On the well-posedness of the Schrödinger–Korteweg–de Vries system

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We prove that the Cauchy problem for the Schrödinger–Korteweg–de Vries system is locally well-posed for the initial data belonging to the Sobolev spaces \( L^2(\mathbb{R}) \times H^{-3/4}(\mathbb{R}) \), and \( H^s(\mathbb{R}) \times H^{-3/4}(\mathbb{R}) \) for the resonant case. The new ingredient is that we use the \( \bar{F}_s \)-type space, introduced by the first author in Guo (2009) [10], to deal with the KdV part of the system and the coupling terms. In order to overcome the difficulty caused by the lack of scaling invariance, we prove uniform estimates for the multiplier. This result improves the previous one by Corcho and Linares (2007) [6].

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1. Introduction

In this paper, we consider the Cauchy problem for the Schrödinger–Korteweg–de Vries (NLS–KdV) system

\[
\begin{aligned}
  i \partial_t u + \partial_x^2 u &= \alpha u v + \beta |u|^2 u, \quad t, x \in \mathbb{R}, \\
  \partial_t v + \partial_x^3 v + \frac{1}{2} \partial_x (v^2) &= \gamma \partial_x (|u|^2), \\
  u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x),
\end{aligned}
\]

(1.1)

where \( u(x, t) \) is a complex-valued function, \( v(x, t) \) is a real-valued function and \( \alpha, \beta, \gamma \) are real constants, \((u_0, v_0)\) are given initial data belonging to \( H^{s_1} \times H^{s_2} \). Our main motivation of this paper is inspired by the work of Corcho and Linares [6], and the work of the first-named author [10].

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The system (1.1) is an important model in fluid mechanics and plasma physics that governs the interactions between short wave and long wave. The case $\beta = 0$ appears in the study of resonant interaction between short and long capillary-gravity waves on water of uniform finite depth, in plasma physics and in a diatomic lattice system (see [6] and reference therein for more introduction).

Before we state our main results, we recall first the early results on this system. M. Tsutsumi [20] obtained global well-posedness (GWP) for data $(u_0, v_0) \in H^{s+1/2}(\mathbb{R}) \times H^s(\mathbb{R})$ with $s \in \mathbb{Z}_-$. In the resonant case ($\beta = 0$) Guo and Miao [8] proved GWP in the natural energy space $H^s(\mathbb{R}) \times H^{s-1/2}(\mathbb{R})$ with $s \in \mathbb{Z}_+$. Bekiranov, Ogawa and Ponce [1] proved local well-posedness (LWP) in $H^s(\mathbb{R}) \times H^{s-1/2}(\mathbb{R})$ for $s \geq 0$, and this result was improved to $L^2 \times H^{-3/4+}$ by Corcho and Linares [6] which seems to be sharp except $L^2 \times H^{-3/4}$ in view of the results for the two single equations in (1.1) (see [16,4,15]). Pecher [17] obtained GWP in $H^s \times H^s$ for $s > 3/5$ ($\beta = 0$) and $s > 2/3$ ($\beta \neq 0$) by using the ideas of I-method [5]. Some generalized interaction equations were considered in [2]. In this paper, we prove the following results:

**Theorem 1.1.** Assume $u_0 \in L^2$, $v_0 \in H^{-3/4}$. Then:

(a) Existence. There exist $T = T(||u_0||_{L^2}, ||v_0||_{H^{-3/4}}) > 0$ and a solution $u$ to the Cauchy problem (1.1) satisfying

$$u \in X_{\tau=-\xi^2}^{0,1/2}(T) \subset C([-T, T] : L^2), \quad v \in H_{\tau=-\xi^3}^{-3/4}(T) \subset C([-T, T] : H^{-3/4}).$$

(b) Uniqueness. The solution mapping $S_T : (u_0, v_0) \rightarrow (u, v)$ is the unique extension of the classical solution $(H^{\infty}, H^{\infty}) \rightarrow C([-T, T] : H^{\infty} \times H^{\infty})$.

(c) Lipschitz continuity. For any $R > 0$, the mapping $(u_0, v_0) \rightarrow (u, v)$ is Lipschitz continuous from

$$((u_0, v_0) \in L^2 \times H^{-3/4}; ||u_0||_{L^2} + ||u_0||_{H^{-3/4}} < R) \rightarrow C([-T, T] : L^2 \times H^{-3/4}).$$

We describe briefly our ideas in proving Theorem 1.1. We also use the scheme as in [6] which is the same spirit as the one by Ginibre, Y. Tsutsumi and Velo [7] for the Zakharov system. The basic idea is that for the second equation in (1.1) we use the $\tilde{F}^3$ space that was used by the first-named author [10] for the KdV equation, but there is an essential difficulty. For the KdV equation, one can assume the initial data $v_0$ has a small norm by using the scaling transform. However, for the NLS-KdV system we don't have such an invariant scaling transform. In order to deal with large initial data, we overcome this difficulty by the following way. We observe that the single nonlinear Schrödinger equation with cubic term $(|u|^2u)$

$$i\partial_t u + \partial_x^2 u = \beta |u|^2 u$$

is $L^2$-subcritical. On the other hand, we also see from [6] that to control the coupling term $uv$ one needs less regularity than $H^{-3/4}$ of $v$. Then we expect that the first equation can be handled without scaling. Thus we scale the system (1.1) according to the second equation as follows: if $(u, v)$ solve the system (1.1) with initial data $(u_0, v_0)$, then we see

$$u_\lambda(t, x) = \lambda^2 u(\lambda^2 t, \lambda x), \quad \phi_1(x) = \lambda^2 u_0(\lambda x),$$

$$v_\lambda(t, x) = \lambda^2 v(\lambda^2 t, \lambda x), \quad \phi_2(x) = \lambda^2 v_0(\lambda x)$$

satisfy the following system

$$\begin{cases}
  i\partial_t u + \lambda^2 \partial_x^2 u = \lambda u v + \lambda^{-1} |u|^2 u, & t, x \in \mathbb{R}, \\
  \partial_t v + \partial_x^3 v + \frac{1}{2} \partial_x (v^2) = \partial_x (|u|^2), \\
  u(0, x) = \phi_1(x), \quad v(0, x) = \phi_2(x).
\end{cases}$$

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\(v_\lambda(t, x) = \lambda^2 v(\lambda^2 t, \lambda x), \quad \phi_2(x) = \lambda^2 v_0(\lambda x)\)

\(\text{satisfy the following system}

\(\begin{cases}
  i\partial_t u + \lambda^2 \partial_x^2 u = \lambda u v + \lambda^{-1} |u|^2 u, & t, x \in \mathbb{R}, \\
  \partial_t v + \partial_x^3 v + \frac{1}{2} \partial_x (v^2) = \partial_x (|u|^2), \\
  u(0, x) = \phi_1(x), \quad v(0, x) = \phi_2(x).
\end{cases}\)
It is easy to see that by taking $0 < \lambda \ll 1$, we have $\|\phi_1\|_{L^2} = \|\phi_2\|_{L^2} = \lambda^{3/2} \|u_0\|_2 \leq R$ and $\|\phi_2\|_{H^{-3/4}} = \|\lambda^2 v_0(\lambda x)\|_{H^{-3/4}} \leq 2 \lambda^{3/4} \|v_0\|_{H^{-3/4}} = c_0 < 1$. Therefore, it reduces to study the system (1.4) under condition that $0 < \lambda \leq 1$ and the following condition
\begin{equation}
\|\phi_1\|_{L^2} \leq R, \quad \|\phi_2\|_{H^{-3/4}} = c_0 < 1, \tag{1.5}
\end{equation}
where $c_0$ is an absolute constant will be defined later. We will prove well-posedness for (1.4)–(1.5) in $[0, T]$ for some $T = T(R, \lambda) > 0$. By the scaling we obtain local well-posedness for the original system (1.1).

In our proof the condition $0 < \lambda \leq 1$ in (1.4) is crucial. Heuristically, the propagation speed for the first equation is $\lambda \xi$, and that for the second equation is $\xi^2$. Then we see that the two waves $u, v$ have a separate speed in high frequency $|\xi| \gtrsim 1$ uniformly for $0 < \lambda \leq 1$. Thus the resonance and coherence cannot be simultaneously large (also see [6] for the case $\lambda = 1$). This is a key to control the coupled wave interactions. Technically, we will prove uniform estimates for the multiplier associated to the coupled terms for all $0 < \lambda \leq 1$. Our proof for the coupled terms is different from those in [6], but with basically the same ideas. We will use the ideas developed by Tao [18], Ionescu and Kenig [11], and the first-named author [9], but in this paper we need to deal with two different wave forms which have independent interest.

At the end of this section we introduce some notations. In Section 2, we prove some $L^2$ bilinear estimates which will be used to prove the bilinear estimates for the coupling terms in Section 3. In Section 4, we prove Theorem 1.1.

**Notations.** Throughout this paper we fix $0 < \lambda \leq 1$. We will use $C$ and $c$ to denote constants which are independent of $\lambda$ and not necessarily the same at each occurrence. For $x, y \in \mathbb{R}$, $x \sim y$ means that there exist $C_1, C_2 > 0$ such that $C_1 |x| \leq |y| \leq C_2 |x|$. For $f \in \mathcal{S}'$ we denote by $\hat{f}$ or $\mathcal{F}(f)$ the Fourier transform of $f$ for both spatial and time variables,
\[
\hat{f}(\xi, \tau) = \int_{\mathbb{R}^2} e^{-ix\xi} e^{-it\tau} f(x, t) \, dx \, dt.
\]
We denote by $\mathcal{F}_x$ the Fourier transform on spatial variable and if there is no confusion, we still write $\mathcal{F} = \mathcal{F}_x$. Let $\mathbb{Z}$ and $\mathbb{N}$ be the sets of integers and natural numbers, respectively. $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. For $k \in \mathbb{Z}_+$ let
\[
I_k = \left\{ \xi : |\xi| \in \left[ 2^{k-1}, 2^{k+1} \right] \right\}, \quad k \geq 1; \quad I_0 = \left\{ \xi : |\xi| \leq 2 \right\}.
\]
Let $\eta_0 : \mathbb{R} \to [0, 1]$ denote an even smooth function supported in $[-8/5, 8/5]$ and equal to 1 in $[-5/4, 5/4]$. For $k \in \mathbb{Z}$ let $\eta_k(\xi) = \eta_0(\xi/2^k) - \eta_0(\xi/2^{k-1})$ if $k \geq 1$ and $\eta_0(\xi) \equiv 0$ if $k \leq -1$, and let $\chi_k(\xi) = \eta_0(\xi/2^k) - \eta_0(\xi/2^{k-1})$. For $k \in \mathbb{Z}$ let $P_k$ denote the operator on $L^2(\mathbb{R})$ defined by
\[
\hat{P_ku}(\xi) = \eta_k(\xi) \hat{u}(\xi).
\]
By a slight abuse of notation we also define the operator $P_k$ on $L^2(\mathbb{R} \times \mathbb{R})$ by the formula $\mathcal{F}(P_ku)(\xi, \tau) = \eta_k(\xi) \mathcal{F}(u)(\xi, \tau)$. For $l \in \mathbb{Z}$ let
\[
P_{\leq l} = \sum_{k \leq l} P_k, \quad P_{\geq l} = \sum_{k \geq l} P_k.
\]
Thus we see that $P_{\leq 0} = P_0$.

For $\phi \in \mathcal{S}'(\mathbb{R})$, we denote by $V(t)\phi = e^{-t\partial_x^3} \phi$ the free solution of linear Airy equation which is defined as
\[
V(t)\phi = e^{-t\partial_x^3} \phi
\]
\[ \mathcal{F}_{\lambda}(V(t)\phi)(\xi) = \exp[i\xi^{2}t]\hat{\phi}(\xi), \quad \forall t \in \mathbb{R}, \]

denote by \( U_{\lambda}(t)\phi = e^{it\lambda^{2}t}\phi \) for \( 0 < \lambda \leq 1 \) the free solution of scaled linear Schrödinger equation which is defined as
\[ \mathcal{F}_{\lambda}(U_{\lambda}(t)\phi)(\xi) = \exp[-i\lambda \xi^{2}t]\hat{\phi}(\xi), \quad \forall t \in \mathbb{R}. \]

We define the Lebesgue spaces \( L_{\ell}^{q}L_{t}^{p} \) and \( L_{\ell}^{q}L_{t}^{p}L_{w}^{q} \) by the norms
\[ \| f \|_{L_{\ell}^{q}L_{t}^{p}} = \left\| \| f \|_{L_{\ell}^{q}(t)} \right\|_{L_{t}^{p}}, \quad \| f \|_{L_{\ell}^{q}L_{t}^{p}L_{w}^{q}} = \left\| \| f \|_{L_{\ell}^{q}(t)} \right\|_{L_{w}^{q}}. \quad (1.6) \]

If \( l = \mathbb{R} \) we simply write \( L_{l}^{q}L_{t}^{p} \) and \( L_{\ell}^{q}L_{w}^{q} \).

We will make use of the \( X^{s,b} \)-type space. Generally, let \( h(\xi) \) be a continuous function, and we define
\[ \| v \|_{X^{s,b}_{\tau = h(\xi)}} = \left\| (\tau - h(\xi))^{\frac{b}{2}}(\xi)^{\frac{d}{2}}v(\xi, \tau) \right\|_{L^{2}(\mathbb{R}^{2})}, \]

where \( \langle \cdot \rangle = (1 + |\cdot|^{2})^{1/2} \). This type of space was first systematically studied by Bourgain [3]. In applications we usually apply \( X^{s,b} \) space for \( b \) close to 1/2. In the case \( b = 1/2 \) one has a good substitute: Besov-type \( X^{s,b} \) space which was first noted by Tataru [19]. For \( k \in \mathbb{Z}_{+} \) we define the frequency dyadically localized \( X^{s,b} \)-type normed spaces \( Y_{k}^{\tau = h(\xi)} \):
\[ Y_{k}^{\tau = h(\xi)} = \left\{ f \in L^{2}(\mathbb{R}^{2}) : f(\xi, \tau) \text{ is supported in } l_{k} \times \mathbb{R} \right\}, \quad \| f \|_{Y_{k}^{\tau = h(\xi)}} = \sum_{j=0}^{\infty} 2^{j/2} \| \eta_{j}\cdot f \|_{L^{2}}. \quad (1.7) \]

Then we define the \( l^{1} \)-analogue of \( X^{s,b} \)-type space \( F_{\tau = h(\xi)}^{s} \) by
\[ \| u \|_{F_{\tau = h(\xi)}^{s}} = \sum_{k \geq 0} 2^{sk} \| \eta_{k}(\xi)\mathcal{F}(u) \|_{Y_{k}^{\tau = h(\xi)}}^{2}. \quad (1.8) \]

In this paper we will use the spaces \( X_{\tau = -\lambda^{2}t}^{s,b} \) and \( F_{\tau = t}^{s} \). In order to avoid some logarithmic divergence, we use the following weaker norm for the low frequency of the KdV equation as in [10],
\[ \| u \|_{\tilde{F}_{\tau = t}^{s}} = \| u \|_{L_{x}^{2}L_{t}^{\infty}}. \]

For \(-3/4 \leq s \leq 0\), we define
\[ \tilde{F}_{\tau = t}^{s} = \left\{ u \in S'(\mathbb{R}^{2}) : \| u \|_{\tilde{F}_{\tau = t}^{s}}^{2} = \sum_{k \geq 1} 2^{sk} \| \eta_{k}(\xi)\mathcal{F}(u) \|_{Y_{k}^{\tau = t}}^{2} + \| P_{\leq 0}(u) \|_{\tilde{F}_{\tau = t}^{s}}^{2} < \infty \right\}. \]

For \( T > 0 \), we define the time-localized spaces \( \tilde{F}_{\tau = t}^{s}(T) \):
\[ \| u \|_{\tilde{F}_{\tau = t}^{s}(T)} = \inf_{w \in \tilde{F}_{\tau = t}^{s}} \left\{ \| P_{\leq 0}u \|_{L_{x}^{2}L_{t}^{\infty}} \leq r \right\} + \| P_{\geq 1}w \|_{\tilde{F}_{\tau = t}^{s}} : w(t) = u(t) \text{ on } [-T, T] \].

Similarly we define \( X_{\tau = h(\xi)}^{s,b}(T) \).
2. $L^2$ bilinear estimates

In this section we prove some $L^2$ bilinear estimates which will be used to prove bilinear estimates for the coupled terms. For $\xi_1, \xi_2 \in \mathbb{R}$ let

$$
\Omega_1(\xi_1, \xi_2) = -\lambda \xi_1^2 + \xi_2^2 + \lambda (\xi_1 + \xi_2)^2, \quad (2.1)
$$

$$
\Omega_2(\xi_1, \xi_2) = -\lambda \xi_1^2 + \lambda \xi_2^2 - (\xi_1 + \xi_2)^3. \quad (2.2)
$$

$\Omega_1$ is the resonance function for the coupled term $uv$, and $\Omega_2$ is the one for $\partial_x |u|^2$. For compactly supported nonnegative functions $f, g, h \in L^2(\mathbb{R} \times \mathbb{R})$ we define for $m = 1, 2$

$$
J_m(f, g, h) = \int_{\mathbb{R}^4} f(\xi_1, \mu_1)g(\xi_2, \mu_2)h(\xi_1 + \xi_2, \mu_1 + \mu_2 + \Omega_m(\xi_1, \xi_2)) \, d\xi_1 \, d\xi_2 \, d\mu_1 \, d\mu_2.
$$

For $k, j \in \mathbb{Z}_+$ we define

$$
D_{k,j}^+ = \{(\xi, \tau): \xi \in I_k, \tau + \lambda \xi^2 \in I_j \},
$$

and for $k \in \mathbb{Z}, j \in \mathbb{Z}_+$ we define

$$
B_{k,j} = \{(\xi, \tau): |\xi| \in [2^{k-1}, 2^{k+1}], \tau - \xi^3 \in I_j \}.
$$

Let $a_1, a_2, a_3 \in \mathbb{R}$. It will be convenient to define the quantities $a_{\text{max}} \geq a_{\text{med}} \geq a_{\text{min}}$ to be the maximum, median, and minimum of $a_1, a_2, a_3$, respectively. Usually we use $k_1, k_2, k_3$ and $j_1, j_2, j_3$ to denote integers, $N_i = 2^{k_i}$ and $L_i = 2^{j_i}$ for $i = 1, 2, 3$ to denote dyadic numbers.

We prove the following lemma.

**Lemma 2.1.** Assume $k_i \in \mathbb{Z}$, $j_i \in \mathbb{Z}_+$, and $f_{k_i,j_i} \in L^2(\mathbb{R} \times \mathbb{R})$ are nonnegative functions supported in $[2^{k_i-1}, 2^{k_i+1}] \times I_{j_i}$, $i = 1, 2, 3$. Then:

(a) For any $k_1, k_2, k_3 \in \mathbb{Z}$ and $j_1, j_2, j_3 \in \mathbb{Z}_+$, $m = 1, 2$,

$$
J_m(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}) \leq C 2^{j_{\text{min}}/2} 2^{k_{\text{min}}/2} \prod_{i=1}^3 \|f_{k_i,j_i}\|_{L^2}. \quad (2.3)
$$

(b) If $k_2 \geq 3$, then

$$
J_1(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}) \leq C 2^{(j_{\text{med}} + j_{\text{max}})/2} 2^{-k_2} \prod_{i=1}^3 \|f_{k_i,j_i}\|_{L^2}. \quad (2.4)
$$

(c) For any $k_1, k_2, k_3 \in \mathbb{Z}$ and $j_1, j_2, j_3 \in \mathbb{Z}_+$,

$$
J_2(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}) \leq C \lambda^{-1/2} 2^{(j_1 + j_2)/2} 2^{-k_3/2} \prod_{i=1}^3 \|f_{k_i,j_i}\|_{L^2}. \quad (2.5)
$$
**Proof.** Let $A_k(\xi) = \int_{\mathbb{R}} |f_{k_i,j_i}(\xi, \mu)|^2 d\mu|^{1/2}$, $i = 1, 2, 3$. Using the Cauchy-Schwarz inequality and the support properties of the functions $f_{k_i,j_i}$, we obtain

$$J_m(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}) \lesssim 2^{j_{\min}/2} \int_{\mathbb{R}^2} A_k(\xi_1) A_k(\xi_2) A_k(\xi_3 + \xi_2) d\xi_1 d\xi_2$$

$$\lesssim 2^{k_{\min}/2} 2^{j_{\min}/2} \prod_{i=1}^3 \|f_{k_i,j_i}\|_{L^2},$$

which is part (a), as desired.

For part (b), in view of the support properties of the functions, it is easy to see that $J_1(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}) \equiv 0$ unless

$$|k_{\max} - k_{\med}| \leq 5, \quad 2^{j_{\max}} \gtrsim |\Omega_1|.$$  \ \ (2.6)

We define two sets

$$A = \left\{ (\xi_1, \xi_2) \in \mathbb{R}^2 : \left| 2\lambda \xi_1 - 3\xi_2 \right| \geq \frac{1}{2} |\xi_2|^2 \right\}$$

and

$$B = \left\{ (\xi_1, \xi_2) : \left| \Omega_1(\xi_1, \xi_2) \right| \geq \frac{1}{2} |\xi_2|^3 \right\} = \left\{ (\xi_1, \xi_2) : \left| \xi_2^2 + 2\lambda \xi_1 + \lambda \xi_2 \right| \geq \frac{1}{2} |\xi_2|^2 \right\},$$

since $|\Omega_1(\xi_1, \xi_2)| = |\xi_2^2 + 2\lambda \xi_1 + \lambda \xi_2|$. We claim that

$$\left[ 2^{k_1-1}, 2^{k_1+1} \right] \times \left[ 2^{k_2-1}, 2^{k_2+1} \right] \subset A \cup B.$$

Indeed, if $(\xi_1, \xi_2) \notin A \cup B$, then

$$|4\xi_2^2 + \lambda \xi_2| \leq \left| 3\xi_2^2 - 2\lambda \xi_1 \right| + \left| \xi_2^2 + 2\lambda \xi_1 + \lambda \xi_2 \right| \leq |\xi_2|^2,$$

which is a contradiction since $|\xi_2| \geq 2$ and $0 < \lambda \leq 1$.

For simplicity of notations we set $f_i = f_{k_i,j_i}$, $i = 1, 2, 3$. Then we get

$$J_1(f_1, f_2, f_3) \lesssim \int_{\mathbb{R}^2} \int_{A} f_1(\xi_1, \mu_1) f_2(\xi_2, \mu_2) f_3(\xi_1 + \xi_2, \mu_1 + \mu_2 + \Omega_1(\xi_1, \xi_2)) d\xi_1 d\xi_2 d\mu_1 d\mu_2$$

$$+ \int_{\mathbb{R}^2} \int_{B} f_1(\xi_1, \mu_1) f_2(\xi_2, \mu_2) f_3(\xi_1 + \xi_2, \mu_1 + \mu_2 + \Omega_1(\xi_1, \xi_2)) d\xi_1 d\xi_2 d\mu_1 d\mu_2$$

$$= I + II.$$

We consider first the contribution of the term $I$. If $(\xi_1, \xi_2) \in A$, then

$$|\partial_{\xi_1} \Omega_1 - \partial_{\xi_2} \Omega_1| = |2\lambda \xi_1 - 3\xi_2| \geq \frac{1}{2} |\xi_2|^2.$$
We will prove that if \( g_i : \mathbb{R} \to \mathbb{R}_+ \) are \( L^2 \) functions supported in \([2^{k_i-1}, 2^{k_i+1}]\), \( i = 1, 2 \), and \( g : \mathbb{R}^2 \to \mathbb{R}_+ \) is an \( L^2 \) function supported in \([2^{k_1-1}, 2^{k_1+1}] \times I_{\max} \), then

\[
\int_{\mathbb{R}^2} g_1(\xi_1)g_2(\xi_2)g(\xi_1 + \xi_2, \Omega_1(\xi_1, \xi_2)) \, d\xi_1 \, d\xi_2 \lesssim 2^{-k_2} \|g_1\|_{L^2} \|g_2\|_{L^2} \|g\|_{L^2}. \tag{2.7}
\]

This suffices for (2.4) by Cauchy–Schwartz inequality. To prove (2.7) we get

\[
\int_{\mathbb{R}^2} g_1(\xi_1)g_2(\xi_2)g(\xi_1 + \xi_2, \Omega_1(\xi_1, \xi_2)) \, d\xi_1 \, d\xi_2 \\
\lesssim \|g_1\|_{L^2} \|g_2\|_{L^2} \|1_A(\xi_1, \xi_2)g(\xi_1 + \xi_2, \Omega_1(\xi_1, \xi_2))\|_{L^2_{\xi_1\xi_2}} \\
\lesssim 2^{-k_2} \|g_1\|_{L^2} \|g_2\|_{L^2} \|g\|_{L^2},
\]

where in the last inequality we used the change of the variables \( x = \xi_1 + \xi_2, y = \Omega_1(\xi_1, \xi_2) \), since the Jacobi is \( |\partial_{\xi_1} \Omega_1 - \partial_{\xi_2} \Omega_1| \geq \frac{1}{2} |\xi_2|^2 \geq 2^{k_2} \) in \( A \).

Now we consider the term \( II \). When \((\xi_1, \xi_2) \in B\), we have that \( |\Omega_1| \geq \frac{1}{2}|\xi_2|^3 \), so from the support properties we get \( j_{\max} \geq 3k_2 - 20 \). Then from (a) we have in this case

\[
II \lesssim 2^{j_{\min}/2} 2^{k_2/2} \prod_{i=1}^3 \|f_{k_i, j_i}\|_{L^2} \lesssim 2^{j_{\min}/2} 2^{j_{\max}/2} 2^{-k_2} \prod_{i=1}^3 \|f_{k_i, j_i}\|_{L^2},
\]

which completes the proof of (b).

Now we prove part (c). We will prove that if \( g_i : \mathbb{R} \to \mathbb{R}_+ \) are \( L^2 \) functions supported in \([2^{k_i-1}, 2^{k_i+1}]\), \( i = 1, 2 \), and \( g : \mathbb{R}^2 \to \mathbb{R}_+ \) is an \( L^2 \) function supported in \([2^{k_1-1}, 2^{k_1+1}] \times I_{\max} \), then

\[
\int_{\mathbb{R}^2} g_1(\xi_1)g_2(\xi_2)g(\xi_1 + \xi_2, \Omega_2(\xi_1, \xi_2)) \, d\xi_1 \, d\xi_2 \lesssim \lambda^{-1/2} 2^{-k_3/2} \|g_1\|_{L^2} \|g_2\|_{L^2} \|g\|_{L^2}. \tag{2.8}
\]

This suffices for (2.5) by Cauchy–Schwartz inequality. To prove (2.8), we notice that

\[
|\partial_{\xi_1} \Omega_1 - \partial_{\xi_2} \Omega_2| = 2\lambda|\xi_1 + \xi_2| \sim 2\lambda 2^{k_3},
\]

thus by change of variables \( \mu_1 = \xi_1 + \xi_2, \mu_2 = \Omega_2(\xi_1, \xi_2) \) we get

\[
\|g(\xi_1 + \xi_2, \Omega_2(\xi_1, \xi_2))\|_{L^2_{\xi_1\xi_2}} \lesssim \lambda^{-1/2} 2^{-k_3/2} \|g\|_{L^2},
\]

which is sufficient for (2.8) by Cauchy–Schwartz inequality. \( \square \)

**Remark 2.2.** It is easy to see from the proof that parts (a) and (b) of Lemma 2.1 also hold if we assume instead \( f_{k_i, j_i} \) is supported in \( I_{k_i} \times I_{j_i} \) for \( k_1, k_2, k_3 \in \mathbb{Z}_+ \). Part (c) of Lemma 2.1 also holds if we assume instead \( f_{k_1, j_1}, f_{k_2, j_2} \) are supported in \( I_{k_1} \times I_{j_1}, I_{k_2} \times I_{j_2} \), respectively, for \( k_1, k_2 \in \mathbb{Z}_+ \).

We restate now Lemma 2.1 in a form that is suitable for the bilinear estimates in the next section.
Corollary 2.3. (a) Let $k_1, k_2, k_3, j_1, j_2, j_3 \in \mathbb{Z}_+$. Assume $f_{k_1,j_1}, f_{k_2,j_2} \in L^2(\mathbb{R} \times \mathbb{R})$ are nonnegative functions that are supported in $\{(\xi, \tau): \xi \in I_{k_1}, \tau + \lambda \xi^2 \in I_{j_1}\}$ and $\{(\xi, \tau): \xi \in I_{k_2}, \tau - \xi^2 \in I_{j_2}\}$, respectively, then

$$\|1_{D^k_{k_3,j_3}}(\xi, \tau)(f_{k_1,j_1} * f_{k_2,j_2})\|_{L^2} \lesssim 2^{\min/2}2^{1\min/2}2^{j_1/2} \prod_{i=1}^{2} \|f_{k_i,j_i}\|_{L^2}. \quad (2.9)$$

Furthermore, if $k_2 > 2$, then we have

$$\|1_{D^k_{k_3,j_3}}(\xi, \tau)(f_{k_1,j_1} * f_{k_2,j_2})\|_{L^2} \lesssim 2^{j_{\text{med}}/2}2^{j_{\text{max}}/2}2^{-k_2} \prod_{i=1}^{2} \|f_{k_i,j_i}\|_{L^2}. \quad (2.10)$$

(b) Let $k_1, k_2, j_1, j_2, j_3 \in \mathbb{Z}_+$ and $k_3 \in \mathbb{Z}$. Assume $f_{k_1,j_1}, f_{k_2,j_2} \in L^2(\mathbb{R} \times \mathbb{R})$ are nonnegative functions that are supported in $\{(\xi, \tau): \xi \in I_{k_1}, \tau + \lambda \xi^2 \in I_{j_1}\}$ and $\{(\xi, \tau): \xi \in I_{k_2}, \tau - \lambda \xi^2 \in I_{j_2}\}$, respectively, then

$$\|1_{B_{k_3,j_3}}(\xi, \tau)(f_{k_1,j_1} * f_{k_2,j_2})\|_{L^2} \lesssim 2^{\min/2}2^{1\min/2} \prod_{i=1}^{2} \|f_{k_i,j_i}\|_{L^2}, \quad (2.11)$$

and also

$$\|1_{B_{k_3,j_3}}(\xi, \tau)(f_{k_1,j_1} * f_{k_2,j_2})\|_{L^2} \lesssim \lambda^{-1/2}2^{j_{1/2}}2^{j_{2/2}}2^{-k_3/2} \prod_{i=1}^{2} \|f_{k_i,j_i}\|_{L^2}. \quad (2.12)$$

**Proof.** We just prove (a), the proof for (b) is similar. Clearly, we have

$$\|1_{D^k_{k_3,j_3}}(\xi, \tau)(f_{k_1,j_1} * f_{k_2,j_2})(\xi, \tau)\|_{L^2} = \sup_{\|f\|_{L^2}=1} \left| \int f \cdot f_{k_1,j_1} * f_{k_2,j_2} d\xi d\tau \right|. \quad (2.13)$$

We denote $f_{k_3,j_3} = 1_{p_{k_3,j_3}}(\xi, \tau) \cdot f(\xi, \tau)$. Define $f_{k_1,j_1}^\xi(\xi, \mu) = f_{k_1,j_1}(\xi, \mu - \lambda \xi^2)$, $f_{k_2,j_2}^\mu(\xi, \mu) = f_{k_2,j_2}(\xi, \mu + \lambda \xi^2)$, $f_{k_3,j_3}^\xi(\xi, \mu) = f_{k_3,j_3}(\xi, \mu - \lambda \xi^2)$. Then for $i = 1, 2, 3$ the functions $f_{k_i,j_i}^\xi$ are supported in $I_k \times I_j$, and $\|f_{k_i,j_i}^\xi\|_{L^2} = \|f_{k_i,j_i}\|_{L^2}$. Using simple changes of variables, we get that

$$\int_{D^k_{k_3,j_3}} f \cdot f_{k_1,j_1} * f_{k_2,j_2} d\xi d\tau = J_1(f_{k_1,j_1}^\xi, f_{k_2,j_2}^\xi, f_{k_3,j_3}^\xi).$$

Then (a) follows from Lemma 2.1(a), (b) and Remark 2.2. \(\square\)

3. Bilinear estimates for the coupling terms

This section is devoted to prove the bilinear estimates for the coupling terms in the $F^\xi$-type space. We first recall an abstract extension lemma. For a continuous function $h(\xi)$, define a group $W_h(t)$ by

$$W_h(t)f = \mathcal{F}_x e^{ith(\xi)} \mathcal{F}_x f.$$

Then $W_h(t)f$ is the solution to the following equation

$$\partial_t u - ih(-i\partial_x)u = 0, \quad u(x, 0) = f(x).$$
Lemma 3.1 (Extension lemma). Assume $h$ is a continuous function. Let $Z$ be any space–time Banach space which obeys the time modulation estimate

$$\|g(t) F(t, x)\|_Z \leq \|g\|_{L^\infty_t} \|F(t, x)\|_Z$$

(3.1)

for any $F \in Z$ and $g \in L^\infty_t$. Moreover, if for all $u_0 \in L^2$ and $g \in L^\infty_t$, then one also has the estimate that for all $k \in Z^+$ and $u \in F^0$

$$\|P_k(u)\|_Z \leq \|\hat{P}_k(u)\|_{Y^k_{\tau=h(t)}}.$$

Proof. We refer the readers to Lemma 3.2 in [10]. □

Proposition 3.2 ($X^k_{\tau=-\lambda \xi^2}$ embedding). Let $k \in Z^+$, $j \in \mathbb{N}$. Assume $u \in S(\mathbb{R}^2)$, then we have

$$\|P_k(u)\|_{L^q_t L^r_x} \lesssim \lambda^{-1/q} \|P_k(u)\|_{Y^k_{\tau=-\lambda \xi^2}},$$

$$\|P_k(u)\|_{L^\infty_t L^2_x} \lesssim \lambda^{-1/2} 2^{-k/2} \|F[P_j(u)]\|_{Y^k_{\tau=-\lambda \xi^2}}.$$ (3.2)

where $(q, r)$ satisfies $2 \leq q, r \leq \infty$ and $2/q = 1 - 1/r$. As a consequence, we get from the definition that for $u \in F^s_{\tau=-\lambda \xi^2}$

$$\|u\|_{L^\infty_t H^s} \lesssim \|u\|_{F^s_{\tau=-\lambda \xi^2}}.$$ (3.3)

Proof. From Lemma 3.1, it suffices to prove

$$\|U_\lambda(t) f\|_{L^q_t L^r_x} \lesssim \lambda^{-1/q} \|f\|_{L^2_x},$$

$$\|U_\lambda(t) f\|_{L^\infty_t L^2_x} \lesssim \lambda^{-1/2} \|f\|_{H_x^{-1/2}},$$

which are well known, for example see [12] and [13]. □

Proposition 3.3 ($X^k_{\tau=\xi^3}$ embedding). Let $k \in Z^+$, $j \in \mathbb{N}$. Assume $u \in F^0$, then we have

$$\|P_k(u)\|_{L^q_t L^r_x} \lesssim \|P_k(u)\|_{Y^k_{\tau=\xi^3}},$$

$$\|P_k(u)\|_{L^q_t L^\infty_x} \lesssim 2^{k/4} \|F[P_k(u)]\|_{Y^k_{\tau=\xi^3}},$$

$$\|P_j(u)\|_{L^2_t L^q_x} \lesssim 2^{-j} \|F[P_j(u)]\|_{Y^j_{\tau=\xi^3}}.$$ (3.4)

where $(q, r)$ satisfies $2 \leq q, r \leq \infty$ and $3/q = 1/2 - 1/r$. As a consequence, we get from the definition that for $u \in F^s_{\tau=\xi^3}$

$$\|u\|_{L^\infty_t H^s} \lesssim \|u\|_{F^s_{\tau=\xi^3}}.$$
Proof. We refer the readers to Proposition 3.3 in [10]. □

The main result of this section is the following lemma.

Lemma 3.4. (a) If \( s_1 \geq 0, s_2 \in (-1, -1/2), s_1 - s_2 < 1, 0 < \theta \ll 1, \) and \( u \in X^{s_1, 1/2+\theta}_{\tau=-\lambda k^2} \), then
\[
\| \psi(t)uv \|_{X^{s_1, 1/2+\theta}_{\tau=-\lambda k^2}} \lesssim \lambda^{-1/2} \| u \|_{X^{s_1, 1/2+\theta}_{\tau=-\lambda k^2}} \| v \|_{F^2_{\tau=\xi^3}}. \tag{3.5}
\]

(b) If \( s_1 \geq 0, s_2 - s_1 < -1/2, \) and \( u, w \in X^{s_1, 1/2+\theta}_{\tau=-\lambda k^2} \) then
\[
\| \psi(t) \partial_x (u \bar{w}) \|_{X^{s_2, -1/2+\theta}_{\tau=\xi^3}} \lesssim \lambda^{-1/2} \| u \|_{X^{s_1, 1/2+\theta}_{\tau=-\lambda k^2}} \| w \|_{X^{s_1, 1/2+\theta}_{\tau=-\lambda k^2}}. \tag{3.6}
\]

Proof. We first prove (a). From the definition, we get that the left-hand side of (3.5) is dominated by
\[
\left\| 2^{s_1 k_3} \sum_{j_3=0}^\infty 2^{-1/2j_3+2\theta j_3} \| 1_{D_{k_3, j_3}} \cdot \psi(t)uv \|_{L^2_{\xi, \tau}} \right\|_{F^2_{j_3}}. \tag{3.7}
\]

Now we begin to estimate
\[
\sum_{j_3} 2^{-1/2j_3+2\theta j_3} \| 1_{D_{k_3, j_3}} \cdot \psi(t)uv \|_{L^2_{\xi, \tau}}. \tag{3.8}
\]

Decomposing \( u, v \), for \( k_1, j_1 \in \mathbb{Z}_+ \) set

\[
\begin{align*}
\tilde{f}_{k_1,j_1}(\xi_1, \tau_1) &= \eta_{k_1}(\xi_1)\eta_{j_1}(\tau_1 + \lambda \xi_1^2) \psi(t/2)u(\xi_1, \tau_1), \\
\tilde{f}_{k_2,j_2}(\xi_2, \tau_2) &= \eta_{k_2}(\xi_2)\eta_{j_2}(\tau_2 - \xi_2^3) \psi(t/2)v(\xi_2, \tau_2),
\end{align*}
\]

then we get for fixed \( k_3 \),
\[
(3.8) \lesssim \sum_{(k_1,k_2,k_3) \in K \ j_3 \geq 0} \sum_{j_1, j_2 \geq 0} 2^{-j_3/2+2\theta j_3} \sum_{j_1, j_2 \geq 0} \| 1_{D_{k_3, j_3}} \cdot \tilde{f}_{k_1,j_1} \ast \tilde{f}_{k_2,j_2} \|_{L^2_{j_3 j_3}}. \tag{3.9}
\]

where \( K = \{(k_1,k_2,k_3) \in \mathbb{Z}^3_+ : |k_{\text{med}} - k_{\text{max}}| < 5\} \), then it is easy to see that \( 1_{D_{k_3, j_3}} \cdot \tilde{f}_{k_1,j_1} \ast \tilde{f}_{k_2,j_2} \equiv 0 \) unless

\[
|k_{\text{med}} - k_{\text{max}}| < 5.
\]

We may also assume that \( k_{\text{max}} \geq 20 \), since for \( k_{\text{max}} \leq 20 \) we can get from Plancherel’s equality that
\[
(3.8) \lesssim \sum_{j_3 \geq 0} 2^{-j_3/2+2\theta j_3} \| 1_{D_{k_3, j_3}} \cdot \tilde{P}_{\leq 20} u \ast \psi(t) \tilde{P}_{\leq 20} v \|_{L^2_{j_3 j_3}} \lesssim \| P_{\leq 20} u \psi(t) \tilde{P}_{\leq 20} v \|_{L^2_{j_3 j_3}} \lesssim \| P_{\leq 20} u \|_{L^\infty} \| \psi(t) \tilde{P}_{\leq 20} v \|_{L^2_{j_3 j_3}},
\]

which suffices to give the bound for this case by Propositions 3.2, 3.3.
Now we assume \( k_{\text{max}} \geq 20 \) and consider (3.9). First we assume that \( k_2 \geq 2 \). Clearly we may also assume that \( j_{\text{max}} \leq 10k_2 \), otherwise, we can apply (2.9), then we have a \( 2^{-5k_2} \) to spare. After these assumptions, we can make use of (2.10) to bound (3.9) by

\[
\sum_{(k_1,k_2,k_3) \in K, \ k_2 \geq 2} \sum_{j_3 \geq 0} 2^{-j_3/2 + 2\theta j_2} \sum_{j_1,j_2 \geq 0} 2^{j_{\text{max}}/2} 2^{j_{\text{med}}/2} 2^{k_2 - k_2} \prod_{i=1}^{2} \| f_{k_i,j_i} \|_{L^2}, \tag{3.10}
\]

when \( j_3 = j_{\text{min}} \), then (3.10) is bounded by

\[
\sum_{(k_1,k_2,k_3) \in K, \ k_2 \geq 2} \sum_{j_3 \geq 0} 2^{j_1/2} 2^{j_2/2} 2^{k_2 - k_2} \prod_{i=1}^{2} \| f_{k_i,j_i} \|_{L^2} \lesssim \sum_{(k_1,k_2,k_3) \in K, \ k_2 \geq 2} 2^{-k_2} \| P_{k_1} u \|_{Y^{k_1}_{\tau = \lambda e^2}} \| P_{k_2} v \|_{Y^{k_2}_{\tau = e^3}}, \tag{3.11}
\]

when \( j_1 = j_{\text{min}} \), then (3.10) is bounded by

\[
\sum_{(k_1,k_2,k_3) \in K, \ k_2 \geq 2} \sum_{j_3 \geq 0} 2^{2\theta j_3} \sum_{j_1,j_2 \geq 0} 2^{j_2/2} 2^{k_2 - k_2} \prod_{i=1}^{2} \| f_{k_i,j_i} \|_{L^2} \lesssim \sum_{(k_1,k_2,k_3) \in K, \ k_2 \geq 2} 2^{2\theta k_2} \sum_{j_1,j_2 \geq 0} 2^{j_1/2} 2^{j_2/2} 2^{k_2 - k_2} \prod_{i=1}^{2} \| f_{k_i,j_i} \|_{L^2} \lesssim \sum_{(k_1,k_2,k_3) \in K, \ k_2 \geq 2} 2^{-k_2 + 2\theta k_2} \| P_{k_1} u \|_{Y^{k_1}_{\tau = \lambda e^2}} \| P_{k_2} v \|_{Y^{k_2}_{\tau = e^3}}, \tag{3.12}
\]

where we use \( j_{\text{max}} \leq 10k_2 \) in the first step. The case \( j_2 = j_{\text{min}} \) is the same as \( j_1 = j_{\text{min}} \), so we omit the details.

Dividing the summation on \( k_1, k_2 \) in the right-hand side of (3.10) into several parts, we get from (3.11) and (3.12)

\[
(3.10) \lesssim \sum_{i=1}^{5} \sum_{k_3 \in A_i(k_3)} 2^{-k_2 + 2\theta k_2} \| P_{k_1} u \|_{Y^{k_1}_{\tau = \lambda e^2}} \| P_{k_2} v \|_{Y^{k_2}_{\tau = e^3}},
\]

where we denote

- \( A_1(k_3) = \{ |k_2 - k_3| \leq 5, \ k_1 \leq k_2 - 10, \ \text{and} \ k_2 \geq 30 \} \);
- \( A_2(k_3) = \{ |k_1 - k_3| \leq 5, \ 2 \leq k_2 \leq k_1 - 10, \ \text{and} \ k_1 \geq 30 \} \);
- \( A_3(k_3) = \{ |k_1 - k_2| \leq 5, \ k_3 \leq k_2 - 10, \ \text{and} \ k_1 \geq 30 \} \);
- \( A_4(k_3) = \{ |k_2 - k_3| \leq 10, \ |k_1 - k_2| \leq 10, \ \text{and} \ k_2 \geq 30 \} \);
- \( A_5(k_3) = \{ k_1, k_2, k_3 \leq 200 \} \).

So we can bound (3.8) by
Now we begin to estimate (3.13) case by case. For case $A_1$, by Cauchy–Schwarz’s inequality we have

\[
\| 2^{s_1 k_1} \sum_{A_1(k_3)} 2^{-k_2 + 20 \theta k_2} \| P_{k_1} u \|_{Y^k_{\tau = -\lambda \xi^2}} \| P_{k_2} v \|_{Y^k_{\tau = -\epsilon^3}} \|_{L^2_{k_3}} \geq \| 2^{s_1 k_1} \sum_{|k_2 - k_3| \leq 5, k_2 \geq 20} \sum_{k_1 \leq k_2 - 10} 2^{-k_2 + 20 \theta k_2} \| P_{k_1} u \|_{Y^k_{\tau = -\lambda \xi^2}} \| P_{k_2} v \|_{Y^k_{\tau = -\epsilon^3}} \|_{L^2_{k_3}} \geq \| 2^{s_1 k_1} \sum_{|k_2 - k_3| \leq 5, k_2 \geq 20} 2^{-k_2 + 21 \theta k_2} \| P_{k_2} v \|_{Y^k_{\tau = -\epsilon^3}} \|_{L^2_{k_3}} \| 2^{s_1 k_1} \| P_{k_1} u \|_{Y^k_{\tau = -\lambda \xi^2}} \|_{L^2_{k_1}},
\]

since $s_1 - s_2 < 1$, $s_1 \geq 0$, $s_2 > -1$ and $0 < \theta \ll 1$, we can bound (3.14) by $\| v \|_{\dot{F}^{1/2} \lambda^{1/2}} \| u \|_{\dot{X}^{1/2 + \theta}}$. The proofs for $A_2$–$A_5$ are similar, we omit them here.

Now we assume that $k_3 \leq 2$, by the assumption $k_{\max} \geq 20$, we have $|k_1 - k_3| \leq 5$, and $k_1, k_2 \geq 10$. For simplicity of notations we assume $k_1 = k_3 = k$, then the left-hand side of (3.8) can be estimated as

\[
\sum_{k \geq 0} 2^{s_1 k} \| \overline{P_k(u)} * \psi(t) \overline{P_{\leq 2}(v)} \|_{L^2} \lesssim \sum_{k \geq 0} 2^{s_1 k} \| P_k(u) \|_{L^\infty_{\tau} L^2_{\xi}} \| \psi(t) P_{\leq 2}(v) \|_{L^2_{\tau} L^\infty_{\xi}}.
\]

This is enough in view of the definition, Proposition 3.2 and Proposition 3.3.

Now we begin to prove (b), form the definition, we only need to prove

\[
\| \partial_x(u \tilde{w}) \|_{\dot{X}^{s_1 - 1/2 + 2\theta} \lambda^{-1/2} \| u \|_{\dot{F}^{s_1} \lambda^{-\lambda \xi^2}} \| w \|_{\dot{F}^{1} \lambda^{-\lambda \xi^2}}.}
\]

By the definition and $\theta \ll 1$, we have

\[
\| \partial_x(u \tilde{w}) \|_{\dot{X}^{s_1 - 1/2 + 2\theta}} \lesssim \left( \sum_{k \geq 1} 2^{2s_2 k} \left( \sum_{j \geq 0} 2^{-j/4} \| 1_{B_{k,j}}(\xi, \tau) i \xi \mathcal{F}(u \tilde{w}) \|_{L^2_{\xi, \tau}} \right)^2 \right)^{1/2} + \left( \sum_{k \leq 0} \left( \sum_{j \geq 0} 2^{-j/4} \| 1_{B_{k,j}}(\xi, \tau) i \xi \mathcal{F}(u \tilde{w}) \|_{L^2_{\xi, \tau}} \right)^2 \right)^{1/2}
\]

\[
:= I + II.
\]

Now let

\[
\check{f}_{k_1,j_1}(\xi_1, \tau_1) = \eta_{k_1}(\xi_1) \eta_{j_1}(\tau_1 + \lambda \xi_1^2) \hat{u}(\xi_1, \tau_1),
\]

\[
\check{g}_{k_2,j_2}(\xi_2, \tau_2) = \eta_{k_1}(\xi_2) \eta_{j_1}(\tau_2 - \lambda \xi_2^2) \hat{w}(\xi_2, \tau_2),
\]

then we have
So we can bound (3.18) by

$$||u||_{F_{\tau=\lambda t^2}} = \left\| 2^{sk_1} \sum_{j_1 \geq 0} 2^{j_1/2} \| f_{k_1,j_1} \|_{L^2} \right\|_{k_1},$$

$$||w||_{F_{\tau=\lambda t^2}} = \left\| 2^{sk_2} \sum_{j_2 \geq 0} 2^{j_2/2} \| g_{k_2,j_2} \|_{L^2} \right\|_{k_2}.$$

Then from the definition, I and II in (5.7) can be bounded as follows

$$I \lesssim \| B_{k_3,j_3} f_{k_1,j_1} \ast g_{k_2,j_2} \|_{L^2_{\lambda t^3 \tau}} \| L_{k_3 \geq 1} \tag{3.18}$$

and

$$II \lesssim \| B_{k_3,j_3} f_{k_1,j_1} \ast g_{k_2,j_2} \|_{L^2_{\lambda t^3 \tau}} \| L_{k_3 \leq 0} \tag{3.19}.$$

We first estimate term I in (3.18). By (2.12) we have

$$\sum_{(k_1,k_2,k_3) \in K} \sum_{j_3 \geq 0} 2^{-j_3/4} \sum_{j_1,j_2 \geq 0} \| B_{k_3,j_3} f_{k_1,j_1} \ast g_{k_2,j_2} \|_{L^2_{\lambda t^3 \tau}} \tag{3.20}$$

Divide the summation on $k_1, k_2$ in the right-hand side of (3.20) into several parts,

$$(3.20) \lesssim \lambda^{-1/2} \sum_{l=1}^{5} \sum_{B_1(k_3)} \| P_{k_1} u \|_{p_{k_1}} \| P_{k_2} w \|_{p_{k_2}},$$

where we denote

$$B_1(k_3) = \{ |k_2 - k_3| \leq 5, k_1 \leq k_2 - 10, \text{ and } k_2 \geq 30 \};$$

$$B_2(k_3) = \{ |k_1 - k_3| \leq 5, k_2 \leq k_1 - 10, \text{ and } k_1 \geq 30 \};$$

$$B_3(k_3) = \{ |k_1 - k_2| \leq 5, 1 \leq k_3 \leq k_2 - 10, \text{ and } k_1 \geq 30 \};$$

$$B_4(k_3) = \{ |k_2 - k_3| \leq 10, |k_1 - k_2| \leq 10, \text{ and } k_2 \geq 30 \};$$

$$B_5(k_3) = \{ k_1, k_2, k_3 \leq 200, k_3 \geq 1 \}.$$

So we can bound (3.18) by

$$I \lesssim \lambda^{-1/2} \sum_{l=1}^{5} \left\| 2^{(s_2+1/2)k_3} \sum_{B_i(k_3)} \| P_{k_1} u \|_{p_{k_1}} \| P_{k_2} w \|_{p_{k_2}} \right\|_{k_3}. \tag{3.21}$$
Now we begin to estimate (3.21) case by case. For case $B_1$, by Cauchy–Schwartz’s inequality we have

$$
\lambda^{-1/2} \left\| 2^{(s_2+1/2)k_3} \sum_{B_1(k_3)} \| P_{k_1}^* u \| y_{k_1} \| P_{k_2}^* W \| y_{k_2} \right\|_{k_3}^2 \\
= \lambda^{-1/2} \left\| 2^{(s_2+1/2)k_3} \sum_{|k_2-k_3| \leq 5, k_2 \geq 20} \| P_{k_2}^* W \| y_{k_2} \right\|^2_{k_3} \sum_{k_1 \leq k_2-10} \| P_{k_1}^* u \| y_{k_1} \right\|_{k_3}^2 \\
\lesssim \lambda^{-1/2} \left\| 2^{(s_2+1/2+\theta)k_3} \sum_{|k_2-k_3| \leq 5, k_2 \geq 20} \| P_{k_2}^* W \| y_{k_2} \right\|^2_{k_3} \| u \|_{F^1_{\tau-\lambda k_2}},
$$

(3.22)
since $s_2 - s_1 < -1/2$, $s_1 \geq 0$ and $0 < \theta \ll 1$, so we can bound (3.22) by $\| u \|_{X^{s_1,1/2+\theta}_{\tau-\lambda k_2}} \| W \|_{X^{s_1,1/2+\theta}_{\tau-\lambda k_2}}$. The proofs for $B_2-B_5$ are similar, we omit them here.

Using the same argument as above, under the restriction $s_2 - s_1 < -1/2$, $s_1 \geq 0$ and $0 < \theta \ll 1$, we can bound the part $II$ in (3.19) by $\| u \|_{X^{s_1,1/2+\theta}_{\tau-\lambda k_2}} \| W \|_{X^{s_1,1/2+\theta}_{\tau-\lambda k_2}}$. Thus we finish the proof. □

4. Proof of the main theorem

In this section we prove Theorem 1.1. We first recall some linear estimates in $X^{s,b}_{\tau=b(\xi)}$ and $\tilde{F}^{s}_{\tau=b(\xi)}$. Let $W_h(t) f = \mathcal{F}^{-1} e^{ith(\xi)} \mathcal{F} f$. The following lemma has been proved by Kenig, Ponce and Vega in [14], and then improved by Ginibre, Y. Tsutsumi and Velo in [7].

Lemma 4.1. Let $s \in \mathbb{R}$, $-1/2 < b' \leq 0 \leq b \leq b' + 1$ and $T \in [0, 1]$. Then for $u_0 \in H^s$ and $F \in X^{s,b}_{\tau=b(\xi)}$ we have

$$\left\| \psi(t) W_h(t) u_0 \right\|_{X^{s,b}_{\tau=b(\xi)}} \lesssim \| u_0 \|_{H^s},$$

$$\| \psi_T(t) \int_0^t W_h(t-t') F(t', \cdot) \, dt' \|_{X^{s,b'}_{\tau=b(\xi)}} \lesssim T^{1-b+b'} \| F \|_{X^{s,b'}_{\tau=b(\xi)}}.$$

Next we prove the linear estimates in $\tilde{F}^s$. The proof was essentially given in [11] for the Benjamin–Ono equation.

Lemma 4.2. (a) Assume $s \in \mathbb{R}$ and $\phi \in H^s$. Then there exists $C > 0$ such that

$$\| \psi(t) W_h(t) \phi \|_{\tilde{F}^s_{\tau=b(\xi)}} \leq C \| \phi \|_{H^s}. \tag{4.1}$$

(b) Assume $s \in \mathbb{R}$, $k \in \mathbb{Z}_+$ and $u$ satisfies $(i + \tau - \xi^3)^{-1} \mathcal{F}(u) \in V^k_{\tau=b(\xi)}$. Then there exists $C > 0$ such that

$$\left\| \mathcal{F} \left[ \psi(t) \int_0^t W_h(t-s)(u(s)) \, ds \right] \right\|_{V^k_{\tau=b(\xi)}} \leq C \left\| (i + \tau - \xi^3)^{-1} \mathcal{F}(u) \right\|_{V^k_{\tau=b(\xi)}} \tag{4.2}.$$

Now we give the estimates for the cubic nonlinear term.
Lemma 4.3. Let \( u, u' \in X^{s,b}_{r=-\lambda} \) with \( 1/2 < b < 1 \) and \( s \geq 0 \). Then for all \( a \geq 0 \) we have that

\[
\| u \|_{X^{s,a}_{r=-\lambda}}^2 \lesssim \lambda^{-1/2} \| u \|_{X^{s,b}_{r=-\lambda}}^3,
\]

\[
\| u \|_{X^{s,a}_{r=-\lambda}}^2 \lesssim \lambda^{-1/2} \left( \| u \|_{X^{s,b}_{r=-\lambda}}^2 + \| u' \|_{X^{s,b}_{r=-\lambda}}^2 \right) \| u - u' \|_{X^{s,b}_{r=-\lambda}}.
\]

Proof. We just prove the first one when \( s = 0 \) for example. By Hölder inequality and Proposition 3.2, we have

\[
\| u \|_{X^{s,a}_{r=-\lambda}}^2 \leq \| u \|_{\tilde{F}^s}^2 = \| u \|_{\tilde{F}^{3/4}}^3 \lesssim C \lambda^{-1/2} \| u \|_{X^{0,b}_{r=-\lambda}}^3.
\]

Thus we finish the proof. \( \square \)

We also need a bilinear estimates. For \( u, v \in \tilde{F}^s \) we define the bilinear operator

\[
B(u, v) = \psi(t/4) \int_0^t W(t - \tau) \partial_x (\psi^2(u) \cdot v(\tau)) d\tau.
\]

The following lemma is due to the first author [10], which is the key to get the global well-posedness for KdV in \( H^{-3/4} \).

Lemma 4.4. (See [10, Proposition 4.2]) Assume \(-3/4 \leq s \leq 0\). Then there exists \( C > 0 \) such that

\[
\| B(u, v) \|_{\tilde{F}^s} \leq C \left( \| u \|_{\tilde{F}^{3/4}} \| v \|_{\tilde{F}^{3/4}} + \| u \|_{\tilde{F}^{-3/4}} \| v \|_{\tilde{F}^{s}} \right)
\]

(4.3)

holds for any \( u, v \in \tilde{F}^s \).

Now we are ready to prove Theorem 1.1. We consider first (1.4) under the condition (1.5). We may assume \( \alpha = \beta = \gamma = 1 \). From Duhamel’s principle, the Cauchy problem (1.4) is equivalent to the following integral equation system

\[
u(t) = \psi_4(t) \frac{1}{2} \int_0^t V(t - t') \partial_x (v^2)(t') dt' + \psi_5(t) \int_0^t V(t - t') \partial_x (u_t^2)(t') dt',
\]

(4.4)

To study the local existence, it suffices to study the following time-localized system

\[
u(t) = \psi_4(t) \frac{1}{2} \int_0^t V(t - t') \partial_x (v^2)(t') dt' + \psi_5(t) \int_0^t V(t - t') \partial_x (u_t^2)(t') dt'.
\]

(4.5)
It is easy to see that if \((u, v)\) solve the system (4.5) for all \(t \in \mathbb{R}\), then \((u, v)\) also solve the system (4.4) for \(|t| \leq T\).

We follow the similar argument as the one given in [6] to construct our solution spaces. We consider the following function space where we seek our solution:

\[
\Sigma_\theta := \left\{ (u, v) \in X^{0,1/2+\theta}_{\tau=-\xi^2} \times \tilde{F}^{-3/4}_{\tau=0}; \|u\|_{X^{0,1/2+\theta}_{\tau=-\xi^2}} \leq M \text{ and } \|v\|_{\tilde{F}^{-3/4}_{\tau=0}} \leq \epsilon_0 \right\},
\]

where \(0 < \theta, \epsilon_0 < 1\) and \(M > 0\) will be chosen later. Furthermore, \(\Sigma_\theta\) is a complete metric space with norm

\[
\| (u, v) \|_{\Sigma_\theta} = \|u\|_{X^{0,1/2+\theta}_{\tau=-\xi^2}} + \|v\|_{\tilde{F}^{-3/4}_{\tau=0}}.
\]

For \((u, v) \in \Sigma_\theta\), we define the maps \(\Phi = \Phi_1 \times \Phi_2\),

\[
\Phi_1(u, v) = \psi_1(t)U_\lambda(t)\phi_1(x) - i\psi_T(t) \int_0^t U_\lambda(t - t')\left[ \lambda(uv)(t') + \lambda^{-1}(|u|^2u)(t') \right] dt'
\]

and

\[
\Phi_2(u, v) = \psi_1(t)\psi(t)\phi_2(x) - \frac{1}{2}\psi(t) \int_0^t V(t - t')\partial_x(v^2)(t') dt' + \psi_T(t) \int_0^t V(t - t')\partial_x(|u|^2)(t') dt'.
\]

Then from Lemmas 4.1–4.4, we get

\[
\| \Phi_1(u, v) \|_{X^{0,1/2+\theta}_{\tau=-\xi^2}} \leq C_0\|\phi_1\|_{L^2} + C_1T^\theta[\lambda\|uv\|_{X^{0,1/2+2\theta}_{\tau=-\xi^2}} + \lambda^{-1}\||u|^2u\|_{X^{0,1/2+2\theta}_{\tau=-\xi^2}}]
\]

\[
\leq C_0\|\phi_1\|_{L^2} + C_1T^\theta[\lambda\|u\|_{X^{1/2+\theta}_{\tau=-\xi^2}} \|v\|_{\tilde{F}^{-3/4}_{\tau=0}} + \lambda^{-3/2}\|u\|_{X^{0,1/2+\theta}_{\tau=-\xi^2}}^3]
\]

\[
\leq C_0\|\phi_1\|_{L^2} + C_1T^\theta[\lambda M\epsilon_0 + \lambda^{-2}M^2]
\]

and

\[
\| \Phi_2(u, v) \|_{\tilde{F}^{-3/4}_{\tau=0}} \leq C_0\|\phi_2\|_{H^{-3/4}} + \frac{1}{2}\left\| \psi\left(\frac{t}{4}\right) \int_0^t V(t - t')\partial_x(v^2)(t') dt' \right\|_{\tilde{F}^{-3/4}_{\tau=0}}
\]

\[
+ \left\| \psi_T(t) \int_0^t V(t - t')\partial_x(|u|^2)(t') dt' \right\|_{X^{-3/4,1/2+\theta}_{\tau=0}}
\]

\[
\leq C_0\|\phi_2\|_{H^{-3/4}} + C_2\|v\|_{\tilde{F}^{-3/4}_{\tau=0}}^2 + C_2T^\theta\|\partial_x|u|^2\|_{X^{-3/4,-1/2+2\theta}_{\tau=0}}
\]
Therefore, for the original system (1.1), we get the local existence of the solution on $[0, T_1]$ for

$$T_1 \sim \left( \frac{\lambda^{3/2} \epsilon_0}{M^2} \right)^{1/\theta} = \left( \frac{1}{\lambda^{3/2} \|u_0\|_{L^2}^{2}} \right)^{1/\theta}.$$ 

Therefore, for the original system (1.1), we get the local existence of the solution on $[0, T]$ for
Theorem 5.1. \[ T = \lambda^3 T_1 \sim T(\|u_0\|_{L^2}, \|v_0\|_{H^{-3/4}}). \]

Therefore, we obtain the existence. By standard arguments, we can prove the rest parts of Theorem 1.1.

5. Further remarks on the resonant case

The condition \( u_0 \in L^2 \) in Theorem 1.1 seems to be sharp in view of the cubic Schrödinger equation. However, it is only needed for the term \(|u|^2u\). In the resonant case \( \beta = 0 \), we actually have

Theorem 5.1. If \( \beta = 0 \) in (1.1), then Theorem 1.1 holds for \( u_0 \in H^3, v_0 \in H^{-3/4} \), when \( s > -1/16 \).

In order to prove Theorem 5.1, we need to derive a new \( L^2 \) bilinear estimate.

Lemma 5.2. Assume \( k_i \in \mathbb{Z}, j_i \in \mathbb{Z}_+ \), and \( f_{k_i,j_i} \in L^2(\mathbb{R} \times \mathbb{R}) \) are nonnegative functions supported in \([2^{k_i-1}, 2^{k_i+1}] \times I_{j_i}, i = 1, 2, 3\). If \( k_3 \leq k_{\text{max}} - 5, j_3 \neq j_{\text{max}} \) and \( 2^{k_3} \ll \lambda 2^{k_{\text{max}}} \), then

\[
J_2(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}) \lesssim \lambda^{-1/2} 2^{(j_{\text{min}} + j_{\text{med}})/2} 2^{-k_{\text{max}}/2} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_{L^2}. \tag{5.1}
\]

Proof. We denote \( f = f_{k_1,j_1}, g = f_{k_2,j_2}, h = f_{k_3,j_3} \). We may assume \( j_2 = j_{\text{max}} \). First we notice that

\[
J_2(f, g, h) = \int_{\mathbb{R}^4} f(\xi_1, \mu_1)g(\xi_2, \mu_2)h(\xi_1 + \xi_2, \mu_1 + \mu_2 + \Omega_2(\xi_1, \xi_2)) d\xi_1 d\xi_2 d\mu_1 d\mu_2.
\]

where

\[
\Omega_2(\xi_1, \xi_2) = -\lambda \xi_1^2 + \lambda \xi_2^2 - (\xi_1 + \xi_2)^3. \tag{5.2}
\]

Changing variables \( \xi_2 = \xi - \xi_1 \) and then \( \mu_2 = \mu - \mu_1 - \Omega_2(\xi_1, \xi - \xi_1) \), we obtain

\[
J_2(f, g, h) = \int_{\mathbb{R}^4} f(\xi_1, \mu_1)g(\xi - \xi_1, \mu - \mu_1 - \Omega_2(\xi_1, \xi - \xi_1)) h(\xi, \mu) d\xi_1 d\xi d\mu_1 d\mu.
\]

In view of the argument in Lemma 2.1, it is sufficient to show

\[
\|1_D g(\xi - \xi_1, -\Omega_2(\xi_1, \xi - \xi_1))\|_{L^2_{\xi_1} L^2_\xi} \lesssim \lambda^{-1/2} 2^{-k_{\text{max}}/2} \|g\|_{L^2}.
\]

where \( D = \{ (\xi, \xi_1) : |\xi| \sim 2^{k_1}, |\xi_1| \sim 2^{k_{\text{max}}}, |\xi - \xi_1| \sim 2^{k_{\text{max}}} \} \). Now we perform changing of variables \( \eta = \xi - \xi_1 \) and \( \tau = -\Omega_2(\xi_1, \xi - \xi_1) \), and the Jacobi is

\[
|\partial_\xi [\Omega_2(\xi_1, \xi - \xi_1)] + \partial_{\xi_1}[\Omega_2(\xi_1, \xi - \xi_1)]| = |3\xi^2 + 2\lambda \xi_1| \sim 2\lambda |\xi_1| \sim 2\lambda 2^{k_{\text{max}}},
\]

where we use \( 2^{k_3} \ll \lambda 2^{k_{\text{max}}} \) in the second step, which is sufficient for (5.1). Thus we finish the proof. \( \square \)
Corollary 5.3. Let $k_1, k_2, j_1, j_2, j_3 \in \mathbb{Z}^+$ and $k_3 \in \mathbb{Z}$. Assume $f_{k_1, j_1}, f_{k_2, j_2} \in L^2(\mathbb{R} \times \mathbb{R})$ are nonnegative functions that are supported in $\{(\xi, \tau) \colon \xi \in I_{k_1}, \tau + \lambda \xi^2 \in I_{j_1}\}$ and $\{(\xi, \tau) \colon \xi \in I_{k_2}, \tau - \lambda \xi^2 \in I_{j_2}\}$, respectively. If $k_3 \leq k_{\text{max}} - 5$, $j_3 \neq j_{\text{max}}$ and $2^{2k_3} \ll \lambda 2^{k_{\text{max}}}$, then

$$\|1_{B_{k_3, j_3}}(\xi, \tau)(f_{k_1, j_1} \ast f_{k_2, j_2})\|_{L^2} \lesssim \lambda^{-1/2}2^{j_1/2}2^{j_2/2}2^{-k_{\text{max}}/2} \prod_{l=1}^2 \|f_{k_l, j_l}\|_{L^2}. \quad (5.3)$$

To prove Theorem 5.1, it remains to prove the following bilinear estimates for the coupling terms.

Proposition 5.4. (a) If $-1/4 < s_1 \leq 0$, $0 < \theta \ll 1$, and $u \in X_{\tau = -\lambda \xi^2}^{s_1/2 + \theta}$, $v \in \tilde{F}_{\tau = \lambda \xi^2}^{-3/4}$ then

$$\|\psi(t)uv\|_{X_{\tau = -\lambda \xi^2}^{s_1/2 + \theta}} \lesssim \lambda^{-1/2}\|u\|_{X_{\tau = -\lambda \xi^2}^{s_1/2 + \theta}} \|v\|_{\tilde{F}_{\tau = \lambda \xi^2}^{-3/4}}. \quad (5.4)$$

(b) If $-1/16 < s_1 \leq 0$ and $u, w \in X_{\tau = -\lambda \xi^2}^{s_1/2 + \theta}$ then

$$\|\psi(t)\partial_x(u\bar{w})\|_{X_{\tau = \lambda \xi^2}^{-3/4, -1/2 + 2\theta}} \lesssim \lambda^{-5/8}\|u\|_{X_{\tau = -\lambda \xi^2}^{s_1/2 + \theta}} \|w\|_{X_{\tau = -\lambda \xi^2}^{s_1/2 + \theta}}. \quad (5.5)$$

Proof. (a) is implied in the proof of Lemma 3.4(a) by taking $s_2 = -3/4$. It remains to prove (b). We need to prove

$$\|\partial_x(u \bar{w})\|_{X_{\tau = \lambda \xi^2}^{-3/4, -1/2 + 2\theta}} \lesssim \lambda^{-5/8}\|u\|_{\tilde{F}_{\tau = -\lambda \xi^2}^{s_1/2 + \theta}} \|w\|_{\tilde{F}_{\tau = -\lambda \xi^2}^{s_1/2 + \theta}}. \quad (5.6)$$

By the definition and $\theta \ll 1$, we have

$$\|\partial_x(u \bar{w})\|_{X_{\tau = \lambda \xi^2}^{-3/4, -1/2 + 2\theta}} \lesssim \left(\sum_{k \geq 0} 2^{2k} \left(\sum_{j=0} 2^{-j/2 + 2\theta} j \|1_{B_{k,j}}(\xi, \tau)\|_{L^2} \right)^2 \right)^{1/2} \|u\|_{\tilde{F}_{\tau = -\lambda \xi^2}^{s_1/2 + \theta}} \|w\|_{\tilde{F}_{\tau = -\lambda \xi^2}^{s_1/2 + \theta}}. \quad (5.7)$$

Then from the argument in Lemma 3.4, $I$ and $II$ in (5.7) can be bounded as follows

$$I \lesssim 2^{k_3/4} \sum_{(k_1, k_2, k_3) \in K} \sum_{j_3 \geq 0} 2^{-j_3/2 + 2\theta j_3} \sum_{j_1, j_2 \geq 0} \|1_{B_{k_3, j_3}} f_{k_1, j_1} * g_{k_2, j_2}\|_{L^2} \quad (5.8)$$

and

$$II \lesssim 2^{k_3} \sum_{(k_1, k_2, k_3) \in K} \sum_{j_3 \geq 0} 2^{-j_3/2 + 2\theta j_3} \sum_{j_1, j_2 \geq 0} \|1_{B_{k_3, j_3}} f_{k_1, j_1} * g_{k_2, j_2}\|_{L^2} \quad (5.9)$$

Now we notice that when $k_3 = k_{\text{max}}$, then $k_3 \geq 1$ so $II = 0$, and for this case, $s_2 = -3/4$ and there is a $2^{-k_{\text{max}}/4}$ to spare.

Now we assume that $k_3 = k_{\text{min}}$. If $j_3 \neq j_{\text{max}}$ and $2^{2k_3} \ll \lambda 2^{k_{\text{max}}}$, then by (5.3) we can improve (3.20) to
\[
\lambda^{-1/2} \sum_{(k_1, k_2, k_3) \in K} 2^{-k_{\max}/2} \left\| P_{k_1} u \right\|_{Y_{k_1}} \left\| P_{k_2} W \right\|_{Y_{k_2}},
\]
which is sufficient. If \(2^{k_3} \gtrsim \lambda^{2k_{\max}}\), then we have the relation \(2^{-k_3} \lesssim \lambda^{-1/2}2^{-k_{\max}/2}\), which is sufficient since (3.20), also see (5.11) below.

Now we turn to the case \(k_3 = k_{\min}\) and \(j_3 = j_{\max}\), which is the worst case since low frequency has large modulation. We need to exploit the property of \(\Omega_2\).

\[
\Omega_2(\xi_1, \xi_2) = -\lambda \xi_1^2 + \lambda \xi_2^2 - (\xi_1 + \xi_2)^3 = - (\xi_1 + \xi_2) \left[ \lambda (\xi_1 - \xi_2) + (\xi_1 + \xi_2)^2 \right].
\]  

(5.10)

where we have \(|\xi_1| \sim 2^{k_1}, |\xi_2| \sim 2^{k_2}\) and \(|\xi_1 + \xi_2| \sim 2^{k_3}\), since \(k_3 = k_{\min}\) and \(2^{k_1}, 2^{k_2} \sim 2^{k_{\max}}\), thus \(|\xi_1 - \xi_2| \sim 2^{k_{\max}}\). First we assume that \(2^{k_3} \gtrsim \lambda^{2k_{\max}}\), then in view of (3.20) and (5.8), we have for \(0 < \theta \ll 1\)

\[
I \gtrsim \lambda^{-1/2} \left\| \sum_{|k_1 - k_2| \leq 5, k_3 \leq k_2} 2^{-k_1/4} \left\| P_{k_1} u \right\|_{Y_{k_1}} \left\| P_{k_2} W \right\|_{Y_{k_2}} \right\|^2_{k_3 \geq 1}
\]

\[
\lesssim \lambda^{-5/8} \left\| \sum_{|k_1 - k_2| \leq 5, k_3 \leq k_2} 2^{-k_2/2} \left\| P_{k_1} u \right\|_{Y_{k_1}} \left\| P_{k_2} W \right\|_{Y_{k_2}} \right\|^2_{k_3 \geq 1}
\]

\[
\lesssim \lambda^{-5/8} \sum_{|k_1 - k_2| \leq 5} 2^{-k_2/8} \left\| P_{k_1} u \right\|_{Y_{k_1}} \left\| P_{k_2} W \right\|_{Y_{k_2}},
\]  

(5.11)

which is sufficient since there is a \(2^{-k_{\max}/8 + \theta_{k_{\max}}}\) to spare. In view of (2.11), we have

\[
I \lesssim \left\| 2^{k_2/2} \sum_{|k_1 - k_2| \leq 5} \sum_{j_1, j_2 \geq 0} 2^{j_1/2} \left\| f_{k_1, j_1} \right\|_{L^2} \left\| g_{k_1, j_1} \right\|_{L^2} \right\|^2_{k_3 \leq 0}
\]

\[
\lesssim \lambda^{-1/2} \left\| 2^{k_2/2} \sum_{|k_1 - k_2| \leq 5} 2^{-k_1/2} \left\| P_{k_1} u \right\|_{Y_{k_1}} \left\| P_{k_2} W \right\|_{Y_{k_2}} \right\|^2_{k_3 \leq 0}
\]

\[
\lesssim \lambda^{-1/2} \left\| 2^{-k_1/2} \left\| P_{k_1} u \right\|_{Y_{k_1}} \left\| P_{k_2} W \right\|_{Y_{k_2}} \right\|^2_{k_3 \leq 0}
\]

where we use \(2^{-k_1} \lesssim \lambda^{-1/2}2^{-k_1/2}\) in the second step, which is sufficient.

Now we assume that \(2^{k_3} \ll \lambda^{2k_{\max}}\), then in view of (5.10) we have

\[
2^{j_3} \sim |\Omega_2| \gtrsim \lambda^{2k_3}2^{k_{\max}}.
\]  

(5.12)

We now consider \(I\), using (2.11), (5.8) and (5.12), we have

\[
I \lesssim \left\| 2^{k_3/4} \sum_{|k_1 - k_2| \leq 5} \sum_{j_3 \geq 0} 2^{-j_3/2 + 2\theta j_3} \sum_{j_1, j_2 \geq 0} 2^{j_1/2} \left\| f_{k_1, j_1} \right\|_{L^2} \left\| g_{k_1, j_1} \right\|_{L^2} \right\|^2_{k_3 \geq 1}
\]

\[
\lesssim \lambda^{-1/8} \left\| 2^{k_3/8 + 2\theta k_3} \sum_{|k_1 - k_2| \leq 5} 2^{-k_2/8 + 2\theta k_2} \left\| P_{k_1} u \right\|_{Y_{k_1}} \left\| P_{k_2} W \right\|_{Y_{k_2}} \right\|^2_{k_3 \geq 1}
\]

\[
\lesssim \lambda^{-1/8} \sum_{|k_1 - k_2| \leq 5} 2^{-k_2/8 + 2\theta k_2} \left\| P_{k_1} u \right\|_{Y_{k_1}} \left\| P_{k_2} W \right\|_{Y_{k_2}},
\]
which is also sufficient. Thus we finish the proof. □

Remark 5.5. From the proof we see that the condition $s > -1/16$ is actually from the worst case where low frequency wave has large modulation. It seems difficult to go below $-1/16$ by this method.

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