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Tilting modules over an algebra by Igusa, Smalø and Todorov

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Abstract

The finiteness of the little finitistic dimension of an artin algebra R is known to be equivalent to the existence of a tilting R-module T such that $\{T\}^\perp = (\mathcal{P}^{<\infty})^\perp$ where $\mathcal{P}^{<\infty}$ is the category of all finitely presented R-modules of finite projective dimension. Moreover, T can be taken finitely generated if and only if $\mathcal{P}^{<\infty}$ is contravariantly finite.

In this paper, we describe explicitly the structure of T for the IST-algebra, a finite-dimensional algebra with $\mathcal{P}^{<\infty}$ not contravariantly finite. We also characterize the indecomposable modules in $\mathcal{P}^{<\infty}$, and all tilting classes over this algebra.

Keywords: Tilting modules; Cotorsion pairs; Representations of quivers

Infinite-dimensional tilting modules naturally occur in the approximation theory of modules over general rings. Surprisingly, they also play an important role in the classical setting of artin algebras.

The point is that the little finitistic dimension of an artin algebra R equals $n < \infty$ if and only if there is an n-tilting R-module T such that $\{T\}^\perp = (\mathcal{P}^{<\infty})^\perp$ where $\mathcal{P}^{<\infty}$ is the category of all finitely presented R-modules of finite projective dimension, and $\mathcal{C}^\perp = \bigcap_{1\leqslant i<\omega} \operatorname{Ker}\operatorname{Ext}^i_R(\mathcal{C}, -)$

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for a class of R-modules $\mathcal{C}$. Moreover, T can be taken finitely generated if and only if $\mathcal{P}^{<\infty}$ is contravariantly finite, [3].

Though in principle T can be computed by an iteration of $(\mathcal{P}^{<\infty})^{\perp}$-approximations of the regular module R, the structure of T remains unknown in general.

The main goal of this paper is to give an explicit description of T in an important case where $\mathcal{P}^{<\infty}$ is not contravariantly finite, namely for the IST-algebra A—the particular path algebra over a field with relations introduced by Igusa, Smalø and Todorov in [11]. A is known to have infinite global dimension, but the little and the big finitistic dimensions of A equal 1, so T is an infinite-dimensional A-module of projective dimension 1. Besides describing T, we also characterize all indecomposable A-modules in $\mathcal{P}^{<\infty}$, and all tilting classes over A.

The paper is organized as follows. After recalling necessary definitions and preliminary results (Section 1), we concentrate on the IST-algebra A, giving an overview of basic facts (Section 2), characterizing all indecomposable A-modules in $\mathcal{P}^{<\infty}$ up to isomorphism (Proposition 16) and computing their τ-translates. Next, we characterize the lattice of all tilting classes in A-Mod (Section 4) and compute corresponding tilting modules for some of these classes (Section 5), among them our tilting module T.

We also give examples of particular infinite-dimensional A-modules that are in the tilting class $\mathcal{T} = (\mathcal{P}^{<\infty})^{\perp}$, but are not isomorphic to a direct limit of finite-dimensional A-modules in $\mathcal{T}$ (Section 4.2).

1. Preliminaries

Let R be a ring (associative and unital) and let us denote by R-Mod (Mod-R) the category of left (right) R-modules, respectively. Let R-mod and mod-R be the corresponding full subcategories of all modules possessing a projective resolution with all projective modules finitely generated. Note that in case when R is noetherian, R-mod and mod-R coincide with the class of finitely generated left and right modules, respectively. For convenience, all modules from now on will be left R-modules if not stated otherwise. Further, let us denote by $\mathcal{P}_n^{<\infty}$ the full subcategory of R-mod consisting of the modules with proj.dim $\leqslant n$ and by $\mathcal{P}^{<\infty}$ the full subcategory of R-mod consisting of the modules of finite projective dimension.

A pair $(\mathcal{A}, \mathcal{B})$ of classes of modules is called a (*hereditary*) *cotorsion pair* if $\mathcal{A} = {}^{\perp}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp}$, where ${}^{\perp}\mathcal{B} = \{X \in R\text{-Mod} \mid \operatorname{Ext}_R^i(X, B) = 0 \text{ for all } B \in \mathcal{B} \text{ and } i \geqslant 1\}$ and $\mathcal{A}^{\perp} = \{X \in R\text{-Mod} \mid \operatorname{Ext}_R^i(A, X) = 0 \text{ for all } A \in \mathcal{A} \text{ and } i \geqslant 1\}$. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to be *cogenerated* by a class of modules $\mathcal{C}$ if the class $\mathcal{A}$ is the smallest possible containing $\mathcal{C}$, that is $\mathcal{A} = {}^{\perp}(\mathcal{C}^{\perp})$ and $\mathcal{B} = \mathcal{C}^{\perp}$. In case $\mathcal{C}$ contains just one module C, we will write $C^{\perp}$ instead of $\{C\}^{\perp}$.

A module T (not necessarily finitely generated) is said to be *n-tilting* for $n < \omega$ if it satisfies the following conditions:

(1) proj.dim $T \leqslant n$,
(2) $\operatorname{Ext}_R^i(T, T^{(\kappa)}) = 0$ for each $i \geqslant 1$ and cardinal κ,
(3) there is an exact sequence $0 \to R \to T_0 \to T_1 \to \cdots \to T_m \to 0$, where $m < \omega$ and $T_j \in \operatorname{Add} T$ for $0 \leqslant j \leqslant m$.

Here, $\operatorname{Add} T$ stands for the class of all modules isomorphic to direct summands of direct sums of copies of T.

A class of modules $\mathcal{T}$ is said to be *n-tilting* if there is an n-tilting module T such that $\mathcal{T} = T^{\perp}$. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to be *n-tilting* if $\mathcal{B}$ is an n-tilting class, or equivalently if it is cogenerated by some n-tilting module. A (n-)tilting class is of *finite type* in the sense of [2] if its corresponding cotorsion pair is cogenerated by some set of modules of R-mod. Note that n-tilting classes of finite type are exactly the classes $\mathcal{S}^{\perp}$ for $\mathcal{S} \subseteq \mathcal{P}_n^{<\infty}$, [17, 2.9].

The tilting theory is closely related to the second finitistic dimension conjecture. Let us denote by $\operatorname{Fdim} R$ and $\operatorname{fdim} R$ the big and the little finitistic dimension of R, respectively; that is, the supremum of the projective dimensions of all modules with proj.dim $< \infty$ or all finitely generated modules with proj.dim $< \infty$, respectively. The first finitistic dimension conjecture stated that $\operatorname{Fdim} R$ and $\operatorname{fdim} R$ coincide whenever R is a finite-dimensional algebra over a field, and it was proved to be false (cf. [16,19]). The second conjecture states that $\operatorname{fdim} R < \infty$ for finite-dimensional algebras and it is still an open problem in general, even though it turned out to be true for several special cases, [18]. In particular, a sufficient but not necessary condition is the contravariant finiteness of $\mathcal{P}^{<\infty}$. The following theorem relating the second conjecture to tilting theory is shown in [3]:

Theorem 1. *Let R be a left noetherian ring and $(\mathcal{A}, \mathcal{B})$ be the cotorsion pair cogenerated by $\mathcal{P}^{<\infty}$. Then* $\operatorname{fdim} R < \infty$ *if and only if $\mathcal{B}$ is a tilting class. Moreover, if T is a tilting module such that $T^{\perp} = \mathcal{B}$, then* $\operatorname{fdim} R = \operatorname{proj.dim} T$.

In the rest of this section, we recall some results concerning modules over artin algebras. A ring R is called an *artin algebra* if its center C is artinian and R is finitely generated as a C-module. We will use the following notation: D will stand for the canonical duality between left and right R-modules. For a finitely generated R-module X, we denote by $\operatorname{Tr} X$ its *transpose*, by $\tau X = D \operatorname{Tr} X$ its *Auslander–Reiten translation*, and by $\tau^{-} = \operatorname{Tr} D$ "inverse" of the translation. For unexplained terminology see [6].

For R-modules X, Y, denote by $\overline{\operatorname{Hom}}_R(X, Y)$ the quotient group of $\operatorname{Hom}_R(X, Y)$ by the subgroup of homomorphisms from X to Y which factor through an injective module. Similarly, let $\underline{\operatorname{Hom}}_R(X, Y)$ be the quotient of $\operatorname{Hom}_R(X, Y)$ by the homomorphisms which factor through projective modules. We will need the following important result:

Theorem 2 *(Auslander–Reiten formulas).* [5,12] *Let R be an artin algebra and let $X, Y \in R$-Mod, X finitely generated. Then there are following isomorphisms functorial in both X and Y*:

(1) $D \operatorname{Ext}^1_R(X, Y) \cong \overline{\operatorname{Hom}}_R(Y, \tau X)$.
(2) $\operatorname{Ext}^1_R(Y, X) \cong D \underline{\operatorname{Hom}}_R(\tau^{-} X, Y)$.

We also need a characterization of the finitely generated modules of projective or injective dimensions at most 1, which immediately follows from [6, IV.1.16]:

Proposition 3. *Let R be an artin algebra and $X \in R$-mod. Then*:

(1) $\operatorname{proj.dim} X \leqslant 1$ *if and only if* $\operatorname{Hom}_R(I, \tau X) = 0$ *for every injective module I.*
(2) $\operatorname{inj.dim} X \leqslant 1$ *if and only if* $\operatorname{Hom}_R(\tau^{-} X, P) = 0$ *for every projective module P.* $\square$

As a straightforward corollary, we get:

Corollary 4. *Let $X \in R$-mod. Then*:

(1) *If* $\operatorname{proj.dim} X \leqslant 1$, *then* $X^{\perp} = \operatorname{Ker} \operatorname{Hom}_R(-, \tau X)$.
(2) *If* $\operatorname{inj.dim} X \leqslant 1$, *then* ${}^{\perp}X = \operatorname{Ker} \operatorname{Hom}_R(\tau^{-} X, -)$.

Finally, we deduce the following lemma for artin algebras, which is useful in Section 4.2. It was introduced in [4] with a different proof:

Lemma 5. *A finitely generated module M belongs to $(\mathcal{P}_1^{<\infty})^{\perp}$ if and only if it is filtered by factors of the injective cogenerator $D(R)$.*

Proof. The if part is obvious, since $(\mathcal{P}_1^{<\infty})^{\perp} \cap R$-mod is closed under factors and extensions. For the only if part, it is enough to prove that $\operatorname{Hom}_R(D(R), M) \neq 0$ for each non-zero $M \in (\mathcal{P}_1^{<\infty})^{\perp} \cap R$-mod. Moreover, it is sufficient to prove this only for M indecomposable non-injective. Assume to the contrary that $\operatorname{Hom}_R(D(R), M)$ is trivial. Then $\operatorname{proj.dim} \tau^{-} M \leqslant 1$ by Proposition 3. Thus $\operatorname{Ext}_R^1(\tau^{-} M, M) = 0$, a contradiction to the existence of an almost split sequence, [6, V.1.15]. $\square$

2. An example by Igusa, Smalø and Todorov

Let us fix an algebraically closed field k and let A be the algebra introduced by Igusa, Smalø and Todorov in [11], shortly *IST-algebra*. It is a path algebra over k over the quiver

$$\Gamma:\quad \begin{tikzcd} 1 \arrow[r, bend left=40, "\gamma"] & 2 \arrow[l, bend left=10, "\beta"'] \arrow[l, bend left=50, "\alpha"] \end{tikzcd}$$

with relations $\alpha\gamma = \beta\gamma = \gamma\alpha = 0$. In our notation, paths are composed as maps from right to left. From now on, all modules will be considered as modules over this algebra if not stated otherwise. Basic properties of A-modules are summarized in [1, Section 5].

Let us denote $\Lambda = A/\langle\gamma\rangle$; then Λ is isomorphic to *Kronecker algebra*, the hereditary algebra $k\Gamma'$ over the following quiver:

$$\Gamma':\quad \begin{tikzcd} 1 & 2 \arrow[l, bend left=10, "\beta"'] \arrow[l, bend left=50, "\alpha"] \end{tikzcd}$$

Modules M with $\gamma M = 0$ will be called *Kronecker modules*, since they are also Λ-modules. Let us denote by P_i, I_i and S_i the indecomposable projective, injective and simple A-module corresponding to the vertex i ($i = 1, 2$), respectively. Then $\dim_k P_1 = 2$, $\dim_k P_2 = 4$ and $\dim_k I_1 = \dim_k I_2 = 3$. Let $\mathcal{P}^{<\infty}$ be the full subcategory of all finitely generated A-modules of finite projective dimension as before, and let $\mathcal{KP}^{<\infty}$ be the full subcategory of $\mathcal{P}^{<\infty}$ having exactly the Kronecker modules in $\mathcal{P}^{<\infty}$ as objects.

We will briefly recall basic facts about the Kronecker modules. A detailed description of the finite-dimensional Λ-modules is done in [6]. More properties of infinite-dimensional Λ-modules can be found in [14], [13] or [9].

The finite-dimensional indecomposable Λ-modules are divided into three families, *preprojective*, *preinjective* and *regular* modules:

(1) The preprojectives Q_n, $n \geqslant 1$, are the modules with the representation $V_1 = k^n$, $V_2 = k^{n-1}$, $f_\beta = (E, 0)^T$ and $f_\alpha = (0, E)^T$, where E is the unit matrix $(n-1) \times (n-1)$.
(2) The preinjectives J_n, $n \geqslant 1$, are the modules with the representation $V_1 = k^{n-1}$, $V_2 = k^n$, $f_\beta = (E, 0)$ and $f_\alpha = (0, E)$.
(3) For the *quasi-simple* regulars R_λ, $\lambda \in k \cup \{\infty\}$, the vector spaces of the representation are $V_1 = V_2 = k$. For $\lambda \in k$, f_β is the multiplication by λ and f_α is the identity map. For $\lambda = \infty$, f_β the identity map and $f_\alpha = 0$.
(4) Every quasi-simple regular module R_λ, $\lambda \in k \cup \{\infty\}$, defines a *tube*; that is, a chain of indecomposable modules

$$R_\lambda = R_{\lambda,1} \subseteq R_{\lambda,2} \subseteq R_{\lambda,3} \subseteq \cdots$$

connected by the almost split sequences $0 \to R_{\lambda,n} \to R_{\lambda,n-1} \oplus R_{\lambda,n+1} \to R_{\lambda,n} \to 0$ in Λ-mod. Any finite-dimensional indecomposable regular module occurs in this way.

Note, that there are no non-zero homomorphisms from preinjectives to preprojectives or regulars, and no non-zero homomorphisms from regulars to preprojectives. Moreover, $\dim_k \operatorname{Hom}_\Lambda(R_\lambda, R_\mu) = \delta_{\lambda,\mu}$ for any $\lambda, \mu \in k \cup \{\infty\}$.

Prüfer modules $R_{\lambda,\infty}$ are defined as the direct limits of the ascending chains:

$$R_{\lambda,1} \subseteq R_{\lambda,2} \subseteq R_{\lambda,3} \subseteq \cdots.$$

Then $\operatorname{Hom}_\Lambda(R_{\lambda,\infty}, R_\mu) = 0$ and $\dim_k \operatorname{Hom}_\Lambda(R_\mu, R_{\lambda,\infty}) = \delta_{\lambda,\mu}$ for any $\lambda, \mu \in k \cup \{\infty\}$.

3. Finitely generated modules of finite projective dimension

3.1. Simple modules and composition series in $\mathcal{P}^{<\infty}$

In fact, $\mathcal{P}^{<\infty}$ is not an abelian category, but it is closed under extensions, kernels of epimorphisms and cokernels of monomorphisms. We will call an object X of $\mathcal{P}^{<\infty}$ *simple in* $\mathcal{P}^{<\infty}$, if it has no proper submodule that is again an object of $\mathcal{P}^{<\infty}$, or equivalently if it has no proper factor again in $\mathcal{P}^{<\infty}$.

For every finitely generated A-module M, there is an exact sequence

$$0 \to P_1^n \to M \to \overline{M} \to 0$$

where $n < \omega$ and $\overline{M}$ is a Kronecker module. As a cosequence, we get:

Lemma 6. [1, Proposition 5.1] *A module M is an object of $\mathcal{P}^{<\infty}$ if and only if it has a finite filtration*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_l = M$$

with the factors M_j/M_{j-1} isomorphic either to P_1 or to R_λ for some $\lambda \in k$.

Note also that the modules P_1 and R_λ, $\lambda \in k$, are then precisely the simples in $\mathcal{P}^{<\infty}$ in our sense.

3.2. The (non-)uniqueness of the composition series

In general, there is no result analogous to the Jordan–Hölder Theorem in $\mathcal{P}^{<\infty}$. Take for example the short exact sequences $0 \to P_1 \xrightarrow{\iota_\lambda} P_2 \to R_\lambda \to 0$. These exist for all $\lambda \in k$.

But the number of the factors isomorphic to P_1 is unique. Consider a function $f : \mathcal{P}^{<\infty} \to \omega$ defined by the formula:

$$f(U) = \dim_k \mathrm{Hom}_A(U, R_\infty).$$

Since P_1 is projective, we have $\mathrm{Ext}^1_A(P_1, R_\infty) = 0$. The module R_∞ has no submodule isomorphic to S_2, so $\mathrm{Ext}^1_A(R_\lambda, R_\infty) = \mathrm{Ext}^1_\Lambda(R_\lambda, R_\infty) = 0$ for each $\lambda \in k$ by [1, 5.3]. Thus, $\mathrm{Ext}^1_A(U, R_\infty) = 0$ for every $U \in \mathcal{P}^{<\infty}$ and $f(V) = f(U) + f(W)$ for each exact sequence $0 \to U \to V \to W \to 0$ of modules from $\mathcal{P}^{<\infty}$. Further, $f(P_1) = 1$ and $f(R_\lambda) = 0$ for each $\lambda \in k$. The function f "counts" the number of factors isomorphic to P_1 in composition series of modules $U \in \mathcal{P}^{<\infty}$, and its definition is independent of the particular composition series.

If we are only concerned with the modules in $\mathcal{KP}^{<\infty}$, then composition series are unique in the sense of Jordan–Hölder. This can be seen by a similar reasoning as for P_1, this time using the functions:

$$g_\mu(U) = \dim_k \mathrm{Hom}_A(U, R_{\mu,\infty}), \quad \mu \in k.$$

Again, $\mathrm{Ext}^1_A(R_\lambda, R_{\mu,\infty}) = 0$ for every $\lambda, \mu \in k$ and $g_\mu(R_\lambda) = \delta_{\lambda,\mu}$. The function g_μ "counts" the factors isomorphic to R_μ and its definition is independent of the particular composition series.

3.3. Determining regular Kronecker modules by matrices

Let $M \in \mathcal{KP}^{<\infty}$. Then we can write

$$M \cong R_{\lambda_1,i_1} \oplus \cdots \oplus R_{\lambda_m,i_m}$$

for some Kronecker regular modules $R_{\lambda_1,i_1}, \ldots, R_{\lambda_m,i_m}$ with $\lambda_1, \ldots, \lambda_m \in k$. In particular, the linear map $x \mapsto \alpha x$ is a bijective map $e_2M \to e_1M$, since this is true for every R_{λ_j,i_j}. Let us denote by α_M^{-1} the inverse map for a given module M and define the map $\chi_M \in \mathrm{End}_k(e_1M)$ by the formula $\chi_M(x) = \beta \cdot \alpha_M^{-1}(x)$.

Let us focus on the matrix J_M of the linear map χ_M in the Jordan canonical form, with respect to some suitable k-basis of the vector space e_1M. When $M \cong R_{\lambda,i}$, then J_M is the Jordan cell of size $i \times i$ corresponding to the eigenvalue λ, that is:

$$J_M = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}.$$

In general, J_M is block-diagonal, built of the Jordan cells corresponding to the direct summands $R_{\lambda_1,i_1}, \ldots, R_{\lambda_m,i_m}$ of M. That is, $J_M = \mathrm{diag}(J_{R_{\lambda_1,i_1}}, \ldots, J_{R_{\lambda_m,i_m}})$.

Let N be another module from $\mathcal{KP}^{<\infty}$. It is easy to see that if the vector spaces e_1M and e_1N have the same dimension and the linear maps χ_M and χ_N are similar, then the modules M and N are isomorphic. Thus we can state:

Lemma 7. *Two modules M, N from $\mathcal{KP}^{<\infty}$ are isomorphic if and only if the Jordan canonical forms of matrices of the linear maps χ_M and χ_N are the same up to the order of Jordan cells.*

3.4. Special modules of finite projective dimension

Definition 8. A module $M \in \mathcal{P}^{<\infty}$ will be called *special* if its composition series in $\mathcal{P}^{<\infty}$ admits exactly one factor isomorphic to P_1 and if it has no submodule isomorphic to any R_λ, $\lambda \in k$. Let us denote by $\mathcal{SP}^{<\infty}$ the full subcategory of $\mathcal{P}^{<\infty}$ consisting of the special modules.

For example, the modules P_1 and P_2 are special. It is easy to see that special modules are indecomposable. Clearly, if $M \in \mathcal{SP}^{<\infty}$ and M' is a non-zero submodule of M belonging to $\mathcal{P}^{<\infty}$, then $M' \in \mathcal{SP}^{<\infty}$ too. All modules in $\mathcal{SP}^{<\infty}$ have even dimension, since by [11] the same is true for all modules in $\mathcal{P}^{<\infty}$. In the next few paragraphs we will show that for each non-zero even $n < \omega$ there is exactly one isomorphism class of modules of dimension n in $\mathcal{SP}^{<\infty}$. We will start by proving the existence.

Lemma 9. *Let $\lambda \in k$, and let $\delta : 0 \to P_1 \to M \to R_\lambda \to 0$ be an exact sequence. Then either δ splits or $M \cong P_2$. Moreover, δ splits if and only if M has a submodule isomorphic to R_λ.*

Proof. There is always an exact sequence $0 \to P_1 \xrightarrow{\iota_\lambda} P_2 \to R_\lambda \to 0$, and since P_2 is projective, we have the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[r] & P_1 \arrow[r, "\iota_\lambda"] \arrow[d, "f"'] & P_2 \arrow[r] \arrow[d] & R_\lambda \arrow[r] \arrow[d, equal] & 0 \\
\delta: & 0 \arrow[r] & P_1 \arrow[r] & M \arrow[r] & R_\lambda \arrow[r] & 0.
\end{tikzcd}$$

Since $\dim_k \operatorname{End}_A(P_1) = \dim_k e_1P_1 = 1$, f is either the zero map or an isomorphism. In the first case δ splits, in the second case $M \cong P_2$. The second assertion holds, because P_2 has no submodule isomorphic to R_λ. $\square$

Proposition 10. *Take $n < \omega$ non-zero even. Then there is a module $M \in \mathcal{SP}^{<\infty}$ of dimension n.*

Proof. We have the module P_1 for $n = 2$. So let $n > 2$. Put $m = \frac{n}{2} - 1$ and choose m distinct elements $\lambda_1, \ldots, \lambda_m$ of the field k. For each λ_j, consider the exact sequence $0 \to P_1 \xrightarrow{\iota_j}$

$P_2 \to R_{\lambda_j} \to 0$. We will construct the desired module M by the following push-out, where $\sigma : P_1^m \to P_1$ is the summation map:

$$\begin{tikzcd}
0 \arrow[r] & P_1^m \arrow[r, "\bigoplus_j \iota_j"] \arrow[d, "\sigma"'] & P_2^m \arrow[r] \arrow[d] & \bigoplus_j R_{\lambda_j} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P_1 \arrow[r, "\iota"] & M \arrow[r, "\pi"] & \bigoplus_j R_{\lambda_j} \arrow[r] & 0.
\end{tikzcd}$$

Suppose that there is a submodule $N \subseteq M$ isomorphic to R_λ for some $\lambda \in k$. But $\operatorname{soc} N \cong S_1$ and $\operatorname{soc} \iota(P_1) \cong S_2$, so $\iota(P_1) \cap N = 0$ and $\pi \restriction N$ is monic. The module $\pi(N)$ being a submodule of $\bigoplus_j R_{\lambda_j}$ and $\pi(N) \cong R_\lambda$, there must be an index j such that $\lambda = \lambda_j$ and $\pi(N) = R_{\lambda_j}$. Then we have the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & P_1 \arrow[r, "\iota_j"] \arrow[d, "\sigma \restriction P_1"'] & P_2 \arrow[r] \arrow[d] & R_{\lambda_j} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P_1 \arrow[r, "\iota"] & \iota(P_1) + N \arrow[r, "\pi \restriction \iota(P_1)+N"] & R_{\lambda_j} \arrow[r] & 0.
\end{tikzcd}$$

The map in the left column, and therefore also the map in the middle column, is an isomorphism. But the first row does not split and the second row does, a contradiction. Thus $M \in \mathcal{SP}^{<\infty}$. $\square$

Next, we would like to prove that every two modules in $\mathcal{SP}^{<\infty}$ of the same dimension are isomorphic. This is obvious for the dimension 2. First, we will prove a lemma which places a restriction on possible forms of cokernels of inclusions of the module P_1 into a chosen module from $\mathcal{SP}^{<\infty}$.

Lemma 11. *Let $M \in \mathcal{SP}^{<\infty}$ and $0 \to P_1 \xrightarrow{\iota} M \xrightarrow{\pi} \bigoplus_{j=1}^m R_{\lambda_j, i_j} \to 0$ be an exact sequence. Then the elements $\lambda_1, \ldots, \lambda_m$ are pairwise distinct.*

Proof. Assume for a contradiction that the converse is true. Without loss of generality, put $\lambda = \lambda_1 = \lambda_2$. Then the module $\bigoplus_{j=1}^m R_{\lambda_j, i_j}$ has a submodule isomorphic to $R_\lambda \oplus R_\lambda$, and this gives rise to the exact sequence $0 \to P_1 \xrightarrow{\iota} M' \xrightarrow{\pi \restriction M'} R_\lambda \oplus R_\lambda \to 0$. Denote $M'_v = \pi^{-1}(R_\lambda)$ where R_λ is the vth component of $R_\lambda \oplus R_\lambda$, $v = 1, 2$. Since

$$0 \to P_1 \to M'_v \to R_\lambda \to 0 \tag{1}$$

does not split, we have $M'_v \cong P_2$ by Lemma 9. Take a generator h of $\iota(P_1)$ corresponding to e_1 in the presentation of P_1 as Ae_1. Let g_1, g_2 be generators of M'_1, M'_2, respectively, corresponding to the element $e_2 \in P_2 = Ae_2$. We see immediately from the non-split exact sequence (1) that $\beta g_v - \lambda \alpha g_v \in \iota(P_1) \setminus \{0\}$ for $v = 1, 2$. Hence

$$h, (\beta g_1 - \lambda \alpha g_1), (\beta g_2 - \lambda \alpha g_2) \in \iota(e_1 P_1).$$

And since $\iota(e_1 P_1)$ is a 1-dimensional k-vector space, we can assume by possibly multiplying g_1 or g_2 by a scalar that

$$\beta g_v - \lambda \alpha g_v = h, \quad v = 1, 2.$$

Finally, denote $g = g_1 - g_2$. It is straightforward to check that the submodule of M generated by g is isomorphic to R_λ, a contradiction. $\square$

The core of the proof of uniqueness is the following proposition, which states that there is no other restriction for the form of a cokernel of the inclusion ι, apart from the one in Lemma 11.

Proposition 12. *Let $M \in \mathcal{SP}^{<\infty}$, $M \ncong P_1$. Put $n = (\dim_k M)/2 - 1$. Then for arbitrary pairwise distinct elements $\lambda_1, \ldots, \lambda_m \in k$ and positive integers $i_1, \ldots, i_m$ such that $i_1 + \cdots + i_m = n$, there is an inclusion $\iota : P_1 \to M$ with* $\operatorname{Coker} \iota \cong \bigoplus_{j=1}^m R_{\lambda_j, i_j}$.

Proof. Start by considering an arbitrary inclusion $\iota' : P_1 \to M$ and denote $C = \operatorname{Coker} \iota' \cong \bigoplus_{j=1}^q R_{\mu_j, i'_j}$. Then by Lemma 7, the module C is determined up to isomorphism by the Jordan canonical form of a matrix of the linear map χ_C. But there is only one Jordan cell for each eigenvalue of χ_C in the Jordan canonical form by Lemma 11. Thus, the cokernel C is in fact determined only by the multiplicities of the eigenvalues of χ_C. Using the following construction, we can increase by 1 a multiplicity of a chosen $\lambda \in k$ as an eigenvalue, or $\lambda \in k$ will become an eigenvalue if it has not been before. And we can do this at the cost of decreasing the multiplicity of the eigenvalue μ_1 by 1. After applying this method a finite number of times, we can "change" the eigenvalues, and thus also the cokernel of an inclusion $P_1 \to M$, to any prescribed form.

Take an exact sequence $0 \to P_1 \xrightarrow{\iota'} M \xrightarrow{\pi} \bigoplus_{j=1}^q R_{\mu_j, i'_j} \to 0$. Let us denote $M_j = \pi^{-1}(R_{\mu_j, i'_j})$. Further, take the "canonical" generators $\bar{g}_{j,v}$ of R_{μ_j, i'_j} satisfying

$$\beta \bar{g}_{j,1} = \mu_j \alpha \bar{g}_{j,1}, \tag{2}$$

$$\beta \bar{g}_{j,v} = \mu_j \alpha \bar{g}_{j,v} + \alpha \bar{g}_{j,v-1}, \quad 1 < v \leqslant i'_j. \tag{3}$$

Let us denote by $g_{j,v}$ some fixed preimages of $\bar{g}_{j,v}$ under π; that is, $\bar{g}_{j,v} = \pi(g_{j,v})$. Since all $\bar{g}_{j,v}$ could have been chosen to lie in $e_2 R_{\mu_j, i'_j}$, we can w.l.o.g. assume that all $g_{j,v}$ are in $e_2 M$. Moreover, Eq. (2) yields that $\beta g_{j,1} - \mu_j \alpha g_{j,1} \in \iota'(e_1 P_1)$. Since the vector space $\iota'(e_1 P_1)$ is only 1-dimensional, we can assume, possibly by multiplying some of the $g_{j,v}$'s by a scalar, that there is a non-zero element $h \in \iota'(e_1 P_1)$ such that $h = \beta g_{j,1} - \mu_j \alpha g_{j,1}$ for each $j \leqslant q$. And it is easy to see from the representation of P_1 that h generates $\iota'(P_1)$.

Take the module $L \subseteq M_1$ generated by $g_{1,1}$. Then $L \cong P_2$ by Lemma 9 and for any fixed $\lambda \in k$, there is an exact sequence $0 \to P_1 \xrightarrow{\vartheta} L \to R_\lambda \to 0$. In fact, we have also the following exact sequence for some regular Kronecker module Y:

$$0 \to P_1 \xrightarrow{\vartheta} M \xrightarrow{\sigma} Y \to 0.$$

Denote $\bar{f}_{j,v} = \sigma(g_{j,v})$ and let h' be a generator of $\vartheta(P_1)$ such that $h' = \beta g_{1,1} - \lambda \alpha g_{1,1}$. Then

$$\beta \bar{f}_{j,v} = \sigma(\beta g_{j,v}) = \sigma(\mu_j \alpha g_{j,v} + \alpha g_{j,v-1} + c_{j,v} h) = \mu_j \alpha \bar{f}_{j,v} + \alpha \bar{f}_{j,v-1} + c_{j,v} \sigma(h)$$

where $c_{j,v} \in k$ are suitable constants, and for convenience we assume $g_{j,0} = 0$ and $\bar{f}_{j,0} = 0$. This comes from the fact that $\beta g_{j,v} - \mu_j \alpha g_{j,v} - \alpha g_{j,v-1} \in \iota'(e_1 P_1)$ by Eqs. (2) and (3) and $\iota'(e_1 P_1)$ is a 1-dimensional k-vector space generated by h. Further:

$$h = \beta g_{1,1} - \mu_1 \alpha g_{1,1} = h' + (\lambda - \mu_1)\alpha g_{1,1}.$$

So we have:

$$c_{j,v}\sigma(h) = c_{j,v}(\lambda - \mu_1)\alpha\sigma(g_{1,1}) = c_{j,v}(\lambda - \mu_1)\alpha \bar{f}_{1,1}$$

and together:

$$\beta \bar{f}_{j,v} = \mu_j \alpha \bar{f}_{j,v} + \alpha \bar{f}_{j,v-1} + c_{j,v}(\lambda - \mu_1)\alpha \bar{f}_{1,1}.$$

The matrix of the linear endomorphism χ_Y of the vector space $e_1 Y$, with respect to the basis $\alpha \bar{f}_{j,v}$, $j \leqslant q$, $v \leqslant i'_j$, and the pairs (j, v) being ordered lexicographically, is of the form

$$\begin{pmatrix} \lambda & * & * & * & * & * & \cdots \\ & \mu_1 & \ddots & & & & \\ & & \ddots & 1 & & & \\ & & & \mu_1 & & & \\ & & & & \mu_2 & 1 & \\ & & & & & \ddots & \ddots \end{pmatrix}$$

where the symbols $*$ in the first row are to be substituted by some suitably chosen elements of k. Comparing the eigenvalues of χ_Y with the eigenvalues of χ_C, we see that we have exactly changed one occurrence of μ_1 for one occurrence of λ. $\square$

Proposition 13. *Let $n < \omega$. Then any two modules in $\mathcal{SP}^{<\infty}$ of dimension n are isomorphic.*

Proof. It is enough to carry out the proof only for $n > 2$ even. Choose an arbitrary $M \in \mathcal{SP}^{<\infty}$ of dimension n. Put $m = \frac{n}{2} - 1$ and choose m pairwise distinct elements $\lambda_1, \ldots, \lambda_m$ of the field k. Then, by the former proposition, there is an exact sequence

$$0 \to P_1 \xrightarrow{\iota} M \xrightarrow{\pi} \bigoplus_{j=1}^{m} R_{\lambda_j} \to 0.$$

Let N be the factor of the module P_2^m, with generators of the individual components of P_2^m denoted $g_1, \ldots, g_m$, determined by the relations $\beta g_i - \lambda_i \alpha g_i = \beta g_{i+1} - \lambda_{i+1}\alpha g_{i+1}$. Then the dimension of N is at most $n = 2m + 2$, since $\dim_k e_1 P_2^m = \dim_k e_2 P_2^m = 2m$, and for both these vector spaces we have $m - 1$ k-independent relations. Further, considering the proof of the preceding proposition, there is an epimorphism $N \to M$ which maps every element g_i to some suitably chosen generator of $\pi^{-1}(R_{\lambda_i})$. Thus, $\dim_k M = \dim_k N = 2m + 2 = n$ and $N \cong M$. And since the module N is independent of the choice of the module M, we have at most one isomorphism class of A-modules in $\mathcal{SP}^{<\infty}$ for each dimension. $\square$

For every $n \geqslant 1$, let us denote by P_n one fixed representative of the objects of $\mathcal{SP}^{<\infty}$ of dimension $2n$. This notation is consistent with the former notation of the indecomposable projectives P_1 a P_2, since these two modules are representatives of the modules in $\mathcal{SP}^{<\infty}$ of dimensions 2 and 4, respectively.

3.5. Auslander–Reiten translation of modules from $\mathcal{P}^{<\infty}$

In view of Corollary 4, it is convenient to determine the Auslander–Reiten translations of the modules in $\mathcal{P}^{<\infty}$. In this subsection, we will prove that the modules R_λ, $\lambda \in k$, are invariant with respect to the translation, while the modules from $\mathcal{SP}^{<\infty}$ are mapped to the Kronecker preprojective modules.

It is well known that the functor $(-)^* = \mathrm{Hom}_A(-, A)$ maps the indecomposable projective (left) A-module $P_i = Ae_i$ to an indecomposable projective right A-module isomorphic to $e_i A$, $i = 1, 2$. And the latter isomorphism assigns to the path $p \in e_i A$ ending at the vertex i the following homomorphism from Ae_i to A:

$$\begin{aligned} p^* : Ae_i &\to A, \\ e_i &\mapsto p \ (\in A). \end{aligned}$$

From now on, we will identify the modules $e_i A$ and P_i^*. In particular, we will denote the homomorphism in P_i^* corresponding to a path $p \in e_i A$ as p^* to distinguish elements of A considered as left or right A-module. Note that the right A-module structure of P_i^* is given by $p^* \cdot q = (pq)^*$ for a path $q \in A$. It is also clear that a homomorphism $f \in P_i^*$ is determined by its value on e_i. Thus, if $f(e_i) = \sum_{j=1}^m a_j p_j$ for some paths $p_1, \ldots, p_m \in A$ and elements $a_1, \ldots, a_m \in k$, then $f = \sum_{j=1}^m a_j p_j^*$.

Lemma 14. *Let $\lambda \in k$. Then $\tau R_\lambda \cong R_\lambda$.*

Proof. The minimal projective presentation of the module R_λ is $0 \to P_1 \xrightarrow{\iota_\lambda} P_2 \to R_\lambda \to 0$, where $\iota_\lambda(e_1) = \beta - \lambda\alpha$. Considering the map $\iota_\lambda^* : P_2^* \to P_1^*$, we see:

$$\big(\iota_\lambda^*(e_2^*)\big)(e_1) = e_2^* \iota_\lambda(e_1) = e_2^*(\beta - \lambda\alpha) = \beta - \lambda\alpha.$$

Thus, $\iota_\lambda^*(e_2^*) = \beta^* - \lambda\alpha^*$. The module P_1^* has a k-basis e_1^*, α^*, β^*. For $M = P_1^* / \mathrm{Im}\, \iota_\lambda^*$, we have $\dim_k Me_1 = \dim_k Me_2 = 1$ and $M\gamma = 0$. Therefore, DM must be a Kronecker quasi-simple regular module. Because $M(\beta - \lambda\alpha) = 0$, it is also $(\beta - \lambda\alpha)DM = 0$, and thus $DM = D\,\mathrm{Tr}\, R_\lambda \cong R_\lambda$. $\square$

Recall that Q_j denotes the jth indecomposable Kronecker preprojective module; that is $\dim_k e_1 Q_j = j$ and $\dim_k e_2 Q_j = j - 1$. For P_1 and P_2, obviously $\tau P_1 = \tau P_2 = 0$.

Lemma 15. *Let $3 \leqslant n < \omega$. Then $\tau P_n \cong Q_{n-2}$.*

Proof. Looking at the first (push-out) diagram in the proof of Proposition 10, we see that there is a projective presentation of P_n of the form

$$0 \to P_1^{n-2} \xrightarrow{\vartheta} P_2^{n-1} \to P_n \to 0. \tag{4}$$

Next, fix $n-1$ pairwise distinct elements $\lambda_1, \dots, \lambda_{n-1}$ of the field k. Let us denote by f_j the residue of the trivial path e_1 in the jth copy of P_1 and by g_l the residue of the path e_2 in the lth copy of P_2. Examining the proof of Proposition 13 (namely, the construction of the module N there which turns out to be isomorphic to P_n), we can assume that ϑ acts as follows:

$$\vartheta(f_j) = (\beta g_j - \lambda_j \alpha g_j) - (\beta g_{j+1} - \lambda_{j+1}\alpha g_{j+1}), \quad 1 \leqslant j \leqslant n-2.$$

Consequently, it is straightforward to see that the presentation (4) is minimal.

For arbitrary A-modules M, N and non-zero natural numbers m, v, there is a canonical bijection between the elements of $\mathrm{Hom}_A(M^m, N^v)$ and the matrices $v \times m$ over $\mathrm{Hom}_A(M, N)$. Let us denote by $i_j : M \to M^m$ the jth inclusion and by $p_l : N^v \to N$ the lth projection. Then this bijection assigns to an homomorphism $h \in \mathrm{Hom}_A(M^m, N^v)$ the matrix $(p_l h i_j)_{l \leqslant v, j \leqslant m}$. Moreover, $i_j^* : (M^*)^m \to M^*$ is the jth projection, $p_l^* : N^* \to (N^*)^v$ is the lth inclusion, and by the similar canonical bijection for right A-module homomorphisms, the element $h^* \in \mathrm{Hom}_A((N^*)^v, (M^*)^m)$ corresponds to the matrix $(i_j^* h^* p_l^*)_{j \leqslant m, l \leqslant v}$.

Now put $M = P_1$, $N = P_2$, $m = n-2$ and $v = n-1$. Then the map ϑ corresponds to the matrix (ϑ_{lj}), where $\vartheta_{lj} = p_l \vartheta\, i_j$. It holds:

$$\vartheta_{lj}(e_1) = p_l \vartheta(f_j) = \begin{cases} \beta - \lambda_l \alpha & \text{for } l = j, \\ -(\beta - \lambda_l \alpha) & \text{for } l = j+1, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that:

$$\begin{aligned} \big(\vartheta_{jj}^*\big(e_2^*\big)\big)(e_1) &= e_2^* \vartheta_{jj}(e_1) = e_2^*(\beta - \lambda_j \alpha) = \beta - \lambda_j \alpha, \\ \big(\vartheta_{j+1,j}^*\big(e_2^*\big)\big)(e_1) &= e_2^* \vartheta_{j+1,j}(e_1) = e_2^*\big(-(\beta - \lambda_{j+1}\alpha)\big) = -(\beta - \lambda_{j+1}\alpha). \end{aligned}$$

Thus:

$$\vartheta_{lj}^*\big(e_2^*\big) = \begin{cases} \beta^* - \lambda_l \alpha^* = e_1^* \cdot (\beta - \lambda_l \alpha) & \text{for } l = j, \\ -(\beta^* - \lambda_l \alpha^*) = -e_1^* \cdot (\beta - \lambda_l \alpha) & \text{for } l = j+1, \\ 0 & \text{otherwise.} \end{cases}$$

For the map $\vartheta^* : (P_2^*)^{n-1} \to (P_1^*)^{n-2}$, let us denote by g_l' the residue of the element e_2^* in the lth copy of P_2^*, and by f_j' the residue of the element e_1^* in the jth copy of P_1^*. We attain the following formulas by composing the results of the former computations:

$$\vartheta^*\big(g_l'\big) = \begin{cases} f_l'(\beta - \lambda_l \alpha) & \text{for } l = 1, \\ f_l'(\beta - \lambda_l \alpha) - f_{l-1}'(\beta - \lambda_l \alpha) & \text{for } 1 < l < n-1, \\ -f_{l-1}'(\beta - \lambda_l \alpha) & \text{for } l = n-1. \end{cases}$$

Since $\lambda_1, \dots, \lambda_{n-1}$ are pairwise distinct, we have $\dim_k(\mathrm{Im}\,\vartheta^*)e_2 = n-1$. Clearly $(\mathrm{Im}\,\vartheta^*)e_1 = 0$. And we have $\dim_k P_1^* e_1 = 1$, $\dim_k P_1^* e_2 = 2$. Thus for the module $L = (P_1^*)^{n-2}/\mathrm{Im}\,\vartheta^*$ we have $\dim_k L e_1 = n-2$ and $\dim_k L e_2 = 2(n-2) - (n-1) = n-3$. Then $\dim_k e_1 DL = n-2$ and $\dim_k e_2 DL = n-3$. Moreover, $DL = D\,\mathrm{Tr}\,P_n$ must be an indecomposable Kronecker module, and by the characterization of such modules we have $DL \cong Q_{n-2}$. □

3.6. Indecomposable modules in $\mathcal{P}^{<\infty}$

We will use the results of the preceding section to characterize the indecomposable modules of $\mathcal{P}^{<\infty}$ up to isomorphism.

Proposition 16. *Let $0 \neq M \in \mathcal{P}^{<\infty}$ be indecomposable. Then one of the following cases holds true*:

(1) *$M \cong R_{\lambda,i}$ for some $\lambda \in k$ and $i \geqslant 1$,*
(2) *$M \cong P_n$ for some $n \geqslant 1$.*

Before we prove the proposition itself, we need some auxiliary lemmas.

Lemma 17. *Let $M \in \mathcal{P}^{<\infty}$ such that M has no submodule isomorphic to R_λ for any $\lambda \in k$. Then M is $\mathcal{SP}^{<\infty}$-filtered.*

Proof. We will prove the lemma by an induction on the number n of composition factors isomorphic to P_1 in a composition series of M in $\mathcal{P}^{<\infty}$. There is nothing to prove for $n = 1$. Let $n > 1$. Take a composition series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_l = M$$

of M such that the last index j for which $M_{j+1}/M_j \cong P_1$ is the greatest possible. Then $M/M_j \in \mathcal{SP}^{<\infty}$ by the assumption and M_j is $\mathcal{SP}^{<\infty}$-filtered by the induction hypothesis. Thus, M is $\mathcal{SP}^{<\infty}$-filtered too. □

Lemma 18. *Let M be a finitely generated $\mathcal{SP}^{<\infty}$-filtered module. Then M is a direct sum of modules from $\mathcal{SP}^{<\infty}$.*

Proof. The modules P_1 a P_2 are projective and every module P_n, $n \geqslant 3$, has a minimal projective presentation of the form $0 \to P_1^{n-2} \to P_2^{n-1} \to P_n \to 0$. Thus, a minimal projective presentation of the module M must be of the form:

$$0 \to P_1^m \to P_1^u \oplus P_2^v \to M \to 0.$$

The module $\operatorname{Tr} M$ is a factor of $(P_1^*)^m$ by definition. Therefore, the module $D\operatorname{Tr} M$ is a submodule of $D(P_1^*)^m = I_1^m$. Since I_1 is a Kronecker module, so is τM.

Let us choose an arbitrary $\lambda \in k \cup \{\infty\}$. Then

$$D\operatorname{Ext}_A^1(P_n, R_\lambda) \cong \operatorname{Hom}_A(R_\lambda, \tau P_n) \cong \operatorname{Hom}_A(R_\lambda, Q_{n-2}) = 0$$

for all $n \geqslant 3$. The first isomorphism follows by Theorem 2 and Proposition 3 and the second by Lemma 15. In particular, $\operatorname{Ext}_A^1(M, R_\lambda) = 0$, and so $\operatorname{Hom}_A(R_\lambda, \tau M) = 0$. Thus, the module τM is preprojective, that is $\tau M \cong \bigoplus_{j=1}^m Q_{i_j}$ for some $i_1, \dots, i_m$. Then

$$M \cong P \oplus \tau^-(\tau M) \cong P \oplus \bigoplus_{j=1}^m P_{i_j+2}$$

for some finitely generated projective module P. □

Proof of Proposition 16. Let $M \in \mathcal{P}^{<\infty}$ be indecomposable. If M is a Kronecker module, we are in the case number 1.

Suppose M is not a Kronecker module and L is a maximal $\mathcal{KP}^{<\infty}$-submodule of M. Since the subcategory $\mathcal{KP}^{<\infty}$ is closed under extensions, M/L has no submodule isomorphic to R_λ, $\lambda \in k$. Then M/L is $\mathcal{SP}^{<\infty}$-filtered by Lemma 17. Further, we have

$$D\,\mathrm{Ext}_A^1(P_n, R_\lambda) \cong \mathrm{Hom}_A(R_\lambda, \tau P_n) \cong \mathrm{Hom}_A(R_\lambda, Q_{n-2}) = 0$$

for all $\lambda \in k$ and $n \geqslant 3$—the first isomorphism by Theorem 2 and Proposition 3 and the second by Lemma 15. In particular, $\mathrm{Ext}_A^1(M/L, L) = 0$ and $M \cong L \oplus M/L$. Thus, $L = 0$ and $M \in \mathcal{SP}^{<\infty}$ by Lemma 18. □

4. Tilting classes

4.1. The lattice of tilting classes

Since $\operatorname{Fdim} A = 1$ by [11], every tilting A-module is 1-tilting. By [8], all 1-tilting classes over any associative unital ring are of finite type. Thus, every tilting class in A-Mod can be obtained as $\mathcal{S}^\perp$, where $\mathcal{S}$ is some subset of objects of $\mathcal{P}^{<\infty}$. Let us denote by $\operatorname{ind}\mathcal{P}^{<\infty}$ a representative subset of the indecomposable modules in $\mathcal{P}^{<\infty}$. Obviously, it is always possible to choose $\mathcal{S}$ as a subset of $\operatorname{ind}\mathcal{P}^{<\infty}$.

Proposition 19. *The class $\mathcal{T} \subseteq A$-Mod is a tilting class if and only if there is a subset $\mathcal{S} \subseteq \operatorname{ind}\mathcal{P}^{<\infty}$ such that $\mathcal{S}^\perp = \mathcal{T}$.*

Let $\mathcal{S} \subseteq \operatorname{ind}\mathcal{P}^{<\infty}$. Let us denote $\bar{\mathcal{S}} = {}^\perp(\mathcal{S}^\perp) \cap \operatorname{ind}\mathcal{P}^{<\infty}$. It is easy to see that $\mathcal{S}^\perp = \bar{\mathcal{S}}^\perp$. We will call a subset $\mathcal{S}$ of $\operatorname{ind}\mathcal{P}^{<\infty}$ *closed* if $\mathcal{S} = \bar{\mathcal{S}}$. Clearly, the lattice of 1-tilting classes is anti-isomorphic to the lattice of closed subsets of $\operatorname{ind}\mathcal{P}^{<\infty}$. A description of the closed subsets follows.

Theorem 20. *A subset $\mathcal{S} \subseteq \operatorname{ind}\mathcal{P}^{<\infty}$ is closed if and only if it satisfies the following conditions*:

(1) *$P_1 \in \mathcal{S}$, $P_2 \in \mathcal{S}$.*
(2) *If $R_{\lambda,i} \in \mathcal{S}$ for some $\lambda \in k$ and $i \geqslant 1$, then $R_{\lambda,j} \in \mathcal{S}$ for every $j \geqslant 1$.*
(3) *If $R_{\lambda,i} \in \mathcal{S}$ for some $\lambda \in k$ and $i \geqslant 1$, then $P_j \in \mathcal{S}$ for every $j \geqslant 1$.*
(4) *If $P_n \in \mathcal{S}$ for some $n \geqslant 3$, then $P_j \in \mathcal{S}$ for every $j \leqslant n$.*

Proof. First, assume $\mathcal{S} \subseteq \operatorname{ind}\mathcal{P}^{<\infty}$ is closed. The necessity of the condition (1) is obvious. For Kronecker regular modules, we have the exact sequences:

$$0 \to R_{\lambda,i} \to R_{\lambda,i-1} \oplus R_{\lambda,i+1} \to R_{\lambda,i} \to 0.$$

Thus, if $R_{\lambda,i} \in \mathcal{S}$, then also $R_{\lambda,i-1}, R_{\lambda,i+1} \in \mathcal{S}$. The condition (2) follows by induction. Further, by Proposition 12 we have

$$0 \to P_1 \to P_j \to R_{\lambda,j-1} \to 0,$$

for each $j \geqslant 3$. This implies the condition (3). Let $n \geqslant 3$ and $M \in P_n^{\perp}$. Then $\mathrm{Hom}_A(M, Q_{n-2}) = 0$ by Corollary 4 and Lemma 15. Thus, $\mathrm{Hom}_A(M, Q_{j-2}) = 0$ for each $3 \leqslant j \leqslant n$, since Q_{n-2} has submodules isomorphic to Q_{j-2}. This means that $M \in P_j^{\perp}$, and $P_j \in {}^{\perp}(P_n^{\perp})$ for each $3 \leqslant j \leqslant n$. This yields condition (4).

Conversely, let $\mathcal{S} \subseteq \mathrm{ind}\,\mathcal{P}^{<\infty}$ satisfy the conditions (1)–(4). Assume that there is some $M \in \bar{\mathcal{S}} \setminus \mathcal{S}$. If $M = R_{\lambda,i}$ for some λ and i, then $R_{\lambda,j} \notin \mathcal{S}$ for each $j \geqslant 1$ by the condition (2). But this implies $R_\lambda \in \mathcal{S}^{\perp}$ using the characterization of $\mathrm{ind}\,\mathcal{P}^{<\infty}$ in Proposition 16. Then also $R_\lambda \in \bar{\mathcal{S}}^{\perp}$, which is a contradiction. Thus, it remains only the case $M = P_n$ for some $n \geqslant 3$. But then $R_{\lambda,i} \notin \mathcal{S}$ for each $\lambda \in k$, $i \geqslant 1$, and $P_j \notin \mathcal{S}$ for each $j \geqslant n$ by the conditions (3) and (4). So $\mathcal{S}$ consists only of some of the modules $P_1, \ldots, P_{n-1}$, again by Proposition 16. But then Corollary 4 and Lemma 15 yield $Q_{n-2} \in \mathcal{S}^{\perp} = \bar{\mathcal{S}}^{\perp}$ and $D\,\mathrm{Ext}_A^1(P_n, Q_{n-2}) \cong \mathrm{Hom}_A(Q_{n-2}, Q_{n-2}) \neq 0$, a contradiction to the assumption $P_n \in \bar{\mathcal{S}}$. □

Corollary 21. $(\mathcal{P}^{<\infty})^{\perp} = \{R_\lambda \mid \lambda \in k\}^{\perp} = \bigcap_{\lambda \in k} \mathrm{Ker}\,\mathrm{Hom}_A(-, R_\lambda)$.

Proof. For the first equality, see [1, 5.4]. Or alternatively, if we take $\mathcal{S} = \{R_\lambda \mid \lambda \in k\}$, then $\bar{\mathcal{S}} = \mathrm{ind}\,\mathcal{P}^{<\infty}$ by the former theorem. Thus $\mathcal{S}^{\perp} = (\mathrm{ind}\,\mathcal{P}^{<\infty})^{\perp} = (\mathcal{P}^{<\infty})^{\perp}$. The second equality follows from Corollary 4 and Lemma 14. □

4.2. Impossibility of reconstructing a tilting class from finitely generated modules by direct limits

This section is inspired by the dual case, where every 1-cotilting class $\mathcal{C}$ over a noetherian ring could be reconstructed from its finitely generated modules by direct limits. That is $\mathcal{C} = \varinjlim(\mathcal{C} \cap R\text{-mod})$, $\mathcal{C}$ being closed under direct limits, since every 1-cotilting module is pure-injective by [7]. So there is a bijective correspondence between the 1-cotilting classes and the torsion-free classes of finitely generated modules containing ${}_R R$, [17].

But an analogous proposition with direct limits is not true for 1-tilting classes over IST-algebra. Take $\mathcal{T} = (\mathcal{P}^{<\infty})^{\perp}$ and $\mathcal{T}^{<\infty} = \mathcal{T} \cap A\text{-mod}$. Then $\mathcal{T} = \varinjlim \mathcal{T}^{<\infty}$ implies that $\varinjlim \mathcal{T}^{<\infty}$ is closed under direct products. This is equivalent to the covariant finiteness of $\mathcal{T}^{<\infty}$ in A-mod by [1], and thus to the contravariant finiteness of $\mathcal{P}^{<\infty}$ in A-mod by [15]. But this is not true for IST-algebra. The aim of this subsection is to give particular examples of modules from $\mathcal{T} \setminus \varinjlim \mathcal{T}^{<\infty}$.

Proposition 22. *Let $\mathcal{T} = (\mathcal{P}^{<\infty})^{\perp}$ and $\mathcal{T}^{<\infty} = \mathcal{T} \cap A$-mod. Then the Prüfer module $R_{\lambda,\infty}$ is a member of $\mathcal{T}$ for each $\lambda \in k$, but* $\mathrm{Hom}_A(M, R_{\lambda,\infty}) = 0$ *for all $M \in \mathcal{T}^{<\infty}$.*

Proof. It is well known that $\mathrm{Hom}_A(R_{\lambda,\infty}, R_\mu) = 0$ for each $\mu \in k$. Therefore, $R_{\lambda,\infty} \in \mathcal{T}$ by Corollary 21.

We have $\mathrm{Hom}_A(I, R_\lambda) = 0$ for an injective cogenerator $I = I_1 \oplus I_2$ by Proposition 3 and Lemma 14. Then also $\mathrm{Hom}_A(I, R_{\lambda,\infty}) = 0$ and $\mathrm{Hom}_A(M, R_{\lambda,\infty}) = 0$ for every factor M of I. Thus, $\mathrm{Hom}_A(M, R_{\lambda,\infty}) = 0$ for each $M \in \mathcal{T}^{<\infty}$ by Lemma 5.

Corollary 23. $R_{\lambda,\infty} \in \mathcal{T} \setminus \varinjlim \mathcal{T}^{<\infty}$ *for each $\lambda \in k$.*

5. Tilting modules

5.1. Constructing more complex preenvelopes

Now we are close to show an explicit structure of a tilting module for the class $(\mathcal{P}^{<\infty})^{\perp}$. First, we need the following general proposition which is valid for any ring R. Let us recall that a module X over an arbitrary ring is said to be FP_2, if it possesses an exact sequence $W_2 \to W_1 \to W_0 \to X \to 0$ with W_0, W_1, W_2 finitely generated projective R-modules. Then whenever X is an FP_2 module, $X^{\perp_1} = \operatorname{Ker}\operatorname{Ext}^1_R(X, -)$ is closed under direct limits, thus also under filtrations and arbitrary direct sums (see e.g. [1]).

Proposition 24. *Let R be an arbitrary ring and $\mathcal{S}$ be a set of FP_2 modules such that $\operatorname{Ext}^1_R(X, Y) = 0$ for any pair of distinct modules $X, Y \in \mathcal{S}$. Further, let $M \in R$-Mod be any module and assume that*

$$0 \to M \to J_X \to C_X \to 0$$

is a special $X^{\perp_1}$-preenvelope with an $\{X\}$-filtered cokernel C_X for each $X \in \mathcal{S}$. Then the second row of the following push-out diagram (the map σ just adding up the components of the direct sum) is a special $\mathcal{S}^{\perp_1}$-preenvelope of M:

$$\begin{tikzcd}
0 \arrow[r] & M^{(\mathcal{S})} \arrow[r] \arrow[d, "\sigma"'] & \bigoplus_{X\in\mathcal{S}} J_X \arrow[r] \arrow[d] & \bigoplus_{X\in\mathcal{S}} C_X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] & J \arrow[r] & \bigoplus_{X\in\mathcal{S}} C_X \arrow[r] & 0.
\end{tikzcd}$$

Proof. It is sufficient to prove that $J \in \mathcal{S}^{\perp_1}$ and $C = \bigoplus_{X\in\mathcal{S}} C_X \in {}^{\perp_1}(\mathcal{S}^{\perp_1}) = \operatorname{Ker}\operatorname{Ext}^1_R(-, \mathcal{S}^{\perp_1})$. But the latter is clear, since the module C is a direct sum of $\mathcal{S}$-filtered modules and ${}^{\perp_1}\mathcal{C}$ is closed under direct sums and filtrations for an arbitrary class of modules $\mathcal{C}$ (see [10, Lemma 1]).

Choose an arbitrary $Y \in \mathcal{S}$. If we take only the component corresponding to the module Y in the first row of the commutative diagram above, and if we denote by σ' the restriction of the map σ to that component, we will get an induced diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r] \arrow[d, "\sigma'"'] & J_Y \arrow[r] \arrow[d] & C_Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & J \arrow[r] \arrow[d] & \bigoplus_{X\in\mathcal{S}} C_X \arrow[r] \arrow[d] & 0. \\
 & & \bigoplus_{X\in\mathcal{S}\setminus\{Y\}} C_X \arrow[r, equal] \arrow[d] & \bigoplus_{X\in\mathcal{S}\setminus\{Y\}} C_X \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

By assumption, $X \in Y^{\perp_1}$ for each $X \in \mathcal{S} \setminus \{Y\}$ and $Y^{\perp_1}$ is closed under filtrations and direct sums, thus $\bigoplus_{X \in \mathcal{S} \setminus \{Y\}} C_X \in Y^{\perp_1}$. But also $J_Y \in Y^{\perp_1}$, therefore $J \in Y^{\perp_1}$. And this is true for any $Y \in \mathcal{S}$, so $J \in \mathcal{S}^{\perp_1}$. $\square$

5.2. Structure of tilting modules for $R_\lambda^\perp$

Construction 25 *($R_\lambda^\perp$-preenvelopes of P_1 and P_2). Let $\lambda \in k$. By Proposition* 12*, there is an exact sequence $0 \to P_1 \to P_{n+1} \xrightarrow{\sigma} R_{\lambda,n} \to 0$ for each $n \geqslant 1$. If we take an inclusion $j : R_{\lambda,n-1} \to R_{\lambda,n}$ for any $n \geqslant 2$, then the module $M = \sigma^{-1}(\mathrm{Im}\, j)$ is clearly an object of $\mathcal{SP}^{<\infty}$ (cf. the remark after Definition* 8*), thus $M \cong P_n$ by Proposition* 13*. Moreover, $P_{n+1}/M \cong R_{\lambda,n}/\mathrm{Im}\, j \cong R_\lambda$. So we have the following exact sequence for any $n \geqslant 1$:*

$$0 \longrightarrow P_n \xrightarrow{\iota_{n+1,n}} P_{n+1} \longrightarrow R_\lambda \longrightarrow 0.$$

Let us denote $\iota_{m,n} = \iota_{m,m-1} \ldots \iota_{n+2,n+1}\iota_{n+1,n}$ and $\iota_{n,n} = 1_{P_n}$ for every $m > n \geqslant 1$. The following squares are obviously commutative for $n \geqslant 2$:

$$\begin{tikzcd} P_1 \arrow[r, "\iota_{n,1}"] \arrow[d, equal] & P_n \arrow[d, "\iota_{n+1,n}"] \\ P_1 \arrow[r, "\iota_{n+1,1}"] & P_{n+1}. \end{tikzcd}$$

Further, $\mathrm{Coker}\, \iota_{n,1}$ *is R_λ-filtered, thus* $\mathrm{Coker}\, \iota_{n,1} \cong R_{\lambda,n-1}$ *by Lemma* 11*. Therefore, we have the exact commutative diagrams with monomorphisms in columns*:

$$\begin{tikzcd} 0 \arrow[r] & P_1 \arrow[r, "\iota_{n,1}"] \arrow[d, equal] & P_n \arrow[r, "\pi_n"] \arrow[d, "\iota_{n+1,n}"] & R_{\lambda,n-1} \arrow[r] \arrow[d, "j_n"] & 0 \\ 0 \arrow[r] & P_1 \arrow[r, "\iota_{n+1,1}"] & P_{n+1} \arrow[r, "\pi_{n+1}"] & R_{\lambda,n} \arrow[r] & 0. \end{tikzcd}$$

Let us denote by T_λ the direct limit of the modules P_n, $n \geqslant 1$, with the inclusions $\iota_{m,n}$, $m \geqslant n \geqslant 1$. We obtain the exact sequence:

$$\delta_1 : 0 \longrightarrow P_1 \xrightarrow{\iota} T_\lambda \xrightarrow{\pi} R_{\lambda,\infty} \longrightarrow 0.$$

Next, take the commutative diagram with the canonical inclusions in columns:

$$\begin{tikzcd} 0 \arrow[r] & P_1 \arrow[r, "\iota_{2,1}"] \arrow[d, equal] & P_2 \arrow[r, "\pi_2"] \arrow[d, "\iota'"] & R_\lambda \arrow[r] \arrow[d, "j'"] & 0 \\ 0 \arrow[r] & P_1 \arrow[r, "\iota"] & T_\lambda \arrow[r, "\pi"] & R_{\lambda,\infty} \arrow[r] & 0. \end{tikzcd}$$

Then $\mathrm{Coker}\, \iota' \cong \mathrm{Coker}\, j' \cong R_{\lambda,\infty}$*, thus we have the exact sequence*:

$$\delta_2 : 0 \longrightarrow P_2 \xrightarrow{\iota'} T_\lambda \xrightarrow{\pi'} R_{\lambda,\infty} \longrightarrow 0.$$

Using this notation, we get:

Proposition 26. *The short exact sequences δ_1 and δ_2 are special $R_\lambda^\perp$-preenvelopes of the indecomposable projective modules P_1 and P_2, respectively.*

Proof. It is sufficient to prove that $T_\lambda \in R_\lambda^\perp$ and $R_{\lambda,\infty} \in {}^\perp(R_\lambda^\perp)$. The latter is clear, since the Prüfer module $R_{\lambda,\infty}$ is R_λ-filtered.

It is enough to show that $\mathrm{Hom}_A(T_\lambda, R_\lambda) = 0$ by Corollary 4 and Lemma 14. Take an arbitrary $f \in \mathrm{Hom}_A(T_\lambda, R_\lambda)$. If we apply the functor $\mathrm{Hom}_A(-, R_\lambda)$ to the exact sequence $0 \to P_1 \xrightarrow{\iota_{2,1}} P_2 \to R_\lambda \to 0$, we obtain

$$0 \longrightarrow \mathrm{Hom}_A(R_\lambda, R_\lambda) \longrightarrow \mathrm{Hom}_A(P_2, R_\lambda) \xrightarrow{\mathrm{Hom}_A(\iota_{2,1}, R_\lambda)} \mathrm{Hom}_A(P_1, R_\lambda).$$

But $\dim_k \mathrm{Hom}_A(R_\lambda, R_\lambda) = 1$, and also $\dim_k \mathrm{Hom}_A(P_i, R_\lambda) = \dim_k e_i R_\lambda = 1$ for $i = 1, 2$. This implies $\mathrm{Hom}_A(\iota_{2,1}, R_\lambda) = 0$. So $f\iota = f\iota'\iota_{2,1} = 0$. Therefore, there is a map $\bar{f}$ such that $f = \bar{f}\pi$. But now $\bar{f} \in \mathrm{Hom}_A(R_{\lambda,\infty}, R_\lambda) = 0$, and thus $f = 0$. □

Theorem 27. *Let $\lambda \in k$ and T_λ be as in Construction 25. Then $T_\lambda \oplus R_{\lambda,\infty}$ is a tilting module corresponding to the tilting class $R_\lambda^\perp$.*

Proof. By the proof of [17, Theorem 29], once we have a tilting class $\mathcal{T}$, we can construct a corresponding tilting module by iterating special $\mathcal{T}$-preenvelopes starting with the regular module ${}_AA$. Since $R_\lambda^\perp$ is a 1-tilting class, we need to construct only the first iteration. We have ${}_AA \cong P_1 \oplus P_2$, and by the former proposition, there is a special $R_\lambda^\perp$-preenvelope of A of the form

$$0 \to A \to T_\lambda \oplus T_\lambda \to R_{\lambda,\infty} \oplus R_{\lambda,\infty} \to 0.$$

The corresponding tilting module is then $T = (T_\lambda \oplus T_\lambda) \oplus (R_{\lambda,\infty} \oplus R_{\lambda,\infty})$. Note that if T' is a module such that $T' \in \mathrm{Add}\, T$ and $T \in \mathrm{Add}\, T'$, then T' is tilting too, and $T^\perp = (T')^\perp$. Putting $T' = T_\lambda \oplus R_{\lambda,\infty}$ gives us the desired result. □

Remark 28. Let us write down a linear representation corresponding to the module T_λ. It is of the shape

$$\begin{tikzcd} V_1 \arrow[r, bend left, "f_\gamma"] & V_2 \arrow[l, "f_\beta"] \arrow[l, bend left, "f_\alpha"] \end{tikzcd}$$

with the linear maps satisfying equations $f_\alpha f_\gamma = f_\beta f_\gamma = f_\gamma f_\alpha = 0$.

Since T_λ is countable-dimensional, we put $V_1 = V_2 = k^{(\omega)}$. Then the linear maps for T_λ are given by the following column-finite matrices:

$$f_\alpha = \begin{pmatrix} 0 & 0 & 0 & \dots \\ 0 & 1 & & \\ 0 & & 1 & \\ \vdots & & & \ddots \end{pmatrix}, \qquad f_\beta = \begin{pmatrix} 0 & 1 & & \\ 0 & \lambda & 1 & \\ 0 & & \lambda & \ddots \\ \vdots & & & \ddots \end{pmatrix}, \qquad f_\gamma = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 0 & & \\ 0 & & 0 & \\ \vdots & & & \ddots \end{pmatrix}.$$

For the sake of completeness, we also write down a representation of the corresponding Prüfer module $R_{\lambda,\infty}$:

$$f_\alpha = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & & \\ 0 & & 1 & \\ \vdots & & & \ddots \end{pmatrix}, \qquad f_\beta = \begin{pmatrix} \lambda & 1 & & \\ 0 & \lambda & 1 & \\ 0 & & \lambda & \ddots \\ \vdots & & & \ddots \end{pmatrix}, \qquad f_\gamma = \begin{pmatrix} 0 & 0 & 0 & \dots \\ 0 & 0 & & \\ 0 & & 0 & \\ \vdots & & & \ddots \end{pmatrix}.$$

Note also that in contrast to Proposition 13, the modules T_λ and T_μ are non-isomorphic for $\lambda \neq \mu$. Otherwise, there would be an inclusion $i : P_1 \to T_\mu$ with the cokernel isomorphic to $R_{\lambda,\infty}$. But this is not possible, since a cokernel of any inclusion $i : P_1 \to T_\mu$ is isomorphic to $R_{\mu,\infty} \oplus M$, where M is a suitable finitely generated Kronecker regular module.

5.3. *Structure of a tilting module for* $(\mathcal{P}^{<\infty})^\perp$

Theorem 29. *Let $X \subseteq k$ be a non-empty subset and put $\mathcal{S} = \{R_\lambda \mid \lambda \in X\}$. For each $\lambda \in X$ take the special preenvelope $0 \to P_1 \xrightarrow{\iota_\lambda} T_\lambda \to R_{\lambda,\infty} \to 0$ from Construction 25, and take the following push-out diagram with the summation map σ:*

$$\begin{tikzcd}
0 \arrow[r] & P_1^{(X)} \arrow[r] \arrow[d, "\sigma"'] & \bigoplus_{\lambda\in X} T_\lambda \arrow[r] \arrow[d] & \bigoplus_{\lambda\in X} R_{\lambda,\infty} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P_1 \arrow[r] & T_X \arrow[r] & \bigoplus_{\lambda\in X} R_{\lambda,\infty} \arrow[r] & 0.
\end{tikzcd}$$

Then $T = T_X \oplus \bigoplus_{\lambda\in X} R_{\lambda,\infty}$ is a tilting module corresponding to the tilting class $\mathcal{S}^\perp$.

Proof. The set $\mathcal{S}$ fulfills the assumptions of Proposition 24. Thus, the exact sequence $0 \to P_1 \to T_X \to \bigoplus_{\lambda\in X} R_{\lambda,\infty} \to 0$ is a special $\mathcal{S}^\perp$-preenvelope of the projective P_1.

Take an arbitrary $\mu \in X$. Then we have the following commutative diagram with isomorphisms in the first and monomorphisms in the other columns:

$$\begin{tikzcd}
0 \arrow[r] & P_1 \arrow[r] \arrow[d, equal] & P_2 \arrow[r] \arrow[d, "\iota'"'] & R_\mu \arrow[r] \arrow[d, "j'"'] & 0 \\
0 \arrow[r] & P_1 \arrow[r, "\iota_\mu"] \arrow[d, "\sigma'"'] & T_\mu \arrow[r] \arrow[d, "\iota''"'] & R_{\mu,\infty} \arrow[r] \arrow[d, "j''"'] & 0 \\
0 \arrow[r] & P_1 \arrow[r] & T_X \arrow[r] & \bigoplus_{\lambda\in X} R_{\lambda,\infty} \arrow[r] & 0.
\end{tikzcd}$$

Thus $T_X/\operatorname{Im}\iota''\iota' \cong \bigoplus_{\lambda\in X} R_{\lambda,\infty}/\operatorname{Im} j''j' \cong \bigoplus_{\lambda\in X} R_{\lambda,\infty}$, and we have the following short exact sequence, which is necessarily a special $\mathcal{S}^{\perp}$-preenvelope of the module P_2:

$$0 \longrightarrow P_2 \xrightarrow{\iota''\iota'} T_X \longrightarrow \bigoplus_{\lambda\in X} R_{\lambda,\infty} \longrightarrow 0.$$

Since $A \cong P_1 \oplus P_2$, the module $T \oplus T$ is tilting corresponding to the tilting class $\mathcal{S}^{\perp}$ [17, proof of Theorem 2.9], and so is T itself. □

With the notation of Theorem 29, we get for $X = k$:

Corollary 30. *$T_k \oplus \bigoplus_{\lambda\in k} R_{\lambda,\infty}$ is a tilting module corresponding to $(\mathcal{P}^{<\infty})^{\perp}$.*

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