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An Integrodifferential Equation for Rigid Heat Conductors with Memory

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1. INTRODUCTION

The purpose of this paper is to study existence, uniqueness, and continuous dependence on parameters for solutions of the following system of integrodifferential equations:

$$\begin{aligned} e(t, x) &= e_0 + \alpha(0)\,\theta(t, x) + \int_{-\infty}^{t} \alpha'(t-\tau)\,\theta(\tau, x)\,d\tau, \\ q(t, x) &= -k(0)\,\nabla\theta(t, x) - \int_{-\infty}^{t} k'(t-\tau)\,\nabla\theta(\tau, x)\,d\tau, \qquad (1) \\ e'(t, x) &= -\nabla \cdot q(t, x) + r(t, x), \end{aligned}$$

where $0 \leqslant t < \infty$, x is a vector in a real n-dimensional set B, prime denotes differentiation with respect to the time variable t, ∇ is the gradient operator with respect to x, and $\nabla \cdot \nabla = \Delta$ is the Laplacian.

For $k(0) = 0$ these equations represent the linearized theory for heat flow in a rigid, isotropic, homogeneous material as proposed by Gurtin and Pipkin [12]. For $k(0) > 0$ the equations represent an alternate linearized theory proposed by Coleman and Gurtin [1]; see also Gurtin [13]. Nunziato [22, 23], Finn and Wheeler [6], and Nachlinger and Wheeler [21] have studied certain aspects of the general nonlinear theory as well as the problem of uniqueness and wave propagation for the linearized problem. Grabmueller [10] gave a very general uniqueness proof for generalized solutions in a Sobelev space and proved existence theorems in certain special situations. Kremer [16] proved existence and uniqueness theorems for generalized distribution solutions. Grabmueller [11] also studied an inverse problem for (1).

The purpose of this note is to show in Section 2 that the appropriate history value problem of form (1) always has a generalized distribution solution which

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is not only unique but also depends continuously on the initial history and on the function $r(t, x)$. From this continuity it follows that the solutions obtained in [10, Theorem 2.1.2] are the natural Hilbert space approximations of the general distribution solutions. The results in Section 2 are used to motivate the analysis in the rest of the paper. Section 3 is concerned with stability results. Conditions (S) and (E) when $k(0) > 0$ and (S′) when $k(0) = 0$ give necessary conditions on $k(t)$ in order that (1) determine a stable model of the heat problem for all configurations B. Such necessary conditions for $k(t)$ have not previously been noted. In Theorem 6 it is shown that stability of $\theta(t, x)$ can be established by establishing stability for each mode separately. Some general consequences of stability are proved in Corollary 7 and Theorem 8. The results in Sections 4 and 5 show that for large classes of smooth $r(t, x)$ and of smooth initial histories the generalized distribution solutions of (1) will actually be classical solutions. The transformation developed in Section 5 can be used to discuss the hyperbolic character of (1) when $k(0) = 0$ in a more satisfactory way than the earlier analysis in [2, 3, 20].

We shall need some standard background material on existence, uniqueness and solution forms for Volterra integral equations on the real line. This material can be found collected in [17]. Background material for semigroups, Sobelev spaces, and functional analysis can be found in Krein [15], Pazy [24], or especially in Friedman [8].

2. Results for Distribution Solutions

The purpose of this section is to transform system (1) into an equivalent Volterra integrodifferential equation in Hilbert space. Background material and assumptions are given as needed. An existence, uniqueness, and continuity theorem is stated in this abstract setting. This continuity result is used to prove an approximation result, Theorem 2, which is the main result of this section.

Equations (1) are linearized about some nominal constant temperature θ_0 which will be taken to be zero. The rigid body will occupy a fixed open region B in n-dimensional space R^n (normally $n = 1, 2$, or 3) and have boundary ∂B. The *energy–temperature relation function* $\alpha(t)$ and the *heat conduction relation* $k(t)$ are both assumed continuous, to have a certain number of continuous derivatives, and to satisfy the following assumptions:

(H1) $\alpha(0) > 0$ and either $k(0) > 0$, $J = 2$ or $k(0) = 0$, $k'(0) > 0$, $J = 3$.

(H2) $\alpha \in C^{J+1}[0, \infty)$ and $k \in C^J[0, \infty)$.

(H3) $\alpha^{(j)} \in L^1(0, \infty)$ for $1 \leqslant j \leqslant J$ and $k^{(j)} \in L^1(0, \infty)$ for $1 \leqslant j \leqslant J - 1$.

Assumption (H3) is stronger than needed for many of the results in the sequel. However, the full force of this assumption is used in returning from (3) or (4) below to system (1) above.

The appropriate *history value problem* for (1), which we will consider here, is to give a function $\theta(t, x)$ on $\{-\infty < t \leqslant 0,\ x \in B\}$ such that $\nabla\theta$ and $\Delta\theta$ make sense and the following integrals are defined:

$$\begin{aligned} g(t, x) &= \int_{-\infty}^{0} \alpha'(t-\tau)\,\theta(\tau, x)\,d\tau, \\ h(t, x) &= \int_{-\infty}^{0} k'(t-\tau)\,\nabla\theta(\tau, x)\,d\tau \\ \frac{\partial g}{\partial t}(t, x) &= \int_{-\infty}^{0} \alpha''(t-\tau)\,\theta(\tau, x)\,d\tau, \\ \nabla \cdot h(t, x) &= \int_{-\infty}^{0} k'(t-\tau)\,\Delta\theta(\tau, x)\,d\tau. \end{aligned} \tag{2}$$

From (1) and (2) it follows that

$$e(t) = e_0 + \alpha(0)\,\theta(t) + \int_0^t \alpha'(t-\tau)\,\theta(\tau)\,d\tau + g(t),$$

$$\nabla \cdot q(t) = -k(0)\,\Delta\theta(t) - \int_0^t k'(t-\tau)\,\Delta\theta(\tau)\,d\tau + \nabla \cdot h(t),$$

and so

$$\begin{aligned} \theta'(t) = &-\frac{\alpha'(0)}{\alpha(0)}\,\theta(t) + \frac{k(0)}{\alpha(0)}\,\Delta\theta(t) + \int_0^t \left\{\frac{k'(t-\tau)}{\alpha(0)}\,\Delta\theta(\tau) - \frac{\alpha''(t-\tau)}{\alpha(0)}\,\theta(\tau)\right\} d\tau \\ &+ \{r(t) + \nabla \cdot h(t) - g'(t)\}/\alpha(0). \end{aligned} \tag{3}$$

Here the dependence on x has been supressed. We have proved the following result.

LEMMA 1. *Let $k(0) > 0$. If an initial history θ is given on $-\infty < t \leqslant 0$ such that functions* (2) *make sense, then the temperature function $\theta(t)$ must satisfy the problem*

$$\theta'(t) = f(t) + C\Delta\theta(t) - a(0)\,\theta(t) + \int_0^t [Cb(t-\tau)\,\Delta\theta(\tau) - a'(-t\,\tau)\,\theta(\tau)]\,d\tau \tag{3'}$$

for all $t \geqslant 0$ with $\theta(0) = \theta_0$ given (from the initial history). Here

$$C = k(0)/\alpha(0), \qquad a(t) = \alpha'(t)/\alpha(0), \qquad b(t) = k'(t)/k(0),$$

and

$$f(t) = [r(t) + \nabla \cdot h(t) - g'(t)]/\alpha(0).$$

The reader should note that the *initial history problem* (1) has been reformulated as an *initial value problem* (3′). The *initial history* $\theta(t)$, $-\infty \leqslant t \leqslant 0$ and the given function $r(t)$ have been used to determine *initial conditions* θ_0 and $f(t)$ in the manner specified above. If $r(t)$ were zero, it would be simplier to retain the initial history formulation. Since we wish to allow $r(t)$ not zero, we think that (3′) will often be a more convenient form.

If $k(0) = 0$, then one can take another derivative in (3) to obtain the equation

$$\begin{aligned}\alpha(0)\,\theta''(t, x)&\\ &= -\alpha'(0)\,\theta'(t) - \alpha''(0)\,\theta(t) + k'(0)\,\Delta\theta(t) \qquad (4)\\ &\quad + \int_0^t [k''(t-\tau)\,\Delta\theta(\tau) - \alpha'''(t-\tau)\,\theta(\tau)]\,d\tau + [r'(t) + \nabla\cdot h'(t) - g''(t)].\end{aligned}$$

The analog of Lemma 1 is the following result.

LEMMA 2. *If $k(0) = 0$ and if the initial history θ is given on $-\infty < t \leqslant 0$ in such a manner that the functions $g, g', g'', h, \nabla\cdot h$, and $\nabla\cdot h'$ make sense, then $\theta(t)$ must satisfy*

$$\begin{aligned}\theta''(t) &= f(t) + C_1\Delta\theta(t) - a(0)\,\theta'(t) - a'(0)\,\theta(t)\\ &\quad + \int_0^t [C_1 b_1(t-\tau)\,\Delta\theta(\tau) - a''(t-\tau)\,\theta(\tau)]\,d\tau\end{aligned} \qquad (4')$$

for all $t \geqslant 0$ with $\theta(0) = \theta_0$ and $\theta'(0) = v_0$ given. Here

$$C_1 = k'(0)/\alpha(0), \qquad b_1(t) = k''(t)/k'(0), \qquad a(t) = \alpha'(t)/\alpha(0),$$

and

$$f(t) = [r'(t) + \nabla\cdot h'(t) - g''(t)]/\alpha(0).$$

As before, the *initial history problem* has been replaced by an *initial value problem.* This time the appropriate *initial conditions* are the quantities θ_0, v_0, and $f(t)$.

We shall study a class of distribution solutions for (3) or for (4) under boundary conditions of the form

$$u(t, x) = 0 \qquad (0 \leqslant t < \infty, x \in \partial B) \qquad (5)$$

or of the form

$$b_1(x)\frac{\partial u}{\partial \nu}(t, x) + b_0(x)\,u(t, x) = 0 \qquad (0 \leqslant t < \infty, x \in \partial B), \qquad (6)$$

where ν is the outward normal on ∂B, $b_0(x)$ and $b_1(x)$ are continuously differentiable on ∂B, and $b_1(x) \neq 0$ for all $x \in \partial B$.

We use the Hilbert space $X_1 = L^2(B)$ of Lebesgue measurable functions ϕ on B such that

$$\|\phi\| = \left\{\int_B |\phi(x)|^2\,dx\right\}^{1/2} < \infty.$$

The space $H^k(B)$ will be the set of all $\phi \in X_1$ such that ϕ has distribution partial derivatives through order k and these derivatives are all in $L^2(B)$. Let $\bar{B} = B U \partial B$ be the closure of the open set B, let $C_0(B)$ be all C^∞-smooth functions ϕ with the compact support in B, and let $H_0{}^k(B)$ be the closure in $H^k(B)$ of $C_0(B)$. Define $X_2 = H_0{}^1(B)$ and $X_3 = X_2 \times X_1$.

If B is a bounded set and if the boundary ∂B is a C^2-smooth surface, then it is well known (see, e.g., [8, Chap. I, Sect. 19] that the operator Δ can be considered as a closed, linear operator on X_1 whose domain is the closure of the set of all functions $\phi \in C^2(\bar{B})$ such that ϕ satisfies boundary condition (5) or alternately boundary condition (6). The operator $A = (k(0)/\alpha(0))\Delta - (\alpha'(0)/\alpha(0))I$ will satisfy the following two conditions:

(H4) A generates a C_0-semigroup on X_1.

(H5) For λ sufficiently large, $\lambda I - A$ has a compact inverse on X_1. Moreover there exist simple eigenvalues $\lambda_m(\lambda_{m+1} < \lambda_m)$ and corresponding eigenfunctions $\phi_m \in C^\infty(\bar{B})$ such that ϕ_m satisfies the given boundary conditions and $\Delta\phi_m = \lambda_m\phi_m$. The set $\{\phi_m\}$ is a complete, orthonormal set.

Assume that (H1)–(H4) are true with $k(0) > 0$, $J = 2$. Let θ_0 and $f(t)$ be given as in Lemma 1 with $f\colon [0, \infty) \to X_1$ continuous. The notion of "solution of (3′) satisfying the given initial and boundary conditions" has several possible interpretations. We could mean a *classical solution*, that is that $\theta(t, x)$ is continuous on $t \geqslant 0$, $x \in \bar{B}$, θ satisfies the initial-boundary conditions at all relevant points (t, x), all requisite partial derivatives of θ w.r.t. t and x exist and are continuous and (3′) is true for all (t, x) in $(0, \infty) \times B$. In order to obtain existence and, in particular, continuity w.r.t. parameters it will be necessary to weaken this notion of solution considerably. By a *distribution solution* of (3′) satisfying initial condition θ_0 we shall mean a function $\theta\colon [0, \infty) \to D(\Delta)$, where $D(\Delta)$ is the domain of Δ in X_1, such that $\theta(t)$ and $\Delta\theta(t)$ are continuous as maps from $[0, \infty)$ to X_1, $\theta(0) = \theta_0$ and (3′) is true for all $t \in R^+$. By a *generalized distribution* solution satisfying initial condition θ_0 we shall mean a continuous function $\theta\colon [0, \infty) \to X_1$ such that $\theta(0) = \theta_0$ and such that there is a sequence θ_k of distribution solutions such that

$$\lim_{k\to\infty} \theta_k(t) = \theta(t),$$

with the limit existing uniformly on compact subsets of the interval $0 < t < \infty$.

The following result is a special case of Theorem 7.3 in [18].

THEOREM 1. *If* (H1)–(H4) *are true with* $k(0) > 0$ *and* $J = 2$, *then the following results are true*:

(a) *For each* $\theta_0 \in D(\Delta)$ *and each* C^1*-function* $f:[0, \infty) \to X_1$ *there is a unique distribution solution* $\theta(t, \theta_0, f)$ *of* (3′) *satisfying* $\theta(0, \theta_0, f) = \theta_0$.

(b) *For each* $\theta_0 \in X_1$ *and each continuous function* $f: [0, \infty) \to X_1$ *there is a unique generalized distribution solution* $\theta(t, \theta_0, f)$ *satisfying* $\theta(0, \theta_0, f) = \theta_0$.

(c) $\theta(t, \theta_0, f)$ *varys continuously with* (t, θ_0, f). *That is, given* $T > 0$, *there exists* $K \geqslant 0$ *such that*

$$\| \theta(t, \theta_0, f)\| \leqslant K(\| \theta_0 \| + \max_{0 \leqslant t \leqslant T} \|f(t)\|) \qquad \text{for all } t \in [0, T].$$

The uniqueness part of Theorem 1 has already been proved by many authors. The most general uniqueness theorem is in [10]. Kremer [16] has proved existence of generalized distribution solutions. The continuity result (c) is new (in the sense that it has not earlier been pointed out that the work in [18] could be applied to (1)). We remark that the constant K in (c) depends on T and of course on the fixed parameters C, $a(t)$, and $b(t)$. The value of K is independent of θ_0 and $f(t)$. Since the existence of K follows from an application of the closed graph theorem, it is not possible to estimate its magnitude. (Similar remarks apply to Theorem 3 below.)

The continuity result (c) can be used to good advantage to justify an approximation scheme suggested by the work of Grabmueller [10]. Indeed, the following result is true.

THEOREM 2. *Suppose* (H1)–(H5) *are true with* $k(0) > 0$ *and* $J = 2$. *Let* θ_0 *and* $f(t)$ *be fixed with* $\theta_0 \in X_1$ *and* $f: [0, \infty) \to X_1$ *continuous. If* $\{\lambda_k\}$ *and* $\{\phi_k\}$ *are the sequences given in* (H5) *and* $\langle \, , \rangle$ *denotes the inner product in* X_1 *define*

$$\gamma_k = \langle \theta_0, \phi_k \rangle, \qquad g_k(t) = \langle f(t)\, \phi_k \rangle,$$

$$\theta_{0m} = \sum_{k=1}^{m} \gamma_k \phi_k, \qquad f_m(t) = \sum_{k=1}^{m} g_k(t)\, \phi_k,$$

and

$$\theta_m(t) = \sum_{k=1}^{m} y_k(t)\, \phi_k,$$

where y_k *is the unique solution of the scalar problem*

$$y_k'(t) = g_k(t) + [\lambda_k C - a(0)]\, y_k(t) + \int_0^t [\lambda_k C b(t-\tau) - a'(t-\tau)]\, y_k(\tau)\, d\tau,$$
$$y_k(0) = \gamma_k. \tag{M$_k$}$$

Then $\theta_m(t) = \theta(t, \theta_{0m}, f_m)$ *is the distribution solution of* (3′) *corresponding to* *initial conditions* θ_{0m} *and* $f_m(t)$. *Moreover on any finite interval* $[0, T]$ *the limit*

$$\lim_{m\to\infty} \theta(t, \theta_{0m}, f_m) = \theta(t, \theta_0, f)$$

exists uniformly in $t \in [0, T]$.

Proof. It was shown in [10] that $\theta_m(t)$ is a distribution solution of (3′) for initial condition θ_{0m} and function $f_m(t)$. Since the sequence $\{\phi_m\}$ is a complete orthonormal set, then

$$\theta_0 = \sum_{k=1}^{\infty} \langle \theta_0, \phi_k \rangle \phi_k = \lim_{m\to\infty} \sum_{k=1}^{m} \langle \theta_0, \phi_k \rangle \phi_k$$
$$= \lim_{m\to\infty} \theta_{0m}.$$

Moreover, for each fixed $t \geqslant 0$,

$$f(t) = \sum_{k=1}^{\infty} \langle f(t), \phi_k \rangle \phi_k = \lim_{m\to\infty} f_m(t). \tag{7}$$

The convergence in (7) is uniform over any closed bounded set $0 \leqslant t \leqslant T$. To see this, note that

$$\|f(t)\|^2 = \sum_{k=1}^{\infty} |\langle f(t), \phi_k \rangle|^2 \tag{8}$$

converges pointwise on $0 \leqslant t < \infty$. Each term in series (8) is positive and continuous. The sum $\|f(t)\|^2$ is also continuous. By a well-known result of Dini (see, for example [25, pp. 447–448]), the convergence in (8) must be uniform on $[0, T]$. This is the same as uniform convergence in (7). Part (c) of Theorem 1 can now be applied to complete the proof of Theorem 2. Q.E.D.

One might paraphase Theorem 2 by saying that the series solutions obtained by separation of variables always converge. Note that since all of the $y_k(t)$ are continuously differentiable and since the eigenvectors ϕ_k are in $C^\infty(\bar{B})$, then the approximations θ_m are actually classical solutions of (3′). These classical solutions are dense among all generalized distribution solutions.

In Theorem 2 the functions θ_m will be called *model approximations*. The function $y_k(t)$ determines the *model modulation* for the kth mode λ_k.

For Eq. (4) under boundary conditions (5) one can prove similar results as follows. If we define $v = \theta' + (a(0)/2)\theta$, $u = \theta$, then (4′) becomes

$$\begin{aligned} u'(t) &= -\frac{a(0)}{2} u(t) + v(t), \\ v'(t) &= \left(C_1 \Delta - a'(0) + \frac{a(0)^2}{4}\right) u(t) - \frac{a(0)}{2} v(t) \\ &\quad + \int_0^t \{C_1 b_1(t-\tau)\, \Delta - a''(t-\tau)\}\, u(\tau)\, d\tau + f(t). \end{aligned} \tag{4''}$$

Let A be the operator defined on X_3 by

$$A\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}-a(0)/2 & 1\\ C_1\Delta-a'(0)+a(0)^2/4 & -a(0)/2\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}.$$

Then the following facts are known if B is bounded and ∂B is smooth:

(H4′) A generates a C_0-semigroup on X_3.

(H5′) For λ sufficiently large, $\lambda I-\Delta$ has a compact inverse on $H_0^1(B)$. There is a sequence $\lambda_m\to-\infty$ and a complete orthonormal set $\{\phi_m\}\subset C^\infty(\bar{B})\cap H_0^1(B)$ such that $\Delta\phi_m=\lambda_m\phi_m$.

The following result follows immediately from [18, Theorem 7.3].

THEOREM 3. *If* (H1)–(H3) *and* (H4′) *are true with* $k(0)=0$, $J=3$, *then the following results are true*:

(a) *For any pair* $(u_0, v_0)\in D(\Delta)\times H_0^1(B)$ *and any* C^1 *function* $f\colon[0,\infty)\to H_0^1(B)$ *there is a unique distribution solution of* $(u(t), v(t))$ *of* (4″) *satisfying the given initial conditions.*

(b) *For each pair* $(u_0, v_0)\in X_3$ *and each continuoux function* $f\colon[0,\infty)\to H_0^1(B)$ *there is a unique generalized distribution solution satisfying the given initial data.*

(c) *The functions* $u(t, u_0, v_0, f)$ *and* $v(t, u_0, v_0, f)$ *vary continuously with the data. That is, given* $T>0$ *there exists* $K>0$ (*independent of* u_0, v_0 *and* f) *such that for* $0\leqslant t\leqslant T$,

$$\|u(t,u_0,v_0,f)\|+\|v(t,u_0,v_0,f)\|\leqslant K\{\|u_0\|+\|v_0\|+\max_{0\leqslant t\leqslant T}\|f(t)\|\}.$$

The next result follows from the continuity (c) above.

THEOREM 4. *Suppose* (H1)–(H3) *and* (H4′)–(H5′) *are true with* $k(0)=0$, $J=3$. *Fix* $(u_0, v_0)\in X_3$ *and* $f\colon[0,\infty)\to H_0^1(B)$, f *continuous. Let* $\langle\,,\rangle_j$ *denote the inner product for* X_2 *or for* X_3 *and define*

$$\langle(u_0,v_0),(\phi_k,\phi_k)\rangle_3=\langle\gamma_k,\delta_k\rangle,\qquad g_k(t)=\langle f(t),\phi_k\rangle_2,$$

$$u_{0m}=\sum_{k=1}^m\gamma_k\phi_k,\qquad v_{0m}=\sum_{k=1}^m\delta_k\phi_k,\qquad f_m(t)=\sum_{k=1}^m g_k(t)\,\phi_k,$$

and

$$u_m(t)=\sum_{k=1}^m y_k(t)\,\phi_k,$$

where $y_k(t)$ is the solution of the scalar problem

$$y_k''(t) = [C_1\lambda_k - a'(0)]\, y_k(t) - a(0)\, y_k'(t)$$
$$+ \int_0^t \{C_1 b_1(t-\tau)\,\lambda_k - a''(t-\tau)\}\, y_k(\tau)\, d\tau + g_k(t),$$
$$y_k(0) = \gamma_k\,, \qquad y_k'(0) = \delta_k\,.$$

Then $u_m(t) = u(t, u_{0m}, v_{0m}, f_m)$ is a distribution solution of (4′) *and for any finite interval* $[0, T]$ *the limit*

$$\lim_{m\to\infty} u_m(t) = u(t, u_0\,, v_0\,, f)$$

exists uniformly in $t \in [0, T]$.

3. Stability Considerations

We now study some stability properties of solutions of (1).

Definition. Consider Eq. (3′) under assumptions (H1)–(H5), $J = 2$, and $k(0) > 0$. Assume $r(t) \equiv 0$. System (3′) is called

(a) *Stable* if for any initial history $\theta\colon (-\infty, 0] \to D(\Delta)$ with $\|\theta(t)\| + \|\Delta\theta(t)\|$ bounded, the distribution solution $\theta(t)$ of (3′) is bounded on $0 \leqslant t < \infty$.

(b) *Unstable* if it is not stable.

(c) *Asymptotically stable* if it is stable and in addition each solution $\theta(t)$ with bounded initial history must have limit $\theta(t) \to 0$ as $t \to \infty$.

We remark that the restriction $r(t) \equiv 0$ is not essential. As will be seen below, once the stability properties of (3′) are determined for $r(t) \equiv 0$ the properties of solutions with $r(t) \not\equiv 0$ will follow. We also remark that in the sequel for any function ϕ, by $\phi^*(s)$ we shall mean the Laplace transformation of ϕ.

Using Theorem 2 above as motivation one might conjecture that the stability of (3′) is related to the stability properties of the infinite set of scalar equations (M_k). Indeed the proof of Theorem 5 below is that a necessary condition for stability of (3′) is that for each mode λ_k the equations (M_k) are stable. Theorem 6 states that under minor additional restrictions this condition is also sufficient. It will be convenient to state stability assumptions for the equations (M_k) in terms of the Laplace transform conditions (10) given below. Any other condition for stability of all modes must necessarily imply that (10) is true.

Theorem 5. *A necessary condition for stability of* (3′) *is that* $k(0) + (k')^*(s) \neq 0$ *for all s in the open right half plane* $\operatorname{Re} s > 0$.

Proof. Suppose $k(0) + (k')^*(s) = 0$ for some s_0 with $\operatorname{Re} s_0 > 0$. Consider the equation

$$\lambda_m = s\left(\frac{\alpha(0) + (\alpha')^*(s)}{k(0) + (k')^*(s)}\right),$$

where λ_m is an eigenvalue of Δ. The function on the right above has a pole at s_0. Thus this function must map any punctured neighborhood of s_0 onto a neighborhood of infinity in the complex plane. Since $\lambda_m \to -\infty$ as $m \to \infty$, then for all m sufficiently large one can find solutions $s = s_m$ of the equation

$$s[\alpha(0) + (\alpha')^*(s)] - \lambda_m[k(0) + (k')^*(s)] = 0$$

with $\operatorname{Re} s_m > 0$.

The function $u_m(t) = \exp(s_m t)\,\phi_m$ has bounded initial history. Indeed since since $\Delta\phi_m = \lambda_m\phi_m$ and since $\operatorname{Re} s_m > 0$,

$$\begin{aligned}\| u_m(t)\| + \| \Delta u_m(t)\| &= (1 + |\lambda_m|)\,\|\phi_m\|\,|\exp(s_m t)| \\ &\leqslant (1 + |\lambda_m|)\,\|\phi_m\| < \infty.\end{aligned}$$

This function is a solution of (3′) when $r(t) \equiv 0$, as is easily checked. Since s_m is a root of the equation above, then

$$\begin{aligned}&u_m'(t) - C\Delta u_m(t) + a(0)\,u_m(t) - \int_{-\infty}^{t} \{Cb(t-\tau)\,\Delta - a'(t-\tau)\}\,u_m(\tau)\,d\tau \\ &\quad = \{s_m - C\lambda_m + a(0)\}\,u_m(t) - \int_0^\infty \{Cb(\tau)\,\Delta - a'(\tau)\}\,e^{-s_m\tau}u_m(t)\,d\tau \\ &\quad = [s_m - C\lambda_m + a(0) - C\lambda_m b^*(s_m) + (a')^*(s_m)]\,u_m(t) = 0 \cdot u_m(t) = 0.\end{aligned}$$

Since $\operatorname{Re} s_m > 0$, then

$$\limsup_{t\to\infty} \| u_m(t)\| = +\infty.$$

Thus the equation is unstable. Q.E.D.

As a practical matter the equation $k(0) + (k')^*(s) = 0$ cannot have solutions with $\operatorname{Re} s = 0$ if stability is to be guaranteed. For if there were such a root, then arbitrarily small perturbations in k could shift this root into the right half plane. This condition is incorporated in the following assumption:

$$k(0) + (k')^*(s) \neq 0 \text{ when } \operatorname{Re} s \geqslant 0 \text{ and } t\,|\alpha''(t)| \text{ and } t\,|k'(t)| \in L^1(0, \infty). \tag{9}$$

THEOREM 6. *If* (9) *is true, then* (3′) *is asymptotically stable if and only if*

$$s[\alpha(0) + (\alpha')^*(s)] - \lambda_m[k(0) + (k')^*(s)] \neq 0 \text{ for } m = 1, 2, 3,\ldots \text{ and } \operatorname{Re} s \geqslant 0. \tag{10}$$

Proof. If (10) is violated for some λ_m for some λ_m and s_m, then the proof used in Theorem 5 will show that $u_m(t) = e^{s_m t}\phi_m$ has bounded initial history, solves (3′) when $r(t) \equiv 0$ and does not tend to zero as $t \to \infty$.

If (10) is true, then the methods of [9] or [19] can be applied. Thus there is a continuous operator-valued function $R(t)$ such that $\| R(t)\| \in L^1(0, \infty)$ and $\| R(t)x \| \to 0$ as $t \to \infty$. Any distribution solution of (3′) can be represented in the form

$$\theta(t) = R(t)\,\theta(0) + \int_0^t R(t-\tau) f(\tau)\, d\tau.$$

Since $r(t) \equiv 0$, then

$$f(t) = \int_{-\infty}^0 [k'(t-\tau)\,\Delta\theta(\tau) - \alpha''(t-\tau)\,\theta(\tau)]\, d\tau/\alpha(0).$$

If N is a bound for the histories $\| \theta(\tau)\|$ and $\| \Delta\theta(\tau)\|$ then

$$\|f(t)\| \leqslant \int_{-\infty}^0 \{| \alpha''(t-\tau)| + | k'(t-\tau)|\}\, N\, d\tau$$

$$= N \int_t^\infty \{| \alpha''(\tau)| + | k'(\tau)|\}\, d\tau \to 0$$

as $t \to \infty$. Since $\| R(t)\| \in L^1(0, \infty)$, and $\| R(\infty)\,\theta(0)\| = 0$, then

$$\| \theta(t)\| \leqslant \| R(t)\,\theta(0)\| + \int_0^t \| R(t-\tau)\|\, \|f(\tau)\|\, d\tau \to 0 \qquad \text{as} \qquad t \to \infty.$$

Moreover,

$$\| \theta(t)\| \leqslant K(\| \theta(0)\| + \max\|\{f(t)\|: 0 \leqslant t < \infty\}), \tag{11}$$

where

$$K = \max\{\| R(t)\|: 0 \leqslant t \leqslant \infty\} + \int_0^\infty \| R(t)\|\, dt. \qquad \text{Q.E.D.}$$

COROLLARY 7. *If* (9) *and* (10) *are true, then for any initial history* $\theta(t)$ *with* $\| \theta(t)\| + \| \Delta\theta(t)\|$ *bounded on* $-\infty \leqslant t \leqslant 0$ *and for any bounded, continuous function* $r(t)$, *the genralized distribution solution of* (3′) *is bounded. If in addition* $\| r(t)\| \to 0$ *as* $t \to \infty$, *then* $\| \theta(t)\| \to 0$, *too.*

Proof. Inequality (11) is true for any distribution solution, even when $r(t) \not\equiv 0$. Since these solutions are dense among all generalized distribution solutions, (11) remains true for the generalized solutions. If $r(t)$ is a C^1 function with limit zero at $t = \infty$, then $f(t) \to 0$. As in the proof of Theorem 6, the solution $\theta(t) \to 0$. Since C^1-smooth functions are dense in the space $\{\phi: [0, \infty) \to$

X_1: ϕ is continuous and $\phi(t) \to 0$ as $t \to \infty\}$ with the uniform topology, then the conclusion follows from the Banach–Steinhaus theorem. Q.E.D.

It is not possible to give information on the rate of convergence to zero in either Theorem 6 or Corollary 7.

In certain situations $\lambda_1 = 0$ is possible. In this case (10) is never true since $\lambda_1 = 0$ and $s = 0$ make the right-hand side in (10) equal to zero. In this case the following result is true.

THEOREM 8. *Suppose* (9) *is true and* (10) *is true except at the point* $s = 0$ *for* λ_1. *If for* $j = 1, 2$,

$$\int_0^\infty \tau^j \mid \alpha''(\tau)| \, d\tau < \infty, \qquad \int_0^\infty \tau^j \mid k'(\tau)| \, d\tau < \infty,$$

and if

$$\alpha'(0) - \int_0^\infty \tau\alpha''(\tau) \, d\tau \neq 0,$$

then there exists an operator-valued function $R(t)$ *such that* R *is continuous,* $\| R(t)x \| \to 0$ *as* $t \to \infty$, $\| R(t)\| \in L^1(0, \infty)$ *and for any bounded initial history* $\theta(t)$ *on* $-\infty \leqslant t \leqslant 0$ *and any* C^1*-smooth function* $r(t)$ *the distribution solution of* (3′) *has the form*

$$\theta(t) = R(t)\,\theta(0) + \langle\theta(0), \phi_1\rangle\phi_1 + \int_0^t \{R(t-\tau)f(\tau) + \langle f(\tau), \phi_1\rangle\phi_1\} \, d\tau. \tag{12}$$

In particular if $r(t) \to 0$ *as* $t \to \infty$, *then*

$$\theta(t) - \langle\theta(0), \phi_1\rangle\phi_1 - \int_0^t \langle f(\tau), \phi_1\rangle\phi_1 \, d\tau \to 0 \qquad (t \to \infty).$$

Proof. The representation of solutions in form (12) is proved by a trival modification of the proof of Theorem 8 in [19]. Since $\|f(t)\| \to 0$ as $t \to \infty$, the limit follows immediately from the properties of $R(t)$. Q.E.D.

Stability results for Eq. (4′) are much harder to obtain. The methods used in [19] do not apply. However, an analog of Theorem 5 is true. Since the proof is essentially the same as that for Theorem 5, it is omitted.

THEOREM 9. *Suppose* (H1)–(H3) *and* (H4′) *and* (H5′) *are true with* $k(0) = 0$, $J = 3$. *If* (4′) *is stable, then it is necessary that*

$$k'(0) + (k'')^*(s) \neq 0 \qquad \textit{when} \quad \text{Re} s > 0.$$

The remark which precedes assumption (9) also applies in an analogous manner to Theorem 9. Therefore if one believes that a reasonable physical model of heat conduction must be stable and that this stability must be preserved under small changes in the heat conduction relation function $k(t)$, then it necessarily follows that

$$\text{(S)} \qquad k(0) + (k')^*(s) \neq 0 \qquad \text{when} \quad \operatorname{Re} s \geqslant 0 \quad (\text{if } k(0) > 0)$$

or

$$\text{(S}'\text{)} \qquad k'(0) + (k'')^*(s) \neq 0 \qquad \text{when} \quad \operatorname{Re} s \geqslant 0 \quad (\text{if } k(0) = 0).$$

These restriction have not been mentioned in the earlier literature on heat conduction in materials with memory, nor is any physical interpretation of (S) or (S′) clear.

If (S) is true, then under mild additional assumptions it can be seen that (3′) is stable if and only (M_k) is stable for all λ_k. This is equivalent to the Laplace transformation condition (10). Although conditions (10) look unwieldy, it is hoped that graphical techniques of the type used in systems theory (see, e.g., [5, 14]) might be useful in verifying (10). For example, assume (9). Then one will have stability for all bounded configurations B if and only if for each $\lambda > 0$

$$\text{(E)} \qquad s\,\frac{\alpha(0) + (\alpha')^*(s)}{k(0) + (k')^*(s)} \neq -\lambda \qquad (\operatorname{Re} s \geqslant 0).$$

Condition (E) is equivalent to the fact that the graph in the complex plane of the function

$$D(\tau) = i\left(\frac{\alpha(0) + (\alpha')^*(i\tau)}{k(0) + (k')^*(i\tau)}\right), \qquad 0 \leqslant \tau < \infty$$

(for $i = -1^{1/2}$), does not hit the negative real axis. (Also note that (S) will be true if the graph $\{(k')^*(i\tau); -\infty < \tau < \infty\}$ does not touch or encircle the point -1.)

4. Classical Solutions of (3′)

For parabolic partial differential equations it is known that whenever the initial data is sufficiently smooth, then the distribution solution is actually a smooth, classical solution. Unfortunately similar results are not known for equations of the form (3′). The purpose of this section is to give a transformation of (3′) into a new equation for which the existence of a large class of classical solutions is easy to prove.

Lemma 3. *Suppose* (H1)–(H3) *are true with* $k(0) > 0$ *and* $J = 2$.

Then Eq. (3′) *is equivalent to the following integrodifferential equation*:

$$\frac{\partial\theta}{\partial t}(t, x) = F(t, x) + C\Delta\theta(t, x) + y(0)\,\theta(t, x) + \int_0^t y'(t-\tau)\,\theta(\tau, x)\,d\tau, \qquad (13)$$

where F is defined as

$$F(t, x) = f(t, x) - \int_0^t D(t-\tau) f(\tau, x)\,d\tau - D(t)\,\theta(0, x),$$

and where $D(t)$ and $y(t)$ satisfy the scalar equations

$$D(t) = b(t) - \int_0^t b(t-\tau)\,D(\tau)\,d\tau,$$

$$y(t) = [b(t) - a(t)] - \int_0^t b(t-\tau)\,y(\tau)\,d\tau.$$

Proof. If $*$ is used to denote convolution integration, then (3′) can be written as

$$\theta' = C\,\Delta\theta - a(0)\theta + Cb * \Delta\theta - a' * \theta + f. \qquad (14)$$

Since $D(t)$ is the unique continuous solution of $D = b - b * D$ then for any continuous function h the unique solution of

$$z = h - b * z$$

is $z = h - D * h$ (see [17, Chap. I]). Let $h = (\theta' + a(0)\theta + a' * \theta - f)/C$ in (14) to see that

$$C\,\Delta\theta = (\theta' + a(0)\theta + a' * \theta - f) - D * (\theta' + a(0)\theta + a' * \theta - f). \quad (15)$$

An integration by parts yields

$$\theta' - D * \theta' = \theta' - D(0)\theta + D\theta(0) - D' * \theta.$$

If this is used in (15), the result can be rearranged to see that

$$\begin{aligned}\theta' &= \{f - D * f - D\theta(0)\} + C\,\Delta\theta + \{D(0) - a(0)\}\theta \\ &\quad + \{D' + a(0)D - a' + D * a'\} * \theta.\end{aligned}$$

The last term in brackets can be written as

$$y'(t) = \{D' + a(0)D - a' + a' * D\} = \{D - a + a * D\}'.$$

Thus y is the solution of the linear equation given above and (3′) can be written in the form (13). Q.E.D.

It is interesting to note that since $b \in L^1(0, \infty)$ then by a classical result of Paley and Wiener (see, e.g., [17, Chap. IV, Sect. 4]) the function $D \in L^1$ if and only if the stability condition (S) is true. Clearly $D \in L^1$ implies $y \in L^1$. If $b' \in L^1$ and if (S) is true, then it is not hard to see that y' is also in L^1.

If the parabolic part of (15) admits a Green's function in B, then a method suggested by the proof of Corollary 2 in Redheffer and Walter [26] can be used here. For a general discussion of Green's functions for parabolic equations see [7, Chap. 3, Sect. 7]. Assume that

(H6) Region B is bounded on ∂B is $C^{2+\delta}$-smooth for some $\delta > 0$.

Thus

$$\frac{\partial u}{\partial t} = C\Delta u, \qquad \text{for } 0 < t < \infty, \quad x \in B,$$

$$u(t, x) = 0, \qquad \text{for } 0 < t < \infty, \quad x \in \partial B$$

admits a Green's function $G(t - \tau, x, \xi)$. Also assume

(H7) The initial history $\theta(t, x)$ is chosen so that $\Delta\theta$ exists and is continuous in $(-\infty, 0] \times B$, $\theta(0, x)$ is continuous in $B \cup \partial B$, $\theta(0, x) = 0$ on ∂B, and the function $\int_{-\infty}^{0} \{k'(t - \tau)\, \Delta\theta(\tau, x) - a''(t - \tau)\, \theta(\tau, x)\}\, d\tau$ is locally Holder continuous in $[0, \infty) \times B$.

(H8) The function $r(t, x)$ is locally Hölder continuous on the set $[0, \infty) \times B$.

THEOREM 10. *Suppose* (H1)–(H3) *are true with* $k(0) > 0$, $J = 2$, k' *and* α'' are locally *Hölder continuous and* (H6)–(H8) *are true. Then the history problem* (13) *with boundary conditions*

$$\theta(t, x) = 0 \qquad \textit{for} \quad 0 < t < \infty, \quad x \in \partial B$$

has a classical solution $\theta(t, x)$ *satisfying the condition*

$$\lim_{t \searrow 0} \theta(t, x) = \theta(0, x) \qquad \textit{for } x \in B \cup \partial B.$$

Proof. Given the Green's function we try to solve the integral equation

$$\begin{aligned} \theta(t, x) = {} & \int_B G(t, x, \xi)\, \theta(0, \xi)\, d\xi + \int_0^t \int_B G(t - \tau, x, \xi)\, F(\tau, \xi)\, d\xi\, d\tau \\ & + \int_0^t \int_B G(t - \tau, x, \xi) \left(y(0)\, \theta(\tau, \xi) + \int_0^\tau y'(\tau - u)\, \theta(u, \xi)\, du \right) d\xi\, d\tau. \end{aligned} \tag{16}$$

Elementary arguments involving the contraction map will show that (16) has a continuous solution $\theta(t, x)$ in the region $[0, \infty) \times (B \cup \partial B)$. It remains to show that this function $\theta(t, x)$ has the required number of continuous derivatives in $\{0 < t < \infty, x \in B\}$.

Since $\theta(0, x)$ is continuous on $B \cup \partial B$, the first term on the right has continuous derivatives. This is a standard result for Green's functions. The second term will also be smooth if the function $F(t, x)$ is locally Hölder continuous in $[0, \infty) \times B$. Recall that $D(t)$ solves the equation

$$D(t) = b(t) - \int_0^t b(t - \tau) D(\tau)\, d\tau.$$

Since $b(t) = k'(t)/k(0)$ is in $C^1[0, \infty)$, then $D(t) \in C^1[0, \infty)$. Since $\theta(0, x)$ is $C^2(B)$ then the term $D(t)\,\theta(0, x)$ is certainly of class $C^1([0, \infty) \times B)$.

Since $D \in C^1[0, \infty)$, then the terms $f - D * f$ will be Hölder continuous whenever f is. Since f has the form

$$f(t, x) = \left\{r(t, x) + \int_{-\infty}^0 [k'(t - \tau)\, \Delta\theta(\tau, x) - \alpha''(t - \tau)\, \theta(\tau, x)]\, d\tau\right\} \Big/ \alpha(0),$$

assumptions (H7) and (H8) guarantee the Holder continuity of f. Thus the second integral on the right in (16) defines a function with the required continuous partial derivatives.

Since the function

$$m(t, x) = y(0)\, \theta(t, x) + \int_0^t y'(t - \tau)\, \theta(\tau, x)\, d\tau \tag{17}$$

is continuous, then the properties of the Green's function G imply that the third integral on the right in (16) is in C^1 for $0 < t < \infty$, $x \in B$. Thus θ is also in C^1. It remains to show that the second term on the right in (17) is locally Hölder continuous in t. However, $b'(t) - a'(t)$ is locally Hölder continuous and so

$$y'(t) = [b'(t) - a'(t)] - b(0)\, y(t) - \int_0^t b'(t - \tau)\, y(\tau)\, d\tau$$

is, too. Thus $y' * \theta$ is also locally Hölder continuous in t. Q.E.D.

5. Hyperbolicity When $k(0) = 0$

In this section we give a transformation similar to the one in the last section, which will transform Eq. (4′) into a form where its hyperbolic character is more easily seen.

LEMMA 4. *If* (H1)–(H3) *are true with* $k(0) = 0$ *and* $J = 3$, *then* (4′) *can be transformed to the new equation*

$$\begin{aligned}\frac{\partial^2\theta}{\partial t^2}(t, x) = y_1(0)\frac{\partial\theta}{\partial t}(t, x) + y_1'(0)\,\theta(t, x) + C_1\Delta\theta(t, x) + F(t, x)\\ + \int_0^t y_1''(t-\tau)\,\theta(\tau, x)\,d\tau,\end{aligned} \tag{18}$$

where

$$\begin{aligned}F(t, x) = f(t, x) - \int_0^t E(t, \tau) f(\tau, x)\,d\tau - E(t)\left[\frac{\partial\theta}{\partial t}(0, x) + a(0)\,\theta(0, x)\right]\\ - E'(t)\,\theta(0, x),\end{aligned}$$

while E *and* y_1 *satisfy the scalar equations*

$$E(t) = b_1(t) - \int_0^t b_1(t-\tau)\,E(\tau)\,d\tau$$

and

$$y_1(t) = [b_1(t) - a(t)] - \int_0^t b_1(t-\tau)\,y_1(\tau)\,d\tau. \tag{19}$$

Proof. Write (4′) in the abbreviated form

$$(\theta + a * \theta)'' = f + C_1(\Delta\theta + b_1 * \Delta\theta).$$

Since $E(t)$ is the solution of $E = b_1 - b_1 * E$, then

$$C_1\,\Delta\theta = (\theta + a * \theta)'' - E * [(\theta + a * \theta)''] - f + E * f.$$

Integration by parts in the last term on the right and rearrangement yields

$$\begin{aligned}\theta'' &= -a(0)\,\theta' - a'(0)\,\theta - a'' * \theta + E(0)\,[\theta' + a(0)\,\theta + a' * \theta]\\ &\quad + E'(0)\,[\theta + a * \theta] + E'' * (\theta + a * \theta) + C_1\Delta\theta + f - E * f\\ &\quad - E(t)\left[\frac{\partial\theta}{\partial t}(0, x) + a(0)\,\theta(0, x)\right] - E'(t)\,\theta(0, x)\\ &= [(E - a + E * a) * \theta]'' + C_1\Delta\theta + F.\end{aligned}$$

Since $y_1 = E - a + E * a$ is the solution of (19), then (4′) can be written in the equivalent form (18).

We remark that by the Paley–Wiener theorem (see [17, Chap. IV, Sect. 4]) $E \in L^1$ if and only if the stability condition (S') is true. Also, $E \in L^1$ implies $y_1 \in L^1$.

From (18) it can be seen that a distrubance at x_0 travels in all directions at speed at most $C^{1/2} = (k'(0)/\alpha(0))^{1/2}$. The disturbance is damped in time if $y_1(0) = (k''(0)/k'(0) - (\alpha'(0)/\alpha(0)) < 0$ and grows in magnitude if $y_1(0) > 0$. As an example of how (18) can be used, we prove the following result in R^3.

THEOREM 11. *Suppose* (H1)–(H3) *are true with* $k(0) = 0$, $J = 3$, *and* $B = R^3$. *Suppose the initial history* θ *is zero for* $-\infty < t \leqslant 0$, $x \in R^3$. *If* $r(t, x)$ *is a smooth function such that* $r(t, x) = 0$ *for* $t \leqslant 0$ *and for* $|x - x_0| \geqslant \delta$, *then the solution* $\theta(t, x)$ *of* (18) *must be zero when* $|x - x_0| \geqslant \delta + C^{1/2}t$.

Proof. It is well known that the equation

$$u_{tt} = C_1 \Delta u + y_1(0)\, u_t - (y_1(0)^2/4)u + \phi(t, x), \qquad u(0, x) = u_t(0, x) = 0,$$

has as solution

$$u(t, x) = \frac{1}{4\pi(C_1)^{1/2}} \int_0^t \tau e^{y_1(0)\tau/2} \left(\iint_\Omega \phi(t - \tau, x + \omega(C_1)^{1/2}\tau)\, d\omega \right) d\tau,$$

where Ω is the unit sphere in R^3 and $d\omega$ is the differential there. Thus the solution of (18) with $\theta(t, x) = 0$ for $t \leqslant 0$, $x \in R^3$ must be a solution of the integral equation

$$\begin{aligned} \theta(t, x) = \frac{1}{4\pi(C_1)^{1/2}} \int_0^t \tau e^{y_1(0)\tau/2} \iint_\Omega \Big\{ & F(t - \tau, x + \omega(C_1)^{1/2}\tau) \\ & + \left(y_1'(0) + \frac{y_1(0)^2}{4} \right) \theta(t - \tau, x + \omega(C_1)^{1/2}\tau) \\ & + \int_0^{t-\tau} y_1''(t - \tau - \sigma)\, \theta(\sigma, x + \omega(C_1)^{1/2}\tau)\, d\sigma \Big\}\, d\omega\, d\tau. \end{aligned} \tag{19}$$

If $r(t, x)$ is a smooth function such that $r(t, x) = 0$ for $|x - x_0| \geqslant \delta$ and for $t \leqslant 0$ then a straightforward contraction mapping argument can be used to see that the solution of (19) is zero for all (t, x) for which $|x - x_0| \geqslant \delta + C^{1/2}t$.

Q.E.D.

The condition "$\theta(t, x) = 0$ if $|x - x_0| \geqslant \delta + C^{1/2}t$" is the statement that (18) has *finite wave speed*. Thus we see that (18) is *hyperbolic* in the sense used by Davis [2, 3]. The assumptions used by Davis imply that $\alpha(t)$ and $k(t)$ are exponential polynomials. This in turn implies that (4') is really the integrated form of a higher-order linear partial differential equation with constant coefficients. The results in [20] avoid the assumption that $\alpha(t)$ and $k(t)$ are exponential polynomials but the hypotheses used there are still hard to verify. The treatment of hyperbolicity given here is much simpler and more natural for (4) than the earlier work in [2, 3, 20].

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