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An Example of Blowup Produced by Equal Diffusions

Hans F. Weinberger

School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455

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1. INTRODUCTION

It is easily seen that if the single ordinary differential equation

$$\frac{du}{dt} = f(u)$$

has the property that each of its solutions remains bounded, then each solution with bounded initial data of the partial differential equation

$$\frac{\partial u}{\partial t} = \Delta u + f(u) \qquad \text{in } D$$

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial D$$

also remains bounded.

Mizoguchi, Ninomiya, and Yanagida [6] recently proved that the analog of this result is not true for systems of equations. More precisely, they showed that there are solutions with bounded initial data of the nonlinear system

$$\begin{aligned} \frac{\partial u}{\partial t} &= d_u\, \Delta u + |u-v|^{p-1}\,(u-v) - u \\ \frac{\partial v}{\partial t} &= d_v\, \Delta v + |u-v|^{p-1}\,(u-v) - v \\ \frac{\partial u}{\partial n} &= \frac{\partial v}{\partial n} = 0 \qquad \text{on } \partial D \end{aligned} \tag{1.1}$$

with $p > 1$ and $0 \leqslant d_u < d_v$ which blow up in finite time, even though all solutions of the system obtained by setting $d_u = d_v = 0$ approach $(0, 0)$.

The example of Mizoguchi, Ninomiya, and Yanagida depends strongly on the fact that the two diffusion constants are different. In fact, we shall

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show near the beginning of Section 3 that when $d_u = d_v$, any solution of the system (1.1) with bounded initial data remains bounded.

It is the purpose of this note to exhibit a system with equal diffusion constants which has the same properties as the example of Mizoguchi, Ninomiya, and Yanagida. We shall prove that the system

$$\begin{aligned} &\frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} + uv(u-v)(u+1) - \delta u && \text{in } (-1, 1) \\ &\frac{\partial v}{\partial t} = \varepsilon \frac{\partial^2 v}{\partial x^2} + vu(v-u)(v+1) - \delta v && \text{in } (-1, 1) \\ &\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0 && \text{at} \quad x = \pm 1 \end{aligned} \tag{1.2}$$

has the property that when $\varepsilon = 0$ and $\delta > 0$, the origin is a global attractor of the positive solutions, but that when ε and δ are both positive, there is a positive solution with bounded initial data which blows up before any prescribed positive time T.

A positive solution of the system (1.2) is also a solution of the system

$$\begin{aligned} &\frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} + u\,|v|\,(|u| - |v|)(|u| + 1) - \delta u && \text{in } (-1, 1) \\ &\frac{\partial v}{\partial t} = \varepsilon \frac{\partial^2 v}{\partial x^2} + v\,|u|\,(|v| - |u|)(|v| + 1) - \delta v && \text{in } (-1, 1) \\ &\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0 && \text{at} \quad x = \pm 1 \end{aligned} \tag{1.3}$$

for which the origin is a global attractor of the whole (u, v) plane when $\varepsilon = 0$ and $\delta > 0$. Therefore, this system has the same properties as the example of Mizoguchi, Ninomiya, and Yanagida, but the diffusion constants are equal.

The proof of Mizoguchi, Ninomiya, and Yanagida depends upon establishing a rather sophisticated inequality between a certain integral and its derivatives. In Section 2 we shall give an explanation of unequal-diffusion blowup by applying a heuristic argument to a simple linear system. The mechanism in this example does not work in the case of equal diffusions.

Section 3 presents a different heuristic argument to show that the system (1.2) should have positive solutions with the desired blowup property, and a computation to back up this argument. These are followed by two theorems which state that when ε and δ are positive, this system has positive solution which blow up in a finite time. Because there are no comparison theorems for the solutions of the system (1.2), the usual maximum principle arguments do not work, and some different techniques are introduced.

The seminal paper of A. M. Turing [7] showed that unequal diffusions can produce an unstable manifold at a constant equilibrium solution which, in the absence of diffusion, is an isolated attractor. In contrast, the result of Mizoguchi, Ninomiya, and Yanagida [6] shows that unequal diffusions can create a stable manifold for the point at infinity, which is, in the absence of diffusion, a repeller. The present paper shows that this phenomenon can also be brought about by by equal diffusions.

It has been shown by Y. Lou and W.-M. Ni [5, Theorem 1.2] that adding diffusion to a diffusionless system with an attractor at infinity can make the basin of attraction of the attractor at infinity grow. (This work is still in progress, in collaboration with T. Nagylaki.)

There is an extensive blowup literature (see, e.g., the review article [4]), in which it is shown that adding diffusion to one or more ordinary differential equations with solutions which blow up may not prevent this blowup. In contrast, the examples (1.1) and (1.2) show that diffusion may actually cause the blowup.

Churbanov [1] proved that there is a system of ordinary differential equations all of whose solutions exist for all time with the property that the corresponding system with equal diffusions has solutions which blow up in a finite time. An explicit example of such a system was recently given by Guedda and Kirane [3].

2. UNBOUNDEDNESS INDUCED BY UNEQUAL DIFFUSIONS

The fact that unequal diffusions can produce unboundedness seems rather strange at first glance. We shall give a simple linear system which displays such a phenomenon. Consider the system

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \alpha\varepsilon\, \Delta u + u - \nu v - \delta u \\
\frac{\partial v}{\partial t} &= \varepsilon\, \Delta v + u - \nu v - \delta v \\
\frac{\partial v}{\partial n} &= 0 \qquad \text{on } \partial D,
\end{aligned}
\tag{2.1}
$$

where

$$\nu > 1, \qquad 0 < \alpha\nu < 1. \tag{2.2}$$

It is not difficult to see that when $\varepsilon=0$ and $\delta>0$, the origin is a global attractor of the resulting system

When $\varepsilon=\delta=0$, the solution $(u(t,x), v(t,x))$ approaches $((\nu[u(0,x)-v(0,x)]/(\nu-1),\ [u(0,x)-v(0,x)]/(\nu-1))$ as $t\to\infty$. If ε is positive but small, then at times of order $o(1/\varepsilon)$ one still expects the solution to be close to this. In particular, the largest values of u and v are approximately equal to multiples of the largest value of $u(0,x)-v(0,x)$. If this function has a strict interior maximum, then as time increases, the diffusion will reduce the maximum value of v by a small amount $\varepsilon\eta$ while, since $u\approx \nu v$, the maximum value of u is reduced by about $\alpha\nu\varepsilon\eta$. The dynamics then drive the new value $(\nu[u(0,x)-v(0,x)]/(\nu-1)-\alpha\nu\varepsilon\eta,\ \ [u(0,x)-v(0,x)]/(\nu-1)-\varepsilon\eta)$ toward the point $(\nu[u(0,x)-v(0,x)+(1-\alpha\nu)\,\varepsilon\eta]]/(\nu-1), [u(0,x)-v(0,x)+(1-\alpha\nu)\,\varepsilon\eta]]/(\nu-1))$.

In other words, while the diffusion reduces the maximum values of both u and v, it increases the maximum of $u-v$. Thus the combination of the diffusion and the dynamics actually increases the maximum values of u and v, contrary to what one might expect. This heuristic argument indicates that one might expect the solution to approach infinity, provided ε and δ are sufficiently small.

To verify this heuristic argument, we look at a solution of the form $u=ae^{\sigma t}\phi_2$, $v=be^{\sigma t}\phi_2$, where ϕ_2 is a nonconstant Neumann eigenfunction of the operator $-\Delta$ with the corresponding positive eigenvalue λ_2, and σ, a, and b are constants to be determined. The statement that this is a solution leads to a matrix eigenvalue problem for σ. It is easily seen that there is a positive σ when

$$-\lambda_2(1-\alpha\nu)\,\varepsilon+\lambda_2^2\alpha\varepsilon^2+[\nu-1+\lambda_2(1+\alpha)]\,\varepsilon\delta+\delta^2<0.$$

Once the condition (2.2) is satisfied, this condition is valid when ε and then δ are chosen sufficiently small. In such cases, the diffusion produces unbounded solutions with bounded initial data. Because the system is linear, one cannot expect finite-time blowup.

3. BLOWUP PRODUCED BY EQUAL DIFFUSIONS

We note that if $\alpha<1$ so that the diffusion constants in (2.1) are unequal, one can always choose ν so that the conditions (2.2) are satisfied, and the diffusions with suitable positive ε and δ takes bounded initial data into unbounded solutions.

The situation is different when the diffusion constants are equal. In this case, Theorem 1 of [8] states that convex invariant sets for the system without diffusion are also invariant sets of the system with equal diffusions.[1] It is easily seen that any set of the form $|u-v| \leqslant c$, $|u-vv| \leqslant d$ is a convex invariant set for the system (2.1) with $\varepsilon = 0$. Since any bounded initial data lie in such a parallelogram for suitable values of c and d, we conclude that bounded initial data give bounded solutions when $\alpha = 1$.

Similarly, the parallelograms $|u-v| \leqslant c$, $|v| \leqslant c^p$ serve to show that when $d_u = d_v$ in the Mizoguchi, Ninomiya, and Yanagida example (1.1), bounded initial data lead to bounded solutions.

We now consider the system

$$\begin{aligned} &u_t = \varepsilon u_{xx} + uv(u-v)(u+1) - \delta u \qquad \text{in } (0, \infty) \times (-1, 1) \\ &v_t = \varepsilon v_{xx} + vu(v-u)(v+1) - \delta v \qquad \text{in } (0, \infty) \times (-1, 1) \\ &u_x(-1, t) = u_x(1, t) = 0 \\ &v_x(-1, t) = v_x(1, t) = 0 \end{aligned} \tag{3.1}$$

with equal diffusion constants. The trajectories in the first quadrant of the system of ordinary differential equations which is obtained by setting $\varepsilon = \delta = 0$ are arcs of the family of hyperbolas $(u+1)(v+1) = c$, along which the solution moves from the line of equilibria $u = v$ toward the nearer of the lines of equilibria $u = 0$ or $v = 0$. In particular, all positive solutions of this system of ordinary differential equations are bounded.

If δ is positive while ε is zero, the flow goes toward the origin on the axes and the line $u = v$, and the function $(u+1)(v+1)$ is a Lyapounov function which serves to show that *the origin is a global attractor of positive solutions.* However, all the bounded convex invariant subsets of the first quadrant with nonempty interior of this system lie in a fixed bounded set.

We consider the specific solution of the system (3.1) with the initial conditions of the form

$$u(x, 0) = v(-x, 0) = c^2 + \sin(\pi x/2). \tag{3.2}$$

We assume that the prescribed constant c is greater than 1, so that the initial data are positive. Because the first quadrant is an invariant set for the system with $\varepsilon = 0$, it is also an invariant set when $\varepsilon > 0$. Thus the solution corresponding to the initial values (3.2) remains positive.

Because the system (3.1) and the initial conditions (3.2) are invariant under the interchange of u and v together with the replacement of x by $-x$,

[1] The importance of convexity in transferring information about the system without diffusion to the system with equal diffusions was already observed by M. I. Freidlin [2], who showed that under certain conditions a convex attractor for a diffusionless system is also an attractor for the system with equal diffusions.

and because the solution of the initial value problem is unique, we conclude that

$$v(x, t) = u(-x, t) \tag{3.3}$$

for all $t \geqslant 0$ where the solution is defined.

We note that the system of ordinary differential equations obtained by setting $\varepsilon = \delta = 0$ in (3.1) makes the solution with the initial values (3.2) approach the discontinuous function

$$U(x) = V(-x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ c^2 & \text{for} \quad x = 0 \\ (c^2 + \sin(\pi x/2) + 1)(c^2 - \sin(\pi x/2) + 1) - 1 & \\ & \text{for} \quad x > 0 \end{cases} \tag{3.4}$$

as $t \to \infty$. We now keep $\delta = 0$ but take ε small and positive. Then up to a time of order $1/\varepsilon$ the solution approaches (U, V). After a longer time, we can expect the diffusion to replace the values of u and v near the discontinuity by the averages of U and of V in a neighborhood, that is, by values near $(2c^2 + c^4)/2 > c^2$. The growth term again tries to reduce u to zero for $x < 0$, while the values for small positive x should approach $[(2c^2 + c^4)/2]^2 - 1$. The averaging process now gives half of this value near $x = 0$. Thus we see that the value at 0 increases without bound. Moreover, one can expect, or at least hope, that the process will accelerate as the jump gets larger, so that the solution will blow up in a finite time, and that this phenomenon will still occur when δ is small and positive.

To check this heuristic argument, we have done some computations to obtain graphs of the components u (solid curve) and v (dashed curve) of the solution of (3.1), (3.2) with $\varepsilon = 1$, $\delta = 0.1$, and $c^2 = 10$ at a sequence of times.[2] (See Fig. 1.)

Note that, as the heuristic argument predicted, the solution at $t = 0.0019$ is close to the discontinuous function (3.4) except for a sharp but not vertical interface near $x = 0$. The interaction of the diffusion and the dynamics near this interface then produces a peak, which is shown at $t = 0.0022$. As time increases, this peak becomes higher, narrower, and closer to $x = 0$ with increasing time, as the last two pictures show. Further computations seem to indicate that the peak becomes infinite at a t slightly above 0.0024.

While it is not too difficult to show that the solution of the problem (3.1), (3.2) is unbounded, the computation challenges us to prove the

[2] A computation with an explicit scheme developed with the help of Howard Levine warned of the appearance of steep fronts which could could only be resolved with an adaptive scheme. The author is grateful to Don French for creating the adaptive finite element scheme used to obtain these pictures.

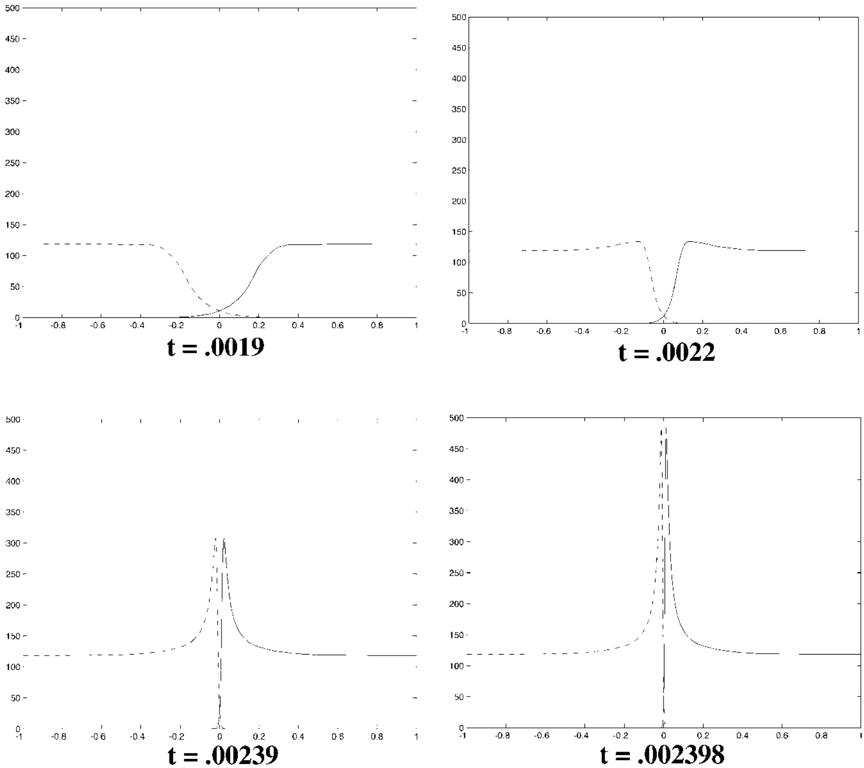

FIGURE 1

stronger result that the solution blows up in a finite time. This is the content of the following theorem, which is our main result.

THEOREM 1. *Let* (u, v) *be the solution of the initial value problem* (3.1), (3.2) *with any prescribed* $\varepsilon > 0$ *and* $\delta \geqslant 0$. *For any positive* T *there is a positive* c_0 *with the property that if* $c \geqslant c_0$, *then* u *and* v *blow up at some* $t^* < T$. *That is, the solution is defined for* $0 \leqslant t < t^*$, *and* $\lim_{t \to t^*} \max u(\cdot, t) = \lim_{t \to t^*} \max v(\cdot, t) = \infty$.

Proof. Suppose that the solution of (3.1), (3.2) is defined on the interval $[0, T]$. Subtracting the first two equations in (3.1) shows that

$$(u-v)_t = \varepsilon(u-v)_{xx} + [uv(u+v+2) - \delta](u-v). \tag{3.5}$$

Moreover, condition (3.3) shows that $u - v$ is zero at $x = 0$, while the initial conditions show that $u - v > 0$ when $t = 0$ and $x > 0$. Since the convex set $0 \leqslant v \leqslant u$ is invariant for the system (3.1) with $\varepsilon = 0$, it is also invariant for the system with $\varepsilon > 0$. We conclude that the function $u(t, x) - v(t, x)$ is positive for $x > 0$.

Since $u - v$ is odd, this fact implies that

$$(u - v)\sin(\pi x/2) \geqslant 0. \tag{3.6}$$

Multiplying both sides of (3.5) by $\sin(\pi x/2)$, integrating with respect to x, and integrating by parts gives the identity

$$\begin{aligned}\frac{d}{dt}\int_{-1}^{1} &(u - v)\sin(\pi x/2)\,dx \\ &= \int_{-1}^{1} [-\varepsilon\pi^2/4 + uv(u + v + 2) - \delta](u - v)\sin(\pi x/2)\,dx.\end{aligned} \tag{3.7}$$

We note that $\min_x u(0, x) = c^2 - 1 > c - 1$. If for some $t_1 \in (0, T]$

$$\min_x u(t, x) = \min_x v(t, x) \geqslant c - 1 \qquad \text{for} \quad 0 \leqslant t \leqslant t_1, \tag{3.8}$$

then $uv \geqslant (c-1)^2$ on this time interval. By using (3.6), (3.8), and the fact that $u + v \geqslant (u - v)\sin(\pi x/2)$, we see that if

$$(c - 1)^2 > (\varepsilon\pi^2/4 + \delta)/2,$$

the equation (3.7) implies the inequality

$$\frac{d}{dt}\int_{-1}^{1} (u - v)\sin(\pi x/2)\,dx \geqslant (c-1)^2 \int_{-1}^{1} [(u - v)\sin(\pi x/2)]^2\,dx \qquad \text{for} \quad 0 \leqslant t \leqslant t_1.$$

Thus if we set

$$w(t) = \frac{1}{\int_{-1}^{1} (u - v)\sin(\pi x/2)\,dx}, \tag{3.9}$$

and apply the Schwarz inequality

$$\left[\int_{-1}^{1} (u - v)\sin(\pi x/2)\,dx\right]^2 \leqslant 2\int_{-1}^{1} [(u - v)\sin(\pi x/2)]^2\,dx,$$

we see that

$$\frac{dw}{dt} \leqslant -(c - 1)^2/2.$$

We integrate this inequality to obtain the bound

$$w(t) \leqslant w(0) - (c-1)^2 t/2 \qquad \text{for} \quad 0 \leqslant t \leqslant t_1. \tag{3.10}$$

Since (3.6) and the definition (3.9) show that $w > 0$, we conclude from (3.10) that $t_1 \leqslant 2w(0)/(c-1)^2$. That is, either the solution stops existing or the inequality (3.8) is violated on $(0, \tau)$ if $\tau > 2w(0)/(c-1)^2$.

Suppose now that t_1 is the largest time for which (3.8) is valid, and that there is a interval $[t_1, t_2]$ of $[0, T]$ such that

$$\min_x u(t, x) \leqslant c - 1 \qquad \text{for} \quad t_1 \leqslant t \leqslant t_2. \tag{3.11}$$

An easy computation shows that

$$(\ln(u+1) - \ln c)_t$$
$$= \varepsilon(\ln(u+1) - \ln c)_{xx} + \varepsilon(\ln(u+1) - \ln c)_x^2 + uv(u-v) - \delta u/(u+1).$$

Since $uv(u-v)$ is an odd function by (3.3), integration and the Neumann boundary conditions show that

$$\frac{d}{dt}\int_{-1}^{1} (\ln(u+1) - \ln c)\, dx = \varepsilon \int_{-1}^{1} (\ln(u+1) - \ln c)_x^2\, dx - \delta \int_{-1}^{1} u/(u+1)\, dx. \tag{3.12}$$

Since $u/(u+1) < 1$, the function

$$q(t) = \int_{-1}^{1} (\ln(u+1) - \ln c)\, dx - 2\delta(T-t) \tag{3.13}$$

is nondecreasing. An easy computation shows that

$$q(0) > 2(\ln c - \delta T). \tag{3.14}$$

We assume that $c > e^{\delta T}$. Then $q(t)$ is positive for all t.

We observe that by (3.11), the fundamental theorem of calculus, and Schwarz's inequality

$$q(t) \leqslant \int_{-1}^{1} (\ln(u+1) - \ln c)\, dx \leqslant 2[\ln(\max u(t, \cdot) + 1) - \ln(\min u(t, \cdot) + 1)]$$
$$\leqslant 2\sqrt{2\int_{-1}^{1} (\ln(u+1) - \ln c)_x^2\, dx}.$$

We see from this together with (3.12) and the inequality $u/(u+1)<1$ that the function $q(t)$ satisfies the inequality

$$\frac{dq}{dt} \geqslant \begin{cases} 0 & \text{for} \quad 0 \leqslant t \leqslant t_1 \\ \varepsilon q^2/8 & \text{for} \quad t_1 \leqslant t \leqslant t_2. \end{cases}$$

We conclude that

$$\frac{1}{q(t)} \leqslant \frac{1}{q(0)} - \varepsilon(t-t_1)/8 \qquad \text{for} \quad t_1 \leqslant t \leqslant t_2. \tag{3.15}$$

Since q is nondecreasing, it remains positive. Therefore, if the solution exists up to the time t_2, we must have $t_2 - t_1 < 8/(\varepsilon q(0))$.

We recall that the solution of (3.1), (3.2) is defined on the interval $[0, T]$. We partition this interval into a (finite or infinite) partition $0 = t_0 < t_1 < t_2 < \cdots \leqslant T$ with the property that

$$\min_x u(t, x) \begin{cases} \geqslant c-1 & \text{for} \quad t \in [t_{2j}, t_{2j+1}] \\ \leqslant c-1 & \text{for} \quad t \in [t_{2j+1}, t_{2j+2}]. \end{cases} \tag{3.16}$$

The derivation of (3.15) leads to the inequality

$$\frac{1}{q(t_{2j+2})} \leqslant \frac{1}{q(t_{2j})} - \varepsilon(t_{2j+2} - t_{2j+1})/8.$$

By adding these inequalities, we find that

$$\frac{1}{q(t_{2j})} \leqslant \frac{1}{q(0)} - \varepsilon \sum_{k=0}^{j-1} (t_{2k+2} - t_{2k+1})/8.$$

The positivity of the left-hand side then leads to the bound

$$\sum (t_{2k+2} - t_{2k+1}) \leqslant \frac{8}{\varepsilon q(0)} \tag{3.17}$$

for the measure of the subset of $[0, T]$ where $\min_x u \leqslant c-1$.

To obtain a bound for the measure of the complement of this set, we observe that the derivation of the inequality (3.10) leads to the inequality

$$w(t_{2j+1}) \leqslant w(t_{2j}) - (c-1)^2 (t_{2j+1} - t_{2j})/2.$$

We see from the equation (3.7) that the function $w(t)\, e^{-[(\varepsilon\pi^2/4)+\delta]t}$ is nonincreasing in t. Therefore, we obtain the recursion

$$\begin{aligned} w(t_{2j+2}) &\leqslant e^{[(\varepsilon\pi^2/4)+\delta](t_{2j+2}-t_{2j+1})} w(t_{2j+1}) \\ &\leqslant e^{[(\varepsilon\pi^2/4)+\delta](t_{2j+2}-t_{2j+1})} [w(t_{2j}) - (c-1)^2\,(t_{2j+1}-t_{2j})/2]. \end{aligned}$$

The usual trick of introducing the new variable

$$r_j = e^{-[(\varepsilon\pi^2/4)+\delta]\sum_0^{j-1}(t_{2i+2}-t_{2i+1})} w(t_{2j})$$

leads to the recursion

$$r_{j+1} \leqslant r_j - e^{-[(\varepsilon\pi^2/4)+\delta]\sum_0^{j-1}(t_{2i+2}-t_{2i+1})} (c-1)^2\,(t_{2j+1}-t_{2j})/2.$$

The inequality (3.17) then yields the recursion

$$r_{j+1} \leqslant r_j - e^{-[2\pi^2+8(\delta/\varepsilon)]/q(0)}(c-1)^2\,(t_{2j+1}-t_{2j})/2,$$

so that

$$r_{j+1} \leqslant w(0) - e^{-[2\pi^2+8(\delta/\varepsilon)]/q(0)} \sum_{i=0}^{j+1} (c-1)^2\,(t_{2i+1}-t_{2i})/2.$$

The fact that the r_j are positive now leads to the bound

$$\sum (t_{2i+1}-t_{2i}) \leqslant 2w(0)\, e^{[2\pi^2+8(\delta/\varepsilon)]/q(0)}/(c-1)^2 \tag{3.18}$$

for the measure of the subset of $[0, T]$ where the first inequality in (3.16) is valid.

A simple computation shows that $w(0)=1/2$. Since the interval $[0, T]$ is the union of the subsets where the two inequalities in (3.16) are valid, we obtain a bound for T by adding the bounds (3.17) and (3.18) for the measures of these two sets; that is,

$$T \leqslant 8/(\varepsilon q(0)) + e^{[2\pi^2+8(\delta/\varepsilon)]/q(0)}/(c-1)^2. \tag{3.19}$$

We see from the inequality (3.14) that the right-hand side converges to 0 as c approaches infinity. Therefore, when c is sufficiently large, the inequality (3.19) is violated. We conclude that the solution must cease to exist before the positive time T.

Since the system (3.1) is semilinear and parabolic, standard local existence and regularity theorems show that the solution can only cease to exist when $\lim_{t\to t^*} \max u(\cdot, t) = \infty$ for some $t^* \leqslant T$, which proves the theorem.

The above proof is easily adapted to yield the following statement for fixed ε and c and small δ.

THEOREM 2. *Let ε be any positive number, and let c be a prescribed number which satisfies the inequality $(c-1)^2 > \varepsilon\pi^2/8$. Then if T is any number greater than the value of the right-hand side of* (3.19) *at $\delta = 0$, there is a positive $\delta_0 = \delta_0(\varepsilon, c, T)$ with the property that if $0 \leqslant \delta \leqslant \delta_0$, the solution of* (3.1), (3.2) *blows up at some $t^* \leqslant T$.*

We remind the reader that positive solutions of the system (3.1) are also solutions of the system (1.3), which has the property that when $\varepsilon = 0$ and $\delta > 0$, the origin is a global attractor, so that the point at infinity is a repeller. Theorems 1 and 2 show that equal diffusions produce a stable manifold at this repeller.

The above theorems are easily extended to the higher-dimensional problem in which the $\partial^2/\partial x^2$ on the right of (3.1) is replaced by the Laplace operator, the interval $[-1, 1]$ is replaced by a domain which is symmetric about $x_1 = 0$, and the function $\sin(\pi x/2)$ in the initial conditions is replaced by an eigenfunction of $-\Delta$ with Neumann boundary conditions which vanishes at $x_1 = 0$ and is positive for $x_1 > 0$.

Our proofs of the above theorems depend strongly on the symmetry of the equations (3.1) and the initial conditions (3.2). These imply the symmetry (3.3), which, in turn, keeps the interface from moving away from $x = 0$. Numerical computation of the solution of (3.1) with $\varepsilon = 1$ and $\delta = 0.1$ with the asymmetric initial data $u(0, x) = 10 + 2\sin(\pi x/2)$, $v = 10 - \sin(\pi x/2)$ seems to indicate that in spite of the lack of symmetry the interface remains at $x = 0$, and that u (but not v) develops a peak which blows up in a finite time. It would be interesting to find a proof of this more general result.

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