Linear Algebra and its Applications 436 (2012) 742–746

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Jacobson's lemma for the generalized Drazin inverse

Guifen Zhuang [a,b], Jianlong Chen [a,*], Jian Cui [a]

[a] Department of Mathematics, Southeast University, Nanjing 210096, China
[b] School of Science, Jiangnan University, Wuxi 214122, China

ARTICLE INFO

Article history:
Received 15 April 2011
Accepted 25 July 2011
Available online 27 August 2011

Submitted by X. Zhan

AMS classification:
15A09
16S10

Keywords:
Generalized Drazin inverse
Drazin inverse
Drazin index
Regular element
Spectral idempotent

ABSTRACT

We study properties of elements in a ring which admit the generalized Drazin inverse. It is shown that the element $1-ab$ is generalized Drazin invertible if and only if so is $1-ba$, and a formula for the generalized Drazin inverse of $1-ba$ in terms of the generalized Drazin inverse and the spectral idempotent of $1-ab$ is provided. Further, recent results relating to the Drazin index can be recovered from our theorems.

1. Introduction

Throughout this paper R will be a ring with unity 1. The notation R^{inv} means the group of all invertible elements of R. Following Koliha and Patricio [15], the *c*ommutant and *d*ouble commutant of an element $a \in R$ are defined by

$$\text{comm}(a) = \{x \in R : ax = xa\};$$

$$\text{comm}^2(a) = \{x \in R : xy = yx \text{ for all } y \in \text{comm}(a)\},$$

respectively. Let $R^{\text{qnil}} = \{a : 1 + ax \in R^{\text{inv}} \text{ for every } x \in \text{comm}(a)\}$, and if $a \in R^{\text{qnil}}$ then a is said to be *quasinilpotent* [13]. Let R^{nil} be the set of all nilpotents of R. Clearly, $R^{\text{nil}} \subseteq R^{\text{qnil}}$.

Recall that an element $a \in R$ is (*von Neumann*) *regular* provided there is an element $b \in R$ such that $a = aba$. In this case b is called an *inner inverse* of a, and denoted by a^-. An element $a \in R$ is said to

* Corresponding author.
E-mail addresses: z377315175@126.com (G. Zhuang), jlchen@seu.edu.cn (J. Chen), seujcui@126.com (J. Cui).

0024-3795/$ - see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2011.07.044

have a *Drazin inverse* [12] if there exists $b \in R$ such that

$$bab = b, \;\; b \in \text{comm}(a), \;\; a - a^2 b \in R^{\text{nil}}.$$

The element b above is unique if it exists and is denoted by a^{D}, and the nilpotency index of $a - a^2 b$ is called the *Drazin index* of a, denoted by $\text{ind}(a)$ [12]. Moreover, $a^{\text{D}} \in \text{comm}^2(a)$ (see [15]). If $\text{ind}(a) = 1$, then a is *group invertible* and the group inverse of a is denoted by $a^{\#}$.

The concept of the generalized Drazin inverse in a Banach algebra was introduced by Koliha [14]. Later, this notion was extended to a ring by Koliha and Patricio [15]. An element $a \in R$ is *generalized Drazin invertible* [15] in case there is an element $b \in R$ satisfying

$$bab = b, \;\; b \in \text{comm}^2(a), \;\; a - a^2 b \in R^{\text{qnil}}.$$

Such b, if it exists, is unique; it is called a *generalized Drazin inverse* of a, and will be denoted by a^{gd}. In a Banach algebra, the condition $b \in \text{comm}^2(a)$ in the above definition can be weakened as $b \in \text{comm}(a)$. Results on generalized Drazin inverses (in rings and Banach algebras) can also be found in [2,4–6,9–11]. We will denote by $R^{\text{D}}, \; R^{\#}$ and R^{gd} the set of all Drazin invertible elements, group invertible elements and generalized Drazin invertible elements in R, respectively.

Jacobson's lemma states that for any $a, \; b \in R$, $1 - ab \in R^{\text{inv}}$ if and only if $1 - ba \in R^{\text{inv}}$ (see [1,7]). In [3,8], the authors generalized Jacobson's lemma to the Drazin invertibility. Motivated by these papers, we consider Jacobson's lemma for the generalized Drazin inverse. It is shown that if $1 - ab$ belongs to R^{gd}, then so does $1 - ba$ and, in addition, an explicit formula relating the generalized Drazin inverse of the two elements is provided.

2. Main results

For the future reference we state two known results.

Lemma 2.1. *Let $a, b \in R$. If $1 - ab \in R^{\text{inv}}$, then $1 - ba \in R^{\text{inv}}$ and $(1 - ba)^{-1} = 1 + b(1 - ab)^{-1}a$.*

Lemma 2.2 [15, Theorem 4.2]**.** *Let $a \in R$. Then $a \in R^{\text{gd}}$ if and only if there exists $p^2 = p \in R$ such that*

$$p \in \text{comm}^2(a), \;\; a + p \in R^{\text{inv}} \;\; \text{and } ap \in R^{\text{qnil}}.$$

In this case, $a^{\text{gd}} = (a + p)^{-1}(1 - p)$ and $p = 1 - aa^{\text{gd}}$, where p is called a spectral idempotent of a and will be denoted by a^{π}.

In [3,8], the authors proved that for any elements $a, \; b \in R$, if $1 - ab \in R^{\text{D}}$ then so is $1 - ba$, which generalized Jacobson's lemma to the Drazin invertibility, and a formula of the Drazin inverse of $1 - ba$ in terms of the Drazin inverse of $1 - ab$ is given in [3]. It is natural to consider whether the same property can be inherited by the generalized Drazin invertibility. However, the double commutativity of elements in rings seems more difficult to deal with than the commutativity, and therefore it is not trivial to pass from the Drazin inverse version of Jacobson's lemma to its generalized counterpart. Now we are in the position to state the main result.

Theorem 2.3. *Let $a, b \in R$. If $\alpha = 1 - ab \in R^{\text{gd}}$, then $\beta = 1 - ba \in R^{\text{gd}}$ with*

$$\beta^{\text{gd}} = 1 + b(1 - \alpha^{\pi}\alpha)^{-1}(\alpha^{\text{gd}} - \alpha^{\pi})a = 1 + b[\alpha^{\text{gd}} - \alpha^{\pi}(1 - \alpha^{\pi}\alpha)^{-1}]a,$$

and

$$\beta^{\pi} = b\alpha^{\pi}(1 - \alpha^{\pi}\alpha)^{-1}a.$$

Proof. Denote $p = \alpha^{\pi}$. Then by Lemma 2.2, $1 - p\alpha \in R^{\text{inv}}$. Let $q = bp(1 - p\alpha)^{-1}a$. Note that $\beta b = b\alpha$ and $a\beta = \alpha a$. Then

$$\beta q = b\alpha p(1 - p\alpha)^{-1}a = bp(1 - p\alpha)^{-1}\alpha a = q\beta.$$

In what follows, by Lemma 2.2, it suffices to show that q is the spectral idempotent of β, i.e., the following conditions hold: (a) $\beta + q \in R^{\text{inv}}$; (b) $\beta q \in R^{\text{qnil}}$; (c) $q^2 = q \in \text{comm}^2(\beta)$.

(a) Write $c = [p(1 - p\alpha)^{-1} - 1]a$. After a calculation in which we substitute $ab = 1 - \alpha$, we obtain

$$(1 - p\alpha)(1 + cb) = \alpha + p - p\alpha - p\alpha^2 = (1 - p\alpha)(\alpha + p),$$

which proves that $1 + cb$ is invertible. By Lemma 2.1, $1 + bc$ is also invertible. Hence

$$\beta + q = 1 - ba + bp(1 - p\alpha)^{-1}a = 1 + bc \in R^{\text{inv}}.$$

(b) We let $z \in R$ with $(\beta q)z = z(\beta q)$. Then

$$[b\alpha p(1 - p\alpha)^{-1}a]z = z[b\alpha p(1 - p\alpha)^{-1}a].$$

Multiplying this equation by a on the left and by b on the right yield

$$ab\alpha p(1 - p\alpha)^{-1}azb = azb\alpha p(1 - p\alpha)^{-1}ab.$$

By substituting $abp = pab = p(1 - p\alpha)$, we can easily obtain

$$(\alpha p)(azb) = (azb)(\alpha p),$$

which implies that $(1 - p\alpha)^{-1}azb \in \text{comm}(\alpha p)$. Since $\alpha p \in R^{\text{qnil}}$, it follows that

$$1 + [(1 - p\alpha)^{-1}azb](\alpha p) = 1 + [(1 - p\alpha)^{-1}az](b\alpha p) \in R^{\text{inv}}.$$

In view of Lemma 2.1, $1 + (\beta q)z = 1 + (b\alpha p)[(1 - p\alpha)^{-1}az] \in R^{\text{inv}}$. Therefore, $\beta q \in R^{\text{qnil}}$.

(c) To this end we first show that $q \in R$ is idempotent. Note that $pab = p(1 - p\alpha)$. Then

$$q^2 = b(1 - p\alpha)^{-1}pab(1 - p\alpha)^{-1}pa = b(1 - p\alpha)^{-1}pa = q.$$

We next show that $q \in \text{comm}^2(\beta)$. Let $y \in R$ be such that $\beta y = y\beta$, i.e., $(ba)y = y(ba)$. Then

$$(ayb)\alpha = a(y\beta)b = a(\beta y)b = \alpha(ayb).$$

So $ayb \in \text{comm}(\alpha)$. From $p \in \text{comm}^2(\alpha)$ it follows that $ayb \in \text{comm}(p)$, which implies $ayb \in \text{comm}(p\alpha)$. It is clear that $ayb \in \text{comm}(1 - p\alpha)$. Then

$$(ayb)p(1 - p\alpha)^{-1} = p(1 - p\alpha)^{-1}(ayb).$$

Multiplying the above equation respectively, by b on the left and by a on the right yield

$$bayq = qyba.$$

Combining this with the equations $bay = yba$ and $\beta q = q\beta$, we have

$$yq(1 - \beta) = (1 - \beta)yq = qy(1 - \beta) = (1 - \beta)qy.$$

It follows that

$$yq(1 - \beta q) = qyq(1 - \beta q) \text{ and } (1 - \beta q)qy = (1 - \beta q)qyq.$$

By (b), $1 - \beta q \in R^{\text{inv}}$, one obtains

$$yq = qyq = qy,$$

which implies that $q \in \text{comm}^2(\beta)$.

Therefore, q is the spectral idempotent of β and $\beta^{\text{gd}} = (\beta + q)^{-1}(1 - q)$.

By virtue of Lemma 2.1, we have

$$(\beta + q)^{-1} = 1 - b(\alpha + p)^{-1}[p(1 - p\alpha)^{-1} - 1]a,$$

where we have used the relations $\beta + q = 1 + bc$ and $\alpha + p = 1 + cb$. Then

$$\begin{aligned}
\beta^{\mathrm{gd}} &= (\beta + q)^{-1}(1 - q) \\
&= [1 - b(\alpha + p)^{-1}(p(1 - p\alpha)^{-1} - 1)a][1 - bp(1 - p\alpha)^{-1}a] \\
&= 1 - bp(1 - p\alpha)^{-1}a + b(\alpha + p)^{-1}(1 - p)(1 - p\alpha)^{-1}a \\
&= 1 + b(1 - \alpha^{\pi}\alpha)^{-1}(\alpha^{\mathrm{gd}} - \alpha^{\pi})a \\
&= 1 + b[\alpha^{\mathrm{gd}} - \alpha^{\pi}(1 - \alpha^{\pi}\alpha)^{-1}]a.
\end{aligned}$$

The proof is complete. □

If the condition $ap \in R^{\mathrm{qnil}}$ is replaced by $ap \in R^{\mathrm{nil}}$, then we can obtain the main result on the Drazin inverse in [3] by applying Theorem 2.3.

Corollary 2.4 [3, Theorem 3.6]. *Let $a, b \in R$. If $1 - ab \in R^{\mathrm{D}}$ with $\mathrm{ind}(1 - ab) = k$, then $1 - ba \in R^{\mathrm{D}}$ with $\mathrm{ind}(1 - ba) = k$ and*

$$(1 - ba)^{\mathrm{D}} = 1 + b[(1 - ab)^{\mathrm{D}} - (1 - ab)^{\pi} r]a,$$

where

$$r = \sum_{i=0}^{k-1}(1 - ab)^i \text{ and } (1 - ab)^{\pi} = 1 - (1 - ab)(1 - ab)^{\mathrm{D}}.$$

Proof. Write $\alpha = 1 - ab$, $\beta = 1 - ba$ and $p = \alpha^{\pi}$. In view of Theorem 2.3, $\beta \in R^{\mathrm{gd}}$ and $\beta^{\pi} = bp(1-p\alpha)^{-1}a$. Let $q = \beta^{\pi}$, then $\beta q = bp\alpha(1-p\alpha)^{-1}a\,(= q\beta)$. By using the relation $abp = (1-p\alpha)p$ we can easily conclude that

$$(\beta q)^n = b(1 - \alpha)^{n-1}(p\alpha)^n(1 - p\alpha)^{-n}a$$

by induction on n. By hypothesis $(p\alpha)^k = 0$, one has $(\beta q)^k = 0$ which implies that $\mathrm{ind}\beta \leqslant k$, i.e.,

$$\mathrm{ind}(1 - ba) \leqslant \mathrm{ind}(1 - ab).$$

By symmetry, $\mathrm{ind}(1 - ab) \leqslant \mathrm{ind}(1 - ba)$. Hence,

$$\mathrm{ind}(1 - ba) = \mathrm{ind}(1 - ab) = k.$$

Since α^{D} exists, $\alpha^{\mathrm{gd}} = \alpha^{\mathrm{D}}$. Moreover,

$$\alpha^{\mathrm{gd}} - \alpha^{\pi}(1 - \alpha^{\pi}\alpha)^{-1} = \alpha^{\mathrm{D}} - p\sum_{i=0}^{k-1}(p\alpha)^i = (1 - ab)^{\mathrm{D}} - (1 - ab)^{\pi} r,$$

where $r = \sum_{i=0}^{k-1}(1 - ab)^i$. Therefore we obtain the required result by applying Theorem 2.3. □

Letting $k = 1$ in Corollary 2.4, we get a result for the group inverse.

Corollary 2.5 [3, Theorem 3.5]. *Let $a, b \in R$. If $1 - ab \in R^{\#}$, then $1 - ba \in R^{\#}$ and*

$$(1 - ba)^{\#} = 1 + b[(1 - ab)^{\#} - (1 - ab)^{\pi}]a,$$

where $(1 - ab)^{\pi} = 1 - (1 - ab)^{\#}(1 - ab)$.

In [3], Castro-González et al. gave a new characterization of the Drazin index to answer a question raised in [16] about the equivalence between the conditions $\mathrm{ind}(a^2a^- + 1 - aa^-) = k$ and $\mathrm{ind}(a + 1 - aa^-) = k$, which implies $a^2a^- + 1 - aa^- \in R^{\mathrm{D}}$ if and only if $a + 1 - aa^- \in R^{\mathrm{D}}$. Here we give a similar result on the generalized Drazin invertibility.

Corollary 2.6. *Let $a \in R$ be a regular element with an inner inverse a^-. If one of the elements $a^2a^- + 1 - aa^-$, $a^-a^2 + 1 - a^-a$, $a + 1 - aa^-$ and $a + 1 - a^-a$ belongs to R^{gd}, then they all do.*

Proof. Let $c = 1 - a$, $d = aa^-$, $e = a$, $f = a^- - aa^-$ and $g = a^-a$. Then we obtain the following equations:

$$a^2a^- + 1 - aa^- = 1 - cd = 1 - ef; \qquad a^-a^2 + 1 - a^-a = 1 - gc;$$

$$a + 1 - a^-a = 1 - fe = 1 - cg; \qquad a + 1 - aa^- = 1 - dc.$$

In view of Theorem 2.3, $1 - xy$ is generalized Drazin invertible if and only if so is $1 - yx$. The proof is complete. □

Combining Corollary 2.4 with the proof of Corollary 2.6, we obtain the result on the Drazin index as in [3].

Corollary 2.7. *Let $a \in R$ be a regular element with an inner inverse a^-. Then the indices of the elements $a^2a^- + 1 - aa^-$, $a^-a^2 + 1 - a^-a$, $a + 1 - aa^-$ and $a + 1 - a^-a$ are equal.*

Acknowledgments

The authors are highly grateful to the referee who has provided many valuable suggestions and simplified the proof of Theorem 2.3. This research is supported by the National Natural Science Foundation of China (10971024), the Specialized Research Fund for the Doctoral Program of Higher Education (200802860024), and the Natural Science Foundation of Jiangsu Province (BK2010393).

References

[1] B.A. Barnes, Common operator properties of the linear operators RS and SR, Proc. Amer. Math. Soc. 126 (1998) 1055–1061.
[2] N. Castro-González, J.J. Koliha, New additive results for the g-Drazin inverse, Proc. Roy. Soc. Edinburgh Sect. A 134 (2004) 1085–1097.
[3] N. Castro-González, C. Mendes-Araújo, P. Patricio, Generalized inverses of a sum in rings, Bull. Aust. Math. Soc. 82 (2010) 156–164.
[4] N. Castro-González, J.Y. Vélez-Cerrada, Elements of rings and Banach algebras with related spectral idempotents, J. Aust. Math. Soc. 80 (2006) 383–396.
[5] D.S. Cvetković-Ilić, The generalized Drazin inverse with commutativity up to a factor in a Banach algebra, Linear Algebra Appl. 431 (2009) 783–791.
[6] D.S. Cvetković-Ilić, D.S. Djordjević, Y.M. Wei, Additive results for the generalized Drazin inverse in a Banach algebra, Linear Algebra Appl. 418 (2006) 53–61.
[7] D.S. Cvetković-Ilić, R.E. Harte, On the algebraic closure in rings, Proc. Amer. Math. Soc. 135 (2007) 3547–3582.
[8] D.S. Cvetković-Ilić, R.E. Harte, On Jacobson's lemma and Drazin invertibility, Appl. Math. Lett. 23 (2010) 417–420.
[9] C.Y. Deng, Y.M. Wei, New additive results for the generalized Drazin inverse, J. Math. Anal. Appl. 370 (2010) 313–321.
[10] D.S. Djordjević, P.S. Stanimirović, On the generalized Drazin inverse and generalized resolvent, Czechoslovak Math. J. 51 (126) (2001) 617–634.
[11] D.S. Djordjević, Y.M. Wei, Additive results for the generalized Drazin inverse, J. Aust. Math. Soc. 73 (2002) 115–125.
[12] M.P. Drazin, Pseudo-inverses in associative rings and semigroup, Amer. Math. Monthly 65 (1958) 506–514.
[13] R.E. Harte, On quasinilpotents in rings, Panam. Math. J. 1 (1991) 10–16.
[14] J.J. Koliha, A generalized Drazin inverse, Glasgow Math. J. 38 (1996) 367–381.
[15] J.J. Koliha, P. Patricio, Elements of rings with equal spectral idempotents, J. Aust. Math. Soc. 72 (2002) 137–152.
[16] P. Patricio, A.V. da Costa, On the Drazin index of regular elements, Cent. Eur. J. Math. 7 (2009) 200–205.