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Note

Weak acyclic coloring and asymmetric coloring games

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Abstract

We introduce the notion of *weak acyclic coloring* of a graph. This is a relaxation of the usual notion of acyclic coloring which is often sufficient for applications. We then use this concept to analyze the (a, b)-*coloring game*. This game is played on a finite graph G, using a set of colors X, by two players Alice and Bob with Alice playing first. On each turn Alice (Bob) chooses a (b) uncolored vertices and properly colors them with colors from X. Alice wins if the players eventually create a proper coloring of G; otherwise Bob wins when one of the players has no legal move. The (a, b)-*game chromatic number* of G, denoted (a, b)-$\chi_g(G)$, is the least integer t such that Alice has a winning strategy when the game is played on G using t colors. We show that if the weak acyclic chromatic number of G is at most k then $(2, 1)$-$\chi_g(G) \leqslant \frac{1}{2}(k^2 + 3k)$.

Keywords: Game chromatic number; Acyclic chromatic number

1. Introduction

The main purpose of this article is to bound the game chromatic number of graphs in terms of their acyclic chromatic number. In Section 3 we give a new, very short proof of a result of Dinski and Zhu [4]. Then we extend our technique to prove our main result (Theorem 5). Our technique does not require the full strength of acyclic coloring. Unlike ordinary coloring, it is not possible to check that a coloring is acyclic by examining local information at every vertex. In Section 2 we introduce a relaxation of acyclic coloring, *weak acyclic coloring*, which is defined in terms of a local condition and is sufficient for our applications. In addition to clarifying our proofs, this approach yields a somewhat stronger bound in terms of a property that can be more easily verified. Indeed the acyclic chromatic number of a graph with weak acyclic chromatic number k is at least k, can be almost twice k, but is $O(k^2)$. This technique was also used in [1] to bound the star chromatic number of a graph.

Our notation is standard except for the following possible exceptions. For a positive integer k, the set $\{1, \ldots, k\}$ is denoted by $[k]$. The set of t-subsets of a set S is denoted by $\binom{S}{t}$. A directed edge from x to y is denoted by $x \to y$.

2. Weak acyclic chromatic number

An *acyclic coloring* of a graph G is a coloring of the vertices of G such that each color class is independent and any two color classes induce an acyclic subgraph. The *acyclic chromatic number* $\chi_a(G)$ of G is the least integer t such that G

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has an acyclic coloring using t colors. The acyclic chromatic number was introduced by Grünbaum [6], who conjectured that the acyclic chromatic number of planar graphs is at most 5. This was proved by Borodin [3]. This parameter has had several useful applications. For example, Raspaud and Sopena [13] proved that if $\vec{G}$ is an oriented graph whose underlying graph G has acyclic chromatic number k, then the oriented chromatic number of $\vec{G}$ is at most $k2^{k-1}$. Thus, the oriented chromatic number of an oriented planar graph is at most 80. This is the best upper bound known. Dinski and Zhu [4] proved that the game chromatic number of a graph with acyclic chromatic number k is at most k^2+k. At the time this provided the best known upper bound (30) on the game chromatic number of planar graphs. Since then Zhu [16] has lowered this bound to 17, but the original result is still very interesting. Most general bounds on game chromatic number are obtained by bounding another parameter, the game coloring number; however, it is not hard to construct families of graphs, see Kierstead and Trotter [10], with bounded acyclic chromatic number and unbounded game coloring number.

Suppose that c is an acyclic coloring of a graph G. Then each pair of distinct color classes $\{X_i, X_j\}$ induces a forest $F_{ij} = G[X_i \cup X_j]$. Let $\vec{F}_{ij}$ be an orientation of F_{ij} such that the outdegree of any vertex is at most one. Then $\vec{G} = \bigcup \vec{F}_{ij}$ is an orientation of G such that every pair of color classes induces a graph with maximum outdegree one. For some applications, this is all that is needed from the definition of acyclic chromatic number. With this motivation we define a coloring c of G to be a *weak acyclic coloring* if each color class is independent and G has an orientation $\vec{G}$ such that each pair of color classes induces a graph with maximum outdegree one. We say that the orientation $\vec{G}$ *witnesses* that c is a weak acyclic coloring. The *weak acyclic chromatic number* $\chi_{\mathrm{wa}}(G)$ of G is the least integer t such that G has a weak acyclic coloring with t colors. Note that if $c : V \to [t]$ is a weak acyclic coloring of G and $\vec{G}$ witnesses this, then all the outneighbors of any vertex have different colors, and so $\Delta^+(\vec{G}) \leqslant t-1$. Given a coloring f and an orientation $\vec{G}$ it only requires local information to determine whether $\vec{G}$ witnesses that f is a weak acyclic coloring: each vertex checks that it has a different color than its neighbors and that its outneighbors have distinct colors. Clearly $\chi_{\mathrm{wa}}(G) \leqslant \chi_{\mathrm{a}}(G)$. If G is an even cycle then the inequality is strict. Here are some more interesting examples. A *double i-wheel* is an i-cycle together with two independent *hubs* and all possible edges between the hubs and the cycle.

Example 1. Let D_i be the double i-wheel with i an odd number greater than 3. Then $\chi_{\mathrm{wa}}(D_i) = 5 = \chi_{\mathrm{a}}(D_i)$, except that $\chi_{\mathrm{wa}}(D_5) = 4$.

Proof. To obtain a 5-acyclic coloring of D_i color the cycle with three colors and use two more colors for the two hubs. Moreover, at least three colors are required for the cycle, these colors cannot be used on the hubs, and unless all the vertices on the cycle receive distinct colors, the hubs must receive distinct colors. So $\chi_{\mathrm{wa}}(D_i) \leqslant \chi_{\mathrm{a}}(D_i) = 5$. If $i = 5$ then one can obtain a weak acyclic coloring with four colors by using three colors on the cycle and the same new color on both hubs: orient the edges between the cycle and the hubs so that each hub points to three cycle-vertices with distinct colors and so that every cycle vertex is pointed to by some hub. When $i > 5$ this improvement is not possible: if we 3-color the cycle then the two hubs can only dominate six vertices; so some cycle vertex must point to both hubs, causing the hubs to receive different colors in any weak acyclic coloring. □

Example 2. Let K_2^t be the complete t-partite graph with two vertices in each part. Then $\chi_{\mathrm{wa}}(K_2^t) = t$, but $\chi_{\mathrm{a}}(K_2^t) = 2t-1$.

Proof. Any proper coloring of K_2^t must assign distinct colors to any two vertices in distinct parts. So $\chi_{\mathrm{wa}}(K_2^t) \geqslant t$. If we use one color for each class we obtain a weak acyclic coloring of K_2^t, since any pair of parts induces a 4-cycle. So $\chi_{\mathrm{wa}}(K_2^t) \leqslant t$. Any acyclic coloring of K_2^t must use two colors on all but at most one part. On the other hand, if we color two vertices from one part with the same color and use distinct colors for the remaining $2t-2$ vertices we obtain an acyclic coloring of K_2^t. So $\chi_{\mathrm{a}}(K_2^t) = 2t-1$. □

We do not know whether there exists a sequence of graphs G_i such that

$$\lim_{i\to\infty} \frac{\chi_{\mathrm{a}}(G_i)}{\chi_{\mathrm{wa}}(G_i)} = \infty.$$

However, it is easy to show that $\chi_{\mathrm{a}}(G)$ is $\mathrm{O}(\chi_{\mathrm{wa}}^2(G))$.

Proposition 3. *Let $G=(V,E)$ be a graph with $\chi_{\mathrm{wa}}(G)=k$. Then*

$$\chi_{\mathrm{a}}(G) \leqslant (k-1)(2k-1)+1 = 2k^2-3k+2.$$

Proof. Let c be a weak acyclic k-coloring of G and let $\vec{G}$ be an orientation of G that witnesses this. For $i \in [k]$, let X_i be the set of vertices colored with i. Define a digraph $\vec{H}_i=(X_i,E_i)$ by

$$\vec{E}_i = \{x \to z : x \to y \text{ and } y \to z \text{ for some } y \in V\}.$$

For each i, the outdegree of $\vec{H}_i$ is at most $k-1$: any $x \in X_i$ has at most $k-1$ outneighbors in $\vec{G}$ and each outneighbor y has at most one outneighbor $z \in X_i$ in $\vec{G}$. It follows that each $\vec{H}_i$ has a proper $2k-1$ coloring c_i. Define a coloring $f : V \to [k-1] \times [2k-1] \cup \{k\}$ by

$$f(x) = \begin{cases} (i, c_i(x)) & \text{if } c(x)=i<k, \\ k & \text{if } c(x)=k. \end{cases}$$

We claim that f is an acyclic coloring of G. Clearly, f is a proper coloring. Suppose that f bicolors some cycle C. Then C is also bicolored by c. So for some $i<k$ every second vertex of C is in X_i. Since c is a weak acyclic coloring, C must be a directed cycle. It follows that C contains three consecutive vertices $x \to y \to z$ with $x, z \in X_i$. So $c(y) \neq c(x) = c(z)$ and, since $x \to z$ in $\vec{H}_i$, it follows that $c_i(x) \neq c_i(z)$. This contradicts the assumption that f bicolors C. $\square$

We note that a very similar argument is used in [1] to prove a similar bound on the star chromatic number in terms of the acyclic chromatic number. That result also holds if acyclic chromatic number is replaced by weak acyclic chromatic number.

3. Game chromatic number

The *coloring game* is played on a finite graph G, using a set of colors X, by two players Alice and Bob with Alice playing first. The players take turns coloring the vertices of G with colors from X so that no two adjacent vertices have the same color. Bob wins if at some time one of the players has no legal move; otherwise Alice wins when the players eventually create a proper coloring of G. The *game chromatic number* of G, denoted $\chi_g(G)$, is the least integer t such that Alice has a winning strategy when the game is played on G using t colors. The game chromatic number was first introduced by Bodlaender [2]. Faigle et al. [5] proved that the game chromatic number of a forest is at most 4. This is best possible as was shown by Bodlaender. Since then many authors have considered game chromatic number and related parameters, including, Dinski and Zhu [4], Kierstead [7,8], Kierstead and Trotter [9,10], Kierstead and Tuza [11], Nešeitřil and Sopena [12], and Zhu [14,15].

Before discussing a more general version of the coloring game, we give a short proof of the Dinski–Zhu theorem which yields a stronger statement in terms of weak acyclic chromatic number.

Theorem 4. *Let $G=(V,E)$ be a graph with $\chi_{\mathrm{wa}}(G)=k$. Then $\chi_g(G) \leqslant k^2+k$.*

Proof. Fix a weak acyclic coloring $c : V \to [k]$ of G and an orientation $\vec{G}$ that witnesses that c is a weak acyclic coloring. Alice and Bob will play the coloring game with the set

$$X = [k] \times [k+1]$$

of k^2+k colors. At a given stage in the game, let C be the set of colored vertices and $g : C \to X$ be the partial coloring that the players have constructed so far. Call a colored vertex v *well colored* if $g(v)$ has the form $(c(v), j)$. Call a vertex v *dangerous* if v is uncolored and v has a colored inneighbor u such that $g(u)$ has the form $(c(v), j)$. We say that u *witnesses* that v is dangerous. Since c is a weak acyclic coloring, a colored vertex can witness that at most one uncolored vertex is dangerous and a well-colored vertex cannot witness that any vertex is dangerous.

It suffices to show that Alice can play so that at the end of each of her plays there are no dangerous vertices. She starts by well coloring any vertex. Now suppose that there were no dangerous vertices at the end of Alice's last play and that Bob has just played by coloring a vertex u. If there are now any dangerous vertices, this must be witnessed by u. So there can be at most one. Alice chooses an uncolored vertex v so that if there is a dangerous vertex, then v is dangerous. So v has at most one inneighbor, u, that has been colored with a color of the form $(c(v), j)$. Since v has at most $k-1$ outneighbors, Alice can legally and well color v with one of the $k+1$ colors in the set $\{(c(v), j) : j \in [k+1]\}$. This colors the only possibly dangerous vertex without creating any new dangerous vertices. So the invariant is maintained. □

In [8] the author introduced the following variant of the coloring game in which Alice and Bob are allowed to make several moves in a row. The (a, b)-*coloring game* is played like the coloring game with the exception that on each turn Alice colors a vertices and Bob colors b vertices. (If there are no uncolored vertices left the players are not required to complete their turns.) Thus, the (1, 1)-coloring game is just the coloring game. The (a, b)-*game chromatic number* of G, denoted (a, b)-$\chi_g(G)$, is the least integer t such that Alice has a winning strategy when the (a, b)-*coloring* game is played on G using t colors. In [8] the author determined exact bounds for the (a, b)-chromatic number on the class of forests for all values of a and b.

Next we show that with a little extra care, the ideas behind the proof of Theorem 4 can be used to show that Alice can do almost twice as well in the (2, 1)-coloring game.

Theorem 5. *Let $G = (V, E)$ be a graph with $\chi_{wa}(G) = k$. Then* $(2, 1)$-$\chi_g(G) \leqslant \frac{1}{2}(k^2 + 3k)$.

Proof. Fix a weak acyclic coloring $c : V \to [k]$ of G and an orientation $\vec{G} = (V, \vec{E})$ that witnesses that c is a weak acyclic coloring. Alice and Bob will play the game with the set

$$X = \binom{[k]}{2} \cup \{\{i, 0\}, \{i, k+1\} : i \in [k]\}$$

of $\frac{1}{2}(k^2 + 3k)$ colors. At a given stage in the game, let C be the set of colored vertices and $g : C \to X$ be the partial coloring that the players have constructed so far. Call a colored vertex v *well colored* if $c(v) \in g(v)$. Call a vertex v *dangerous* if v is uncolored and v has a colored inneighbor u such that $c(v) \in g(u)$. We say that u *witnesses* that v is dangerous. Since c is a weak acyclic coloring and $|g(u)| = 2$, a colored vertex can witness that at most two uncolored vertices are dangerous and a well-colored vertex can witness that at most one vertex is dangerous. Moreover, if a vertex is well colored with a color of the form $\{i, 0\}$ or $\{i, k+1\}$ then it cannot witness that any vertex is dangerous.

It suffices to show that Alice can play so that at the end of each of her plays there are no dangerous vertices. She starts by well coloring any two vertices v and w with the colors $\{c(v), 0\}$ and $\{c(w), 0\}$. Now suppose that there were no dangerous vertices at the end of Alice's last play and that Bob has just played by coloring a vertex u. If there are now any dangerous vertices, this must be witnessed by u. So there can be at most two. Alice chooses two uncolored vertices v and w so that any dangerous vertices are in the set $D = \{v, w\}$. So each $x \in D$ has at most one inneighbor u colored with a set of the form $g(u) = \{c(x), h\}$. Alice will properly and well color each $x \in D$ in such a way that no new dangerous vertices are created. So Alice must color x with a color $T = \{c(x), i\}$, where i is one of the $k+1$ colors in $\{0, 1, \ldots, k+1\} - \{c(x)\}$. For the choice T to be proper, it suffices to insure that

$$i \notin \{h\} \cup \{j : y \in N^+(x) \cap C \text{ and } g(y) = \{c(x), j\}\}.$$

To insure that no new dangerous vertices are created, it suffices to insure that

$$i \notin \{c(y) : y \in N^+(x) \cap U\}.$$

Since

$$|\{h\}| + |N^+(x) \cap C| + |N^+(x) \cap U| = |N^+(x)| + 1 \leqslant k,$$

these additional constraints rule out at most k of the $k+1$ possible choices for i. So Alice can color the only possibly dangerous vertices without creating any new dangerous vertices. Thus, the invariant is maintained. □

4. Open questions

Here are some interesting problems left unresolved by this article. For a graph parameter f and a graph class $\mathscr{C}$, let $f(\mathscr{C}) = \max_{G\in\mathscr{C}} f(G)$. Let W_k be the class of graphs G with $\chi_{\mathrm{wa}}(G) \leqslant k$.

1. Improve the bounds $2k - 1 \leqslant \chi_{\mathrm{a}}(W_k) \leqslant 2k^2 - 3k + 2$.
2. Improve the bounds $2k - 1 \leqslant \chi_g(W_k) \leqslant k^2 + k$.
3. Improve the bounds $(3k - 1)/2 \leqslant (2, 1)\text{-}\chi_g(W_k) \leqslant \frac{1}{2}(k^2 + 3k)$.
4. Find general bounds on $(t, 1)\text{-}\chi_g(W_k)$.
5. Let $\chi_{\mathrm{o}}(G)$ be the oriented chromatic number of G . Is $\chi_{\mathrm{o}}(W_k) \leqslant k2^k$?
6. The list acyclic chromatic number of the class P of planar graphs is known to be at most
7. Is the list weak acyclic chromatic number of P less than 7?

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