frac{nz}{2} = \frac{1}{n}\sum_{k=0}^{n-1}\frac{\sinh z}{\cosh z-\cos(2\pi k/n)}. \tag{42}$$

Then $\coth(\mathrm{i}z) = -\mathrm{i}\cot z, \sinh(\mathrm{i}z) = \mathrm{i}\sin z$ and $\cosh \mathrm{i}z = \cos z$ give

$$\cot\frac{nz}{2} = \frac{1}{n}\sum_{k=0}^{n-1}\frac{\sin z}{\cos(2\pi k/n) - \cos z}, \tag{43}$$

which is a generalization of the identity $\cot(z/2) = \sin z/(1 - \cos z)$. This is the first result.

Integrating (41) on $x > 0$, for fixed n, gives

$$\frac{2}{n}\ln\sinh\frac{nx}{2} = \frac{1}{n}\sum_{k=0}^{n-1}\ln[\cosh x - \cos(2\pi k/n)] + \frac{1}{n}\ln\kappa_n, \tag{44}$$

where $\ln\kappa_n/n$ is the integration constant. Rearrangement yields

$$\sinh^2\frac{nx}{2} = \kappa_n\prod_{k=0}^{n-1}[\cosh x - \cos(2\pi k/n)]. \tag{45}$$

Dividing both sides of this equation by e^{nx} and letting $x \to \infty$ shows that $\kappa_n = 2^{2-n}$, whence

$$2^{1-n}(\cosh nx - 1) = \prod_{k=0}^{n-1}[\cosh x - \cos(2\pi k/n)]. \tag{46}$$

Again, extending this identity to the complex plane by analytic continuation

$$2^{1-n}(\cosh nz - 1) = \prod_{k=0}^{n-1}[\cosh z - \cos(2\pi k/n)] \tag{47}$$

and then replacing z by $\mathrm{i}z$ gives

$$2^{1-n}(\cos nz - 1) = \prod_{k=0}^{n-1}[\cos z - \cos(2\pi k/n)]. \tag{48}$$

This is the second result. □

Corollary 8. *As an indication of the variety of results that can be derived, it is interesting to observe that, for* $a > 0$, $y > 0$, *and for* $m = 1, 3, \ldots,$

$$\sum_{k=0}^{\infty}(-1)^k J_{mk}(y/2)Y_{mk}(y/2) = \frac{1}{2}J_0(y/2)Y_0(y/2) + \frac{1}{2m}Y_0(y) + \frac{1}{m}\sum_{k=1}^{(m-1)/2}Y_0[y\cos(\pi k/m)], \tag{49}$$

$$\sum_{k=0}^{\infty}(-1)^k I_{mk}(ay/2)K_{mk}(ay/2) = \frac{1}{2}I_0(ay/2)K_0(ay/2) + \frac{1}{2m}K_0(ay) + \frac{1}{m}\sum_{k=1}^{(m-1)/2}K_0[ay\cos(\pi k/m)]. \tag{50}$$

In addition to the Bessel functions already defined, these series involve the Bessel function of the second kind $Y_\nu(z)$, *and the modified Bessel function of the second kind* $K_\nu(z)$. *These functions are defined for arbitrary order* ν *and complex argument* z *in* [1].

Proof. The first identity follows from (23) and

$$\int_1^{\infty} \frac{J_v(xy)\,\mathrm{d}x}{\sqrt{x^2-1}} = -\frac{\pi}{2} J_{v/2}(y/2) Y_{v/2}(y/2) \quad (y>0), \tag{51}$$

$$\int_1^{\infty} \frac{\cos(xy)\,\mathrm{d}x}{\sqrt{x^2-1}} = -\frac{\pi}{2} Y_0(y) \quad (y>0), \tag{52}$$

which are the tabulated results (6.552 6.) and (3.753 4.), respectively, in [2]. The latter (with $y>0$) is the $v=0$ case of (9.1.24b) in [1]. The second identity follows from (23) and

$$\int_0^{\infty} \frac{J_v(xy)\,\mathrm{d}x}{\sqrt{x^2+a^2}} = I_{v/2}(ay/2) K_{v/2}(ay/2) \quad (\Re a>0,\ y>0, \Re v>-1), \tag{53}$$

$$\int_0^{\infty} \frac{\cos(xy)\,\mathrm{d}x}{\sqrt{x^2+a^2}} = K_0(ay) \quad (\Re a>0, y>0), \tag{54}$$

which are the tabulated results (6.552 1.) and (3.754 2.), respectively, in [2]. Again, the interchange in the order of integration and summation is justified by uniform convergence. □

4. Summary

The central result is a Neumann series that expresses an infinite sum of modified Bessel functions as a finite sum of exponentials. This in turn leads to several generalized series of Bessel functions that confirm, extend and add to known identities in handbooks and the literature.

Acknowledgements

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References

[1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, Dover, New York, 1972.
[2] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series, and Products, Corrected and Enlarged Edition, Academic Press, Orlando, FL, 1980.
[3] E.R. Hansen, A Table of Series and Products, Prentice-Hall, Englewood Cliffs, NJ, 1975.
[4] E.C. Titchmarsh, Introduction to the Theory of Fourier Integrals, 3rd Edition, Chelsea, New York, 1986.
[5] G.N. Watson, A Treatise on the Theory of Bessel Functions, 2nd Edition, Cambridge University Press, Montpelier, Vermont, 1980.