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Relaxation Results for a Class of Variational Integrals*

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The relaxation problem for functionals of the form $\int_\Omega f(u, Du)\,dx$ with $f(s, z)$ not necessarily continuous with respect to s is studied.

INTRODUCTION

After the proof of a semicontinuity result for functionals of the form

$$F(u, \Omega) = \int_\Omega f(u, Du)\,dx$$

with $f(s, z)$ not necessarily continuous with respect to s (see [5]), the interest of relaxation problems for such functionals was pointed out by E. De Giorgi in a lecture held in Paris, November 1983 (see [4]). In this paper we give a first result in this direction; more precisely we deal with the following problem:

Find hypotheses on f under which it is possible to give an integral representation formula for the greatest functional $\bar{F}(u, \Omega)$ which is lower semicontinuous with respect to the $L^1_{\text{loc}}(\Omega)$-topology and less than or equal to $F(u, \Omega)$.

This problem, under continuity assumptions on f, has been considered by several authors (see, e.g., [1, 2, 6, 9]). In the present paper (see Sec-

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tion 3) we prove that, for a large class of functions f, there exists a function $\phi_f: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that:

(i) $\phi_f(\cdot, 0)$ is lower semicontinuous;

(ii) for every $z \in \mathbb{R}^n$, $\phi_f(\cdot, z)$ is measurable;

(iii) for a.a. $s \in \mathbb{R}$, $\phi_f(s, \cdot)$ is convex;

(iv) for every $u \in W^{1,p}(\Omega)$ we have

$$\bar{F}(u, \Omega) = \int_\Omega \phi_f(u, Du)\, dx.$$

In Section 4 we give an explicit way to construct the function ϕ_f.

1. Integrands

In this section we study some properties of functions from $\mathbb{R} \times \mathbb{R}^n$ into $[0, +\infty]$. Denote by $\mathscr{M}$ the class of all Lebesgue measurable functions $b: \mathbb{R} \to [0, +\infty]$; the class $\mathscr{M}$ is ordered in the usual way by setting

$$b_1 \underset{\mathscr{M}}{\leqslant} b_2 \Leftrightarrow b_1(s) \leqslant b_2(s) \qquad \text{for a.e. } s \in \mathbb{R}.$$

LEMMA 1.1. *Let I be a set of indices, and for every $i \in I$ let $b_i \in \mathscr{M}$. Then there exists a function $b \in \mathscr{M}$ (which we denote by $\mathscr{M}\text{-}\sup_{i \in I} b_i$) such that*:

(i) *$b_i \leqslant_{\mathscr{M}} b$ for every $i \in I$;*

(ii) *if $\beta \in \mathscr{M}$ and $b_i \leqslant_{\mathscr{M}} \beta$ for every $i \in I$, then $b \leqslant_{\mathscr{M}} \beta$.*

Proof. See [10, Proposition II-4-1, p. 43]. ∎

In the following we say that a function $f: \mathbb{R}^n \to \mathscr{M}$ is $\mathscr{M}$-convex if

$$f(\lambda z_1 + (1-\lambda) z_2) \underset{\mathscr{M}}{\leqslant} \lambda f(z_1) + (1-\lambda) f(z_2)$$

for every $z_1, z_2 \in \mathbb{R}^n$, $\lambda \in [0, 1]$.

From Lemma 1.1 we obtain immediately:

COROLLARY 1.2. *Let I be a set of indices, and for every $i \in I$ let $f_i: \mathbb{R}^n \to \mathscr{M}$ be a $\mathscr{M}$-convex function. Then the function $f: \mathbb{R}^n \to \mathscr{M}$ defined by $f(z) = \mathscr{M}\text{-}\sup_{i \in I} f_i(z)$ is $\mathscr{M}$-convex.*

DEFINITION 1.3. Let $f: \mathbb{R} \times \mathbb{R}^n \to [0, +\infty]$ be a function; we say that

(i) f is a *general integrand* if for every $z \neq 0$ the function $f(\cdot, z)$ belongs to $\mathscr{M}$;

(ii) f is an *integrand* if f is a general integrand, $f(\cdot, 0)$ is a Borel function and for a.a. $s \in \mathbb{R}$ the function $f(s, \cdot)$ is continuous;

(iii) f is a *convex general integrand* if for every $z \in \mathbb{R}^n$ the function $f(\cdot, z)$ belongs to $\mathcal{M}$ and the function $z \mapsto f(\cdot, z)$ from $\mathbb{R}^n$ into $\mathcal{M}$ is $\mathcal{M}$-convex;

(iv) f is a *convex integrand* if f is an integrand and for a.a. $s \in \mathbb{R}$ the function $f(s, \cdot)$ is convex on $\mathbb{R}^n$;

(v) f is a *convex l.s.c. general integrand* if f is a convex general integrand and the function $f(\cdot, 0)$ is lower semicontinuous on $\mathbb{R}$;

(vi) f is a *convex l.s.c. integrand* if f is a convex integrand and the function $f(\cdot, 0)$ is lower semicontinuous on $\mathbb{R}$.

In the following, by null set we mean a set of Lebesgue measure zero.

DEFINITION 1.4. Given two functions f_1, f_2 from $\mathbb{R} \times \mathbb{R}^n$ into $[0, +\infty]$ we write

$$\text{(i)}\quad f_1 \preccurlyeq f_2 \Leftrightarrow \begin{cases} (\mathrm{i}_1) & \text{for every } s \in \mathbb{R}\ f_1(s, 0) \leqslant f_2(s, 0); \\ (\mathrm{i}_2) & \text{for every } z \in \mathbb{R}^n \text{ there exists a null subset } N_z \\ & \text{of } \mathbb{R} \text{ such that} \\ & f_1(s, z) \leqslant f_2(s, z) \text{ for every } s \in \mathbb{R} - N_z; \end{cases}$$

$$\text{(ii)}\quad f_1 \lll f_2 \Leftrightarrow \begin{cases} (\mathrm{ii}_1) & \text{for every } s \in \mathbb{R}\ f_1(s, 0) \leqslant f_2(s, 0); \\ (\mathrm{ii}_2) & \text{there exists a null subset } N \text{ of } \mathbb{R} \text{ such that} \\ & f_1(s, z) \leqslant f_2(s, z) \text{ for every } s \in \mathbb{R} - N,\ z \in \mathbb{R}^n. \end{cases}$$

Moreover we call f_1, f_2 generally equivalent (and we write $f_1 \sim f_2$) if $f_1 \preccurlyeq f_2$ and $f_2 \preccurlyeq f_1$; analogously, we call f_1, f_2 equivalent (and we write $f_1 \approx f_2$) if $f_1 \lll f_2$ and $f_2 \lll f_1$.

Remark 1.5. Let f_1, f_2 be integrands; then $f_1 \preccurlyeq f_2$ if and only if $f_1 \lll f_2$.

Remark 1.6. From Corollary 1.2 it follows that for every function f from $\mathbb{R} \times \mathbb{R}^n$ into $[0, +\infty]$ there exists a convex general integrand (which we denote by $\operatorname{co}(f)$) defined by

$$\operatorname{co}(f)(\cdot, z) = \mathcal{M}\text{-}\sup\{\phi(\cdot, z) \colon \phi \text{ convex general integrand}, \phi \preccurlyeq f\}.$$

DEFINITION 1.7. Let $1 \leqslant p < +\infty$ and let $f \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a function; we say that f satisfies hypothesis (H_p) if for a suitable constant $c \geqslant 0$

$$(\mathrm{H}_p)\quad 0 \leqslant f(s, z) \leqslant c(1 + |s|^p + |z|^p) \text{ for every } s \in \mathbb{R},\ z \in \mathbb{R}^n.$$

If $p=+\infty$ we say that f satisfies hypothesis (H_∞) if for every $M\geqslant 0$ there exists $c_M\geqslant 0$ such that

$$(H_\infty)\quad 0\leqslant f(s,z)\leqslant c_M \text{ for every } (s,z)\in\mathbb{R}\times\mathbb{R}^n \text{ with } |s|\leqslant M,\ |z|\leqslant M.$$

In the following we denote by $⨍_A g\,dx$ the average of a function g on a set A.

PROPOSITION 1.8. *Let f be a convex general integrand satisfying hypothesis* (H_∞). *Then there exists a convex integrand $\bar{f}$ such that $\bar{f}\sim f$.*

Proof. For every $s\in\mathbb{R}$, $z\in\mathbb{R}^n$ we define

$$\bar{f}(s,z)=\begin{cases}\limsup\limits_{\varepsilon\to 0^+} ⨍_{s-\varepsilon}^{s+\varepsilon} f(t,z)\,dt & \text{if } z\neq 0,\\ f(s,0) & \text{if } z=0.\end{cases}$$

By Lebesgue's differentiation theorem we have $\bar{f}\sim f$; so to achieve the proof it suffices to prove that for a.a., $s\in R$, the function $\bar{f}(s,\cdot)$ is convex. Let N be a null subset of $\mathbb{R}$ such that

$$\lim_{\varepsilon\to 0^+} ⨍_{s-\varepsilon}^{s+\varepsilon} f(t,0)\,dt=f(s,0)\qquad \text{for every } s\in\mathbb{R}-N \tag{1.1}$$

and let $s\in\mathbb{R}-N$, z, z_1, $z_2\in\mathbb{R}^n$, $\lambda\in[0,1]$, $\varepsilon>0$. Since f is $\mathscr{M}$-convex, we have for $z=\lambda z_1+(1-\lambda)z_2$

$$⨍_{s-\varepsilon}^{s+\varepsilon} f(t,z)\,dt\leqslant\lambda ⨍_{s-\varepsilon}^{s+\varepsilon} f(t,z_1)\,dt+(1-\lambda)⨍_{s-\varepsilon}^{s+\varepsilon} f(t,z_2)\,dt$$

and so, taking the limit as $\varepsilon\to 0^+$ and using (1.1)

$$\bar{f}(s,z)\leqslant\lambda\bar{f}(s,z_1)+(1-\lambda)\bar{f}(s,z_2). \qquad ∎$$

Let $f\colon\mathbb{R}\times\mathbb{R}^n\to[0,+\infty]$ be a function; consider the class

$$\mathscr{F}_f=\{\phi \text{ convex l.s.c. integrand: } \phi\preccurlyeq f\}.$$

PROPOSITION 1.9. *Assume f to be a general integrand that satisfies hypothesis* (H_∞). *Then there exists $\phi_f\in\mathscr{F}_f$ such that $\phi\preccurlyeq\phi_f$ for every $\phi\in\mathscr{F}_f$.*

Proof. Define for every $z\in\mathbb{R}^n$

$$g(\cdot,z)=\begin{cases}\mathscr{M}\text{-}\sup\{\phi(\cdot,z)\colon\phi\in\mathscr{F}_f\} & \text{if } z\neq 0,\\ \sup\{\phi(\cdot,0)\colon\phi\in\mathscr{F}_f\} & \text{if } z=0.\end{cases}$$

Then g is a general integrand, $g \preccurlyeq f$ and $g(\cdot, 0)$ is lower semicontinuous. Since for every $\phi \in \mathscr{F}_f$ the function $\phi(\cdot, 0)$ is lower semicontinuous, by the Lindelöf covering theorem $g(\cdot, 0)$ is the supremum of a countable family $(\phi_h(\cdot, 0))_{h \in \mathbb{N}}$ with $\phi_h \in \mathscr{F}_f$.Then $g(\cdot, 0) = \mathscr{M}\text{-sup}\{\phi_h(\cdot, 0): h \in \mathbb{N}\}$ and so

$$g(\cdot, z) = \mathscr{M}\text{-sup}\{\phi(\cdot, z): \phi \in \mathscr{F}_f\} \qquad \text{for every} \quad z \in \mathbb{R}^n.$$

Therefore, by Corollary 1.2, g is a convex general integrand; thus by Proposition 1.8 there exists $\phi_f \in \mathscr{F}_f$ such that $\phi_f \sim g$. Let $\phi \in \mathscr{F}_f$; by definition of g we have $\phi \preccurlyeq g$ and so $\phi \preccurlyeq \phi_f$. ∎

2. Statement of the Results

In the following Ω will denote a fixed bounded open subset of $\mathbb{R}^n$. We denote by $\mathscr{A}$ the class of all open subsets of Ω; $\mathscr{P}$ the class of all piecewise affine functions on $\mathbb{R}^n$, i.e., the class of all Lipschitz functions u for which there exists a finite family $(\Omega_i)_{i \in I}$ of open subsets of $\mathbb{R}^n$ such that u is affine on each Ω_i and $\text{meas}(\mathbb{R}^n - \bigcup_{i \in I} \Omega_i) = 0$; $u_{s,z}$ the affine function $u_{s,z}(x) = s + \langle z, x \rangle$ ($s \in \mathbb{R}$, $z \in \mathbb{R}^n$); $I_\sigma(s)$ the open interval $]s - \sigma, s + \sigma[$ ($\sigma > 0$, $s \in \mathbb{R}$); $B_\rho(x)$ then open ball $\{y \in R^n: |x - y| < \rho\}$ ($\rho > 0$, $x \in \mathbb{R}^n$); 1_A the characteristic function of A, i.e.,

$$1_A(x) = \begin{cases} 1 & \text{if} \quad x \in A, \\ 0 & \text{otherwise;} \end{cases}$$

$|A|$ the Lebesgue measure of the set A. Let f be a general integrand; then we can define the functional

$$F(u, A) = \int_A f(u, Du)\, dx \tag{2.1}$$

for every $u \in \mathscr{P}$, $A \in \mathscr{A}$. If f is an integrand we have (see [5, Lemma 3]) that the functional F can be defined by (2.1) for every $u \in W^{1,1}_{\text{loc}}(\Omega)$, $A \in \mathscr{A}$.

Remark 2.1. Let f_1, f_2 be two general integrands; then it is immediate to see that

$$f_1 \preccurlyeq f_2 \Leftrightarrow \forall u \in \mathscr{P}, \qquad \forall A \in \mathscr{A} \int_A f_1(u, Du)\, dx \leqslant \int_A f_2(u, Du)\, dx.$$

If f_1, f_2 are integrands we have (see [5, Lemma 2])

$$f_1 \lll f_2 \Leftrightarrow \forall u \in W^{1,1}_{\text{loc}}(\Omega), \qquad \forall A \in \mathscr{A} \int_A f_1(u, Du)\, dx \leqslant \int_A f_2(u, Du)\, dx.$$

In the following, if (X, τ) is a topological space, $E \subseteq X$, $F: E \to \bar{\mathbb{R}}$, we denote by $\Gamma(\tau)F$ the function defined on the τ-closure $\bar{E}$ of E by

$$\Gamma(\tau)\,F(x) = \liminf_{y \to x} F(y) = \sup_{U \in \mathscr{I}(x)} \inf_{y \in U} F(y),$$

where $\mathscr{I}(x)$ stands for the family of all τ-neighborhoods of x in X. We can now state our relaxation results.

THEOREM 2.2. *Let f be a general integrand satisfying hypothesis (H_∞) and let $F(u, A)$ be the functional defined on $\mathscr{P} \times \mathscr{A}$ by (2.1). Then, for every $u \in \mathscr{P}$ and $A \in \mathscr{A}$*

$$\Gamma(L^1_{\mathrm{loc}}(A))\,F(u, A) = \int_A \phi_f(u, Du)\,dx,$$

where ϕ_f is the function defined in Proposition 1.9.

THEOREM 2.3. *Let f be an integrand and let $p \in [1, +\infty]$. For every $A \in \mathscr{A}$ and $u \in W^{1,p}(A)$ define*

$$F(u, A) = \int_A f(u, Du)\,dx.$$

Suppose that

(i) *f satisfies hypothesis (H_p);*

(ii) *the function $\phi_f(\cdot, 0)$ is continuous.*

Then, for every $A \in \mathscr{A}$ and $u \in W^{1,p}(A)$,

$$\Gamma(L^1_{\mathrm{loc}}(A))\,F(u, A) = \int_A \phi_f(u, Du)\,dx.$$

3. Proof of the Results

Let f be a general integrand satisfying hypothesis (H_∞) and let F be the functional defined on $\mathscr{P} \times \mathscr{A}$ by (2.1); we denote briefly by $\bar{F}(u, A)$ the functional $\Gamma(L^1_{\mathrm{loc}}(A))\,F(u, A)$. The following theorem holds.

THEOREM 3.1. *For every $u \in \mathscr{P}$ the set function $\bar{F}(u, \cdot)$ is the restriction to $\mathscr{A}$ of a regular Borel measure which is absolutely continuous with respect to the Lebesgue measure.*

Proof. It follows as in Section 3 of [2]. ▮

PROPOSITION 3.2. *For every* $x_0 \in \mathbb{R}^n$, $u \in L^1_{\text{loc}}(\Omega)$, $A \in \mathscr{A}$ *we have*

$$\bar{F}(u, A) = \bar{F}(u \circ \tau_{x_0}, \tau_{x_0}^{-1}(A)),$$

where $\tau_{x_0}\colon \mathbb{R}^n \to \mathbb{R}^n$ *is the map* $\tau_{x_0}(x) = x + x_0$.

Proof. It follows from the fact that the same property holds for F. ▮

LEMMA 3.3. *Let* $g \in L^1_{\text{loc}}(\mathbb{R})$ *and let* $\lambda > 0$; *for every* $s \in \mathbb{R}$ *and* $h \in \mathbb{N}$ *define*

$$M_h(s) = \frac{1}{2h} \sum_{i=-h}^{h-1} g\left(s + \frac{i\lambda}{h}\right), \qquad M(s) = \fint_{I_\lambda(s)} g(t)\, dt.$$

Then we have $M_h \to M$ *in* $L^1_{\text{loc}}(\mathbb{R})$.

Proof. If g is continuous, then the assertion follows immediately from Riemann's integrability theorem. In the general case, let $a > 0$ and for every $\varepsilon > 0$, let g_ε be a continuous function such that

$$\| g_\varepsilon - g \|_{L^1(-a-\lambda, a+\lambda)} \leqslant \varepsilon.$$

Set $M_h^\varepsilon(s) = (1/2h) \sum_{i=-h}^{h-1} g_\varepsilon(s + (i\lambda/h))$ and $M^\varepsilon(s) = \fint_{I_\lambda(s)} g_\varepsilon(t)\, dt$. Then for every $\varepsilon > 0$ we have

$$\begin{aligned}
&\limsup_h \|M_h - M\|_{L^1(-a,a)} \\
&\quad \leqslant \limsup_h \left[\|M^\varepsilon - M\|_{L^1(-a,a)} + \|M^\varepsilon - M_h^\varepsilon\|_{L^1(-a,a)} + \|M_h^\varepsilon - M_h\|_{L^1(-a,a)} \right] \\
&\quad \leqslant \limsup_h \left[\frac{a\varepsilon}{\lambda} + \frac{1}{2h} \sum_{i=-h}^{h-1} \int_{-a}^{a} \left| g\left(s + \frac{i\lambda}{h}\right) - g_\varepsilon\left(s + \frac{i\lambda}{h}\right) \right| ds \right] \\
&\quad \leqslant \frac{a\varepsilon}{\lambda} + \|g - g_\varepsilon\|_{L^1(-a-\lambda, a+\lambda)} \leqslant \frac{a\varepsilon}{\lambda} + \varepsilon.
\end{aligned}$$

From the arbitrariness of $\varepsilon > 0$ we get $\lim_h \|M_h - M\|_{L^1(-a,a)} = 0$. ▮

Define now for every $s \in \mathbb{R}$, $z \in \mathbb{R}^n$

$$\bar{f}(s, z) = \limsup_{\rho \to 0^+} \frac{\bar{F}(u_{s,z}, B_\rho(0))}{|B_\rho(0)|}.$$

THEOREM 3.4. *The function* $\bar{f}$ *satisfies the following conditions*:

(i) *hypothesis* (H_∞) *holds for* $\bar{f}$;

(ii) *for every* $z \in \mathbb{R}^n$ *the function* $\bar{f}(\cdot, z)$ *is measurable on* $\mathbb{R}$;

(iii) *for every* $u \in \mathscr{P}$, $A \in \mathscr{A}$ $\bar{F}(u, A) = \int_A \bar{f}(u, Du)\, dx$;

(iv) *the function* $\bar{f}(\cdot, 0)$ *is lower semicontinuous on* $\mathbb{R}$;

(v) *for a.a.* $s \in \mathbb{R}$ *the function* $\bar{f}(s, \cdot)$ *is convex on* $\mathbb{R}^n$.

Proof. Property (i) follows immediately from the definition of $\bar{f}$ and from the fact that hypothesis (H_∞) holds for f. Let $z \in \mathbb{R}^n$; for every $\rho > 0$ the function $s \mapsto \bar{F}(u_{s,z}, B_\rho(0))/|B_\rho(0)|$ is lower semicontinuous on $\mathbb{R}$; then for every $h \in \mathbb{N}$ the function

$$\alpha_h(s) = \sup\left\{\frac{\bar{F}(u_{s,z}, B_\rho(0))}{|B_\rho(0)|} : 0 < \rho < \frac{1}{h}\right\}$$

is lower semicontinuous on $\mathbb{R}$, and so from the equality

$$\bar{f}(s, z) = \inf\{\alpha_h(s) : h \in \mathbb{N}\}$$

we get (ii). From Theorem 3.1, for every $s \in \mathbb{R}$, $z \in \mathbb{R}^n$, $A \in \mathscr{A}$, we have

$$\bar{F}(u_{s,z}, A) = \int_A g_{s,z}(x)\, dx,$$

where $g_{s,z}$ is a suitable function belonging to $L^1(\Omega)$. Now, let $s \in \mathbb{R}$, $z \in \mathbb{R}^n$, $A \in \mathscr{A}$; since $g_{s,z} \in L^1(\Omega)$, for a.a. $x \in \Omega$

$$g_{s,z}(x) = \lim_{\rho \to 0^+} \fint_{B_\rho(x)} g_{s,z}(y)\, dy;$$

thus by using Proposition 3.2, we have

$$\begin{aligned}\int_A \bar{f}(s + \langle z, x\rangle, z)\, dx &= \int_A \left[\limsup_{\rho \to 0^+} \fint_{B_\rho(0)} g_{s+\langle z,x\rangle, z}(y)\, dy\right] dx \\ &= \int_A \left[\limsup_{\rho \to 0^+} \fint_{B_\rho(x)} g_{s,z}(y)\, dy\right] dx \\ &= \int_A g_{s,z}(x)\, dx = \bar{F}(u_{s,z}, A).\end{aligned}$$

Therefore, by using Theorem 3.1, property (iii) is proved. To prove property (iv) consider $s_h, s \in \mathbb{R}$ with $s_h \to s$; then $u_{s_h,0} \to u_{s,0}$ and so, by using property (iii), for every $A \in \mathscr{A}$ we have

$$|A|\, \bar{f}(s, 0) = \bar{F}(u_{s,0}, A) \leqslant \liminf_h \bar{F}(u_{s_h,0}, A) = |A| \liminf_h \bar{f}(s_h, 0).$$

Finally we prove (v). Let $s \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^n$ with $z_1 \neq z_2$, $\lambda \in]0, 1[$ and set $z = \lambda z_1 + (1-\lambda) z_2$; suppose that $z \neq 0$, $z_1 \neq 0$, $z_2 \neq 0$. We shall prove that

$$\bar{f}(s, z) \leqslant \lambda \bar{f}(s, z_1) + (1-\lambda) \bar{f}(s, z_2). \tag{3.1}$$

Let $z_0 = (z_2 - z_1)/|z_2 - z_1|$; for every $h \in \mathbb{N}$, $j \in \mathbb{Z}$ set

$$\Omega_{hj}^1 = \left\{ y \in \mathbb{R}^n : \frac{j-1}{h} < \langle z_0, y \rangle < \frac{j-1+\lambda}{h} \right\},$$

$$\Omega_{hj}^2 = \left\{ y \in \mathbb{R}^n : \frac{j-1+\lambda}{h} < \langle z_0, y \rangle < \frac{j}{h} \right\},$$

$$\Omega_h^1 = \bigcup_{j \in \mathbb{Z}} \Omega_{hj}^1, \quad \Omega_h^2 = \bigcup_{j \in \mathbb{Z}} \Omega_{hj}^2.$$

Note that, in the weak* topology of $L^\infty(\Omega)$, the sequences $(1_{\Omega_h^1})$ and $(1_{\Omega_h^2})$ converge to λ and to $(1-\lambda)$, respectively. Let (u_h) be the sequence of functions of $\mathscr{P}$ defined by

$$u_h(y) = \begin{cases} s + c_{hj}^1 + \langle z_1, y \rangle & \text{if } \ y \in \Omega_{hj}^1, \\ s + c_{hj}^2 + \langle z_2, y \rangle & \text{if } \ y \in \Omega_{hj}^2, \end{cases} \tag{3.2}$$

where $c_{hj}^1 = ((j-1)(1-\lambda)/h)\,|z_2 - z_1|$ and $c_{hj}^2 = -(jh/h)\,|z_2 - z_1|$. For every $y \in \Omega_{hj}^1$,

$$|u_h(y) - u_{s,z}(y)| = |c_{hj}^1 + \langle z_1 - z, y \rangle| = (1-\lambda)\,|z_2 - z_1| \left| \frac{j-1}{h} - \langle z_0, y \rangle \right|$$
$$\leqslant \frac{\lambda(1-\lambda)}{h}\,|z_2 - z_1|.$$

Analogously, for every $y \in \Omega_{hj}^2$,

$$|u_h(y) - u_{s,z}\ (y)| \leqslant \frac{\lambda(1-\lambda)}{h}\,|z_2 - z_1|;$$

therefore (u_h) converges to $u_{s,z}$ in $L^\infty(\mathbb{R}^n)$ and weakly* in $W^{1,\infty}(\Omega)$. By Lusin's theorem, for every $\delta > 0$ there exist an open set $A_\delta \subseteq \mathbb{R}$ with $|A_\delta| < \delta$ and two continuous functions g_1, g_2 from $\mathbb{R}$ into $[0, +\infty[$ such that for every $s \in \mathbb{R} - A_\delta$ we have

$$\bar{f}(s, z_1) = g_1(s) \qquad \text{and} \qquad \bar{f}(s, z_2) = g_2(s).$$

Since $\bar{f}$ satisfies hypothesis (H_∞), there exists a constant c such that

$$\bar{F}(u_{s,z}, B_\rho(0)) \leqslant \liminf_h \int_{B_\rho(0)} \bar{f}(u_h, Du_h)\,dx$$
$$= \liminf_h \left[\int_{B_\rho(0) \cap \Omega_h^1} \bar{f}(u_h, z_1)\,dx + \int_{B_\rho(0) \cap \Omega_h^2} \bar{f}(u_h, z_2)\,dx \right]$$
$$\leqslant \liminf_h \left[\int_{B_\rho(0) \cap \Omega_h^1} g_1(u_h)\,dx + \int_{B_\rho(0) \cap \Omega_h^2} g_2(u_h)\,dx + c\,|B_\rho(0) \cap u_h^{-1}(A_\delta)| \right]. \tag{3.3}$$

It is not difficult to see that there exists a constant $c_1 > 0$ independent of h such that for every $h \in \mathbb{N}$

$$|B_\rho(0) \cap u_{s,z}^{-1}(A_\delta)| \leqslant c_1 \delta \qquad \text{and} \qquad |B_\rho(0) \cap u_h^{-1}(A_\delta)| \leqslant c_1 \delta.$$

Then, by using (3.3), for a suitable constant $c_2 > 0$

$$\bar{F}(u_{s,z}, B_\rho(0))$$
$$\leqslant \lambda \int_{B_\rho(0)} g_1(s + \langle z, x\rangle)\,dx + (1-\lambda) \int_{B_\rho(0)} g_2(s + \langle z, x\rangle)\,dx + cc_1\delta$$
$$\leqslant \lambda \int_{B_\rho(0)} \bar{f}(s + \langle z, x\rangle, z_1)\,dx + (1-\lambda) \int_{B_\rho(0)} \bar{f}(s + \langle z, x\rangle, z_2)\,dx + c_2\delta$$
$$= \left(\frac{|z_1|}{|z|}\right)^n \int_{B_{\rho(|z|/|z_1|)}(0)} \bar{f}(s + \langle z_1, x\rangle, z_1)\,dx$$
$$+ (1-\lambda)\left(\frac{|z_2|}{|z|}\right)^n \int_{B_{\rho(|z|/|z_2|)}(0)} \bar{f}(s + \langle z_2, x\rangle, z_2)\,dx + c_2\delta.$$

Since δ is arbitrary, we obtain

$$\fint_{B_\rho(0)} \bar{f}(s + \langle z, x\rangle, z)\,dx \leqslant \lambda \fint_{B_{\rho(|z|/|z_1|)}(0)} \bar{f}(s + \langle z_1, x\rangle, z_1)\,dx$$
$$+ (1-\lambda) \fint_{B_{\rho(|z|/|z_2|)}(0)} \bar{f}(s + \langle z_2, x\rangle, z_2)\,dx$$

and so, taking the limit as $\rho \to 0^+$, we get (3.1). We prove now that for every $s \in \mathbb{R}$ there exists

$$\gamma(s) = \lim_{\substack{z \to 0 \\ z \neq 0}} \bar{f}(s, z).$$

We argue by contradiction: suppose that two sequences (z_h^1) and (z_h^2) exist which converge to 0 such that for a suitable $s \in \mathbb{R}$,

$$\lim_h \bar{f}(s, z_h^1) < \lim_h \bar{f}(s, z_h^2). \tag{3.4}$$

By (3.1) and by hypothesis (H_∞) the function $\bar{f}(s,\cdot)$ is continuous on $\mathbb{R}^n - \{0\}$, so we can assume that (z_h^1) and (z_h^2) are such that for every $h \in \mathbb{N}$ the straight line 1_h joining z_h^1 to z_h^2 does not contain the origin. Then, by (3.1), the restriction of $\bar{f}(s, \cdot)$ to 1_h is convex and so for every h sufficiently large there exists z_h^3 with $|z_h^3| = 1$ and such that

$$\bar{f}(s, z_h^3) \geqslant \bar{f}(s, z_h^1) + |z_h^3 - z_h^1| \frac{\bar{f}(s, z_h^2) - \bar{f}(s, z_h^1)}{|z_h^2 - z_h^1|}.$$

Taking the limit as $h \to +\infty$ we have, by (3.4),

$$\lim_h \bar{f}(s, z_h^3) = +\infty$$

which contradicts hypothesis (H_∞). Now, define

$$\tilde{f}(s, z) = \begin{cases} \bar{f}(s, z), & \text{if } \quad z \neq 0, \\ \gamma(s), & \text{if } \quad z = 0; \end{cases}$$

we have proved that for every $s \in \mathbb{R}$ the function $\tilde{f}(s, \cdot)$ is convex on $\mathbb{R}^n$, so to conclude the proof of property (v) it is enough to prove that $\bar{f}(s, 0) = \tilde{f}(s, 0)$ for a.a. $s \in \mathbb{R}$. We prove first that

$$\bar{f}(s, 0) \leqslant \tilde{f}(s, 0) \qquad \text{for} \quad \text{a.a. } s \in \mathbb{R}. \tag{3.5}$$

Let $s \in \mathbb{R}$, $\varepsilon > 0$, and let (u_h) be the sequence defined in (3.2) with $z_1 = (-\varepsilon, 0, ..., 0)$, $z_2(\varepsilon, 0, ..., 0)$, $\lambda = \frac{1}{2}$. Let Q be the cube of $\mathbb{R}^n$ given by $Q = \{x \in \mathbb{R}^n \colon |x_i| < 1 \text{ for } i = 1, ..., n\}$; then, setting $v_h(x) = u_h(x) + (\varepsilon/4h)$, we have

$$\begin{aligned} 2^n \bar{f}(s, 0) = \bar{F}(u_{s,0}, Q) &\leqslant \liminf_h \bar{F}(v_h, Q) \\ &= \liminf_h \left[\int_{\Omega \cap \Omega_h^1} \tilde{f}(v_h, z_1)\, dx + \int_{\Omega \cap \Omega_h^2} \tilde{f}(v_h, z_2)\, dx \right. \\ &\leqslant \omega(\varepsilon) + \liminf_h \int_Q \tilde{f}(v_h, 0)\, dx, \end{aligned} \tag{3.6}$$

where $\lim_{\varepsilon \to 0^+} \omega(\varepsilon) = 0$ and the last inequality follows from the convexity of $\bar{f}(s, z)$ with respect to z and from hypothesis (H_∞). By (3.6), it follows that

$$2^n\bar{f}(s, 0) \leqslant \omega(\varepsilon) + \liminf_h \sum_{j=-h+1}^{h} \left\{ \int_{\Omega \cap \Omega_{hj}^1} \bar{f}(s + (j - \tfrac{3}{4}/h)\, \varepsilon - \varepsilon x_1, 0)\, dx \right.$$

$$\left. + \int_{Q \cap \Omega_{hj}^2} \bar{f}(s - (j - \tfrac{1}{4}/h)\, \varepsilon + \varepsilon x_1, 0)\, dx \right\}$$

$$= \omega(\varepsilon) + 2^{n-1} \liminf_h \frac{4h}{\varepsilon} \int_{s-(\varepsilon/4h)}^{s+(\varepsilon/4h)} \bar{f}(t, 0)\, dt$$

$$= \omega(\varepsilon) + 2^n \liminf_h \fint_{s-(\varepsilon/4h)}^{s+(\varepsilon/4h)} \bar{f}(t, 0)\, dt. \tag{3.7}$$

Therefore, if s is a Lebesgue point of $\bar{f}(\cdot, 0)$ we have

$$\bar{f}(s, 0) \leqslant 2^{-n}\omega(\varepsilon) + \bar{f}(s, 0)$$

and thus, since ε is arbitrary, we get (3.5). Finally, we prove that

$$\bar{f}(s, 0) \leqslant f(s, 0) \qquad \text{for a.a. } s \in \mathbb{R}. \tag{3.8}$$

Let $s \in \mathbb{R}$, $\lambda \in]0, 1[$, let (u_h) be the sequence defined in (3.2) with $z_1 = (1, 0, ..., 0)$, $z_2 = (0, 0, ..., 0)$, and let Q be the cube of $\mathbb{R}^n$ given by $Q = \{x \in \mathbb{R}^n\colon |x_i| < 1 \text{ for } i = 1, ..., n\}$. Then, setting $u(x) = s + \langle \lambda z_1, x \rangle$, we have, for a suitable constant $c > 0$ and a suitable function $\omega(\lambda)$ with $\lim_{\lambda \to 0^+} \omega(\lambda) = 0$

$$\fint_{I_\lambda(s)} \bar{f}(t, 0)\, dt$$

$$\leqslant \fint_{I_\lambda(s)} [\omega(\lambda) + \bar{f}(t, \lambda z_1)]\, dt$$

$$= \omega(\lambda) + \fint_{I_1(0)} \bar{f}(s + \lambda x_1, \lambda z_1)\, dx_1 = \omega(\lambda) + \fint_Q \bar{f}(u, Du)\, dx$$

$$= \omega(\lambda) + \frac{1}{|Q|} \bar{F}(u, Q) \leqslant \omega(\lambda) + \frac{1}{|Q|} \liminf_h \bar{F}(u_h, Q)$$

$$= \omega(\lambda) + \frac{1}{|Q|} \liminf_h \left[\int_{Q \cap \Omega_h^1} f(u_h, z_1)\, dx + \int_{Q \cap \Omega_h^2} f(u_h, 0)\, dx \right]$$

$$\leqslant \omega(\lambda) + \frac{1}{|Q|} \liminf_h \left[c\, |Q \cap \Omega_h^1| + \sum_{j=1-h}^{h} f\left(s - \frac{j\lambda}{h}, 0\right) |Q \cap \Omega_{hj}^2| \right]$$

$$= \omega(\lambda) + c\lambda + (1 - \lambda) \liminf \frac{1}{2h} \sum_{i=-h}^{h-1} f\left(s + \frac{i\lambda}{h}, 0\right). \tag{3.9}$$

Using Lemma 3.3 we obtain that there exist a null subset N of $\mathbb{R}$ and a subsequence (h_k) such that

$$\lim_k \frac{1}{2h_k} \sum_{i=-h_k}^{h_k-1} f\left(s+\frac{i\lambda}{h_k}, 0\right) = ⨍_{I_\lambda(s)} f(t, 0)\, dt$$

for every $\lambda \in \mathbb{Q} \cap]0, 1[$ and $s \in \mathbb{R} - N$. Therefore by (3.9), for every $\lambda \in \mathbb{Q} \cap]0, 1[$ and $s \in \mathbb{R} - N$

$$⨍_{I_\lambda(s)} \bar{f}(t, 0)\, dt \leqslant \omega(\lambda) + c\lambda + (1-\lambda) ⨍_{I_\lambda(s)} f(t, 0)\, dt;$$

so, in the limit as $\lambda \to 0^+$ we get (3.8), and the proof of Theorem 3.4 is achieved. ∎

Proof of Theorem 2.2. In Theorem 3.4 we have proved that for every $u \in \mathscr{P}$, $A \in \mathscr{A}$

$$\bar{F}(u, A) = \int_A \bar{f}(u, Du)\, dx$$

with $\bar{f}$ convex l.s.c. integrand. Since $\bar{F} \leqslant F$, by Remark 2.1 we have that $\bar{f} \in \mathscr{F}_f$ and so $\bar{f} \preccurlyeq \phi_f$. On the other hand, the functional $\Phi(u, A) = \int_A \phi_f(u, Du)\, dx$ is $L^1_{\text{loc}}(A)$-lower semicontinuous (see [5], Proposition 2.7 and Theorem 1) and $\Phi \leqslant F$; then $\Phi \leqslant \bar{F}$ and so $\phi_f \preccurlyeq \bar{f}$. Therefore $\phi_f \sim \bar{f}$ and thus for every $u \in \mathscr{P}$, $A \in \mathscr{A}$ we have

$$\bar{F}(u, A) = \int_A \phi_f(u, Du)\, dx. \quad ∎$$

Proof of Theorem 2.3. For every $A \in \mathscr{A}$ and $u \in W^{1,p}(A)$ define

$$G(u, A) = \Gamma(L^1_{\text{loc}}(A))\, F(u, A);$$

$$\Phi(u, A) = \int_A \phi_f(u, Du)\, dx.$$

Since ϕ_f is a convex integrand, there exists a null subset N of $\mathbb{R}$ such that, for every $s \notin N$ the function $\phi_f(s, \cdot)$ is convex on $\mathbb{R}^n$. Set for every $s \in \mathbb{R}$, $z \in \mathbb{R}^n$

$$\alpha(s, z) = \begin{cases} \phi_f(s, 0), & \text{if } s \in N, \\ \phi_f(s, z), & \text{otherwise.} \end{cases}$$

We have $\alpha \approx \phi_f$; moreover, by hypotheses i) and ii) of Theorem 2.3, the function α satisfies all conditions of Theorem 2.1 of [3]. Let $A \in \mathscr{A}$ and let

$u \in W^{1,p}(A)$; Let B be an open set such that $\bar{B} \subseteq A$ and let $v \in W^{1,p}(R^n)$ with compact support in A such that $u = v$ a.e. on B. Let (v_h) be a sequence in $\mathscr{P}$ strongly converging to v in $W^{1,p}(\mathbb{R}^n)$ (in the case $p = +\infty$ we choose (v_h) converging to v in $L^\infty(\mathbb{R}^n)$, with $Dv_h(x) \to Dv(x)$ a.e. in $\mathbb{R}^n$ and such that $\sup_h \|Dv_h\|_{L^\infty} < +\infty$). Then, by Theorem 2.2 and by Theorem 2.1 of [3], we have

$$\begin{aligned} G(u, B) = G(v, B) &\leqslant \liminf_h G(v_h, B) = \liminf_h \Phi(v_h, B) \\ &= \liminf_h \int_B \alpha(v_h, Dv_h)\, dx = \int_B \alpha(v, Dv)\, dx \\ &= \int_B \phi_f(v, Dv)\, dx = \int_B \phi_f(u, Du)\, dx. \end{aligned}$$

Since the function $G(u, \cdot)$ is the restriction to the open subsets of A of a regular Borel measure on A (see Section 3 of [2]), we have

$$G(u, A) \leqslant \int_A \phi_f(u, Du)\, dx. \tag{3.10}$$

On the other hand, the functional $\Phi(\cdot, A)$ is $L^1_{\text{loc}}(A)$-lower semicontinuous (see [5, Theorem 1]); therefore

$$\Phi(u, A) \leqslant G(u, A)$$

which, together with (3.10), completes the proof. ∎

4. Further Remarks

We begin this section by giving an explicit characterization of the function ϕ_f. In the following, given a measurable function $\gamma: \mathbb{R} \to \bar{\mathbb{R}}$, we define (see [7, p. 159]) for every $s_0 \in \mathbb{R}$,

$$\gamma^-(s_0) = \sup\{t \in \mathbb{R}\colon \text{the set } \{s\colon \gamma(s) < t\} \text{ has density 0 at } s_0\}.$$

Lemma 4.1. *Let $\gamma \in L^\infty_{\text{loc}}(\mathbb{R})$ be a nonnegative function. Then, for every $s_0 \in \mathbb{R}$ and every $\rho > 0$ we have*

$$\inf \left\{ \liminf_h \fint_{I_{\rho/h}(0)} \gamma(s + s_h)\, ds \right\} \leqslant \gamma^-(s_0)$$

where the infimum is taken over all sequences (s_h) converging to s_0.

Proof. Let $s_0 \in \mathbb{R}$ and let $t > \gamma^-(s_0)$; then the set $E = \{s: \gamma(s) < t\}$ has the property

$$\limsup_{\sigma \to 0^+} \frac{|E \cap I_\sigma(s_0)|}{|I_\sigma(s_0)|} > 0.$$

Therefore, for a suitable sequence (σ_h) converging to 0 and a suitable $\delta > 0$, we have

$$|E \cap I_{\sigma_h}(s_0)| \geqslant 2\delta\sigma_h \qquad \text{for every} \quad h \in \mathbb{N}. \tag{4.1}$$

By (4.1), for every $h \in \mathbb{N}$ there exists $s_h \in E \cap I_{\sigma_h}(s_0)$ such that

$$\lim_{\tau \to 0^+} \frac{|E \cap I_{\sigma_h}(s_0) \cap I_\tau(s_h)|}{|I_\tau(s_h)|} = 1. \tag{4.2}$$

Let $\rho > 0$; for every $k \in \mathbb{N}$ we have

$$\begin{aligned} ⨍_{I_{\rho/k}(0)} \gamma(s + s_h)\, ds &= \frac{k}{2\rho}\left[\int_{I_{\rho/k}(s_h) \cap E \cap I_{\sigma_h}(s_0)} \gamma(s)\, ds + \int_{I_{\rho/k}(s_h) - (E \cap I_{\sigma_h}(s_0))} \gamma(s)\, ds\right] \\ &\leqslant t + \|\gamma\|_{L^\infty}\left[1 - \frac{|I_{\rho/k}(s_h) \cap E \cap I_{\sigma_h}(s_0)|}{|I_{\rho/k}(s_h)|}\right]. \end{aligned}$$

By (4.2), for a suitable sequence (k_h) of integers, we get

$$\liminf_h ⨍_{I_{\rho/k_h}(0)} \gamma(s + s_h)\, ds \leqslant t;$$

hence the thesis follows from the arbitrariness of $t > \gamma^-(s_0)$. ∎

Given a function $g: \mathbb{R} \times \mathbb{R}^n \to \overline{\mathbb{R}}$ we denote by $g^{**}(s, z)$ the greatest function convex in z which is less than or equal to $g(s, z)$. Let $f: \mathbb{R} \times \mathbb{R}^n \to [0, +\infty]$ be a function; by using Remark 1.6 we define

$$\gamma(s) = \operatorname{co}(f)(s, 0);$$

$$f_1(s, z) = \begin{cases} \operatorname{co}(f)(s, z), & \text{if} \quad z \neq 0, \\ \liminf\limits_{t \to s}[\gamma^-(t) \wedge f(t, 0)], & \text{if} \quad z = 0; \end{cases}$$

$$f_2(s, z) = \begin{cases} \operatorname{co}(f_1)(s, z), & \text{if} \quad z \neq 0, \\ f_1(s, 0) & \text{if} \quad z = 0. \end{cases}$$

THEOREM 4.2. *If f is a general integrand satisfying hypothesis* (H_∞), *then* $\phi_f \sim f_2$.

Proof. We shall prove Theorem 4.2 in several steps.

Step 1. For every $\phi \in \mathscr{F}_f$ we have $\phi \preccurlyeq f_1$. By definition of the class $\mathscr{F}_f$ and by Remark 1.6, it is enough to prove that

$$\phi(s, 0) \leqslant \gamma^-(s) \qquad \text{for every } s \in \mathbb{R}. \tag{4.3}$$

Let $s \in \mathbb{R}$, $\varepsilon > 0$ and let (s_h) be a sequence converging to s; let Q be the cube of $\mathbb{R}^n$ given by $Q = \{x \in \mathbb{R}^n\colon |x_i| < 1 \text{ for } i = 1,..., n\}$ and let (u_h) be the sequence defined in (3.2) with $z_1 = (-\varepsilon, 0,..., 0)$, $z_2 = (\varepsilon, 0,..., 0)$, $\lambda = \frac{1}{2}$. Set $w_h = u_h + (\varepsilon/4h) + s_h - s$; then, by using the L^1_{loc}-lower semicontinuity of the functional $\int_Q \phi(u, Du)\, dx$ (see [5] Theorem 1) and by arguing as in (3.6) we get

$$\begin{aligned} |Q|\, \phi(s, 0) &\leqslant \liminf_h \int_Q \phi(w_h, Dw_h)\, dx \\ &\leqslant \liminf_h \left[\int_{Q \cap \Omega_h^1} f_1(w_h, z_1)\, dx + \int_{Q \cap \Omega_h^2} f_1(w_h, z_2)\, dx \right] \\ &\leqslant \omega(\varepsilon) + \liminf_h \int_Q \gamma(w_h)\, dx, \end{aligned}$$

where in the last inequality we have used Proposition 1.8. Therefore, with the same calculations used in (3.7) we obtain

$$\phi(s, 0) \leqslant 2^{-n} \omega(\varepsilon) + \liminf_h \fint_{I_{\varepsilon/4h}(0)} \gamma(t + s_h)\, dt.$$

Since (s_h) is arbitrary, by Lemma 4.1

$$\phi(s, 0) \leqslant 2^{-n} \omega(\varepsilon) + \gamma^-(s),$$

and so, taking the limit as $\varepsilon \to 0^+$, we get (4.3).

Step 2. For every $\phi \in \mathscr{F}_f$ we have $\phi \preccurlyeq f_2$. It follows immediately from the definition of the class $\mathscr{F}_f$ and from Remark 1.6.

Step 3. The function f_2 is a convex l.s.c. general integrand. By Proposition 1.8 there exists a convex integrand $\phi \sim \text{co}(f)$; define

$$\phi_1(s, z) = \begin{cases} \phi(s, z), & \text{if } z \neq 0, \\ f_1(s, 0), & \text{if } z = 0; \end{cases}$$

$$\phi_2(s, z) = \begin{cases} \phi_1^{**}(s, z), & \text{if } z \neq 0, \\ f_1(s, 0), & \text{if } z = 0. \end{cases}$$

By the definition of f_1 we have

$$f_1(s, 0) \leqslant \phi(s, 0) \qquad \text{for a.a. } s \in \mathbb{R}. \tag{4.4}$$

By [8, Proposition 7, p. 329] and by (4.4), the function ϕ_2 is a convex l.s.c. integrand. Fix $z \in \mathbb{R}^n$; we obtain easily

$$\operatorname{co}(f_1)(s, z) = \operatorname{co}(\phi_1)(s, z) = \phi_1^{**}(s, z) \qquad \text{for a.a. } s \in \mathbb{R};$$

thus $\phi_2 \sim f_2$ and so the proof of Step 3 is achieved.

Step 4. We have $f_2 \sim \phi_f$. By Step 2 it follows that $\phi_f \preccurlyeq f_2$. On the other hand, if ϕ_2 is the convex l.s.c. integrand defined in Step 3, we have $\phi_2 \preccurlyeq f$; thus, by definition of ϕ_f, we get $\phi_2 \preccurlyeq \phi_f$ and so $f_2 \preccurlyeq \phi_f$. ∎

Remark 4.3. Let f be an integrand satisfying hypothesis (H_∞). We can define

$$\beta(s) = f^{**}(s, 0);$$

$$g_1(s, z) = \begin{cases} f^{**}(s, z), & \text{if } z \neq 0, \\ \liminf\limits_{t \to s} [\beta^-(t) \wedge f(t, 0)] & \text{if } z = 0; \end{cases}$$

$$g_2(s, z) = \begin{cases} g_1^{**}(s, z), & \text{if } z \neq 0, \\ g_1(s, 0), & \text{if } z = 0. \end{cases}$$

It is easy to prove that $\beta = \gamma$ a.e. on $\mathbb{R}$, $g_1 \sim f_1$, $g_2 \sim f_2$. Therefore, since g_2 is a convex l.s.c. integrand (see [8, Proposition 7, p. 329]), we have $g_2 \approx \phi_f$.

EXAMPLE 4.4. *Let* $A(s) = (a_{ij}(s))$ be a measurable symmetric $n \times n$ matrix such that

$$0 \leqslant \sum_{i,j} a_{ij}(s)\, z_i z_j \leqslant \Lambda\, |z|^2 \qquad \text{for every } s \in \mathbb{R},\ z \in \mathbb{R}^n.$$

Let $g\colon \mathbb{R} \to \mathbb{R}$ be a Borel function such that

$$0 \leqslant g(s) \leqslant c(1 + |s|^2) \qquad \text{for every} \quad s \in \mathbb{R}.$$

The function

$$f(s, z) = \langle A(s)\, z, z \rangle + g(s)$$

is a convex integrand satisfying condition (H_2). Denote by $\bar{g}$ the function

$$\bar{g}(s) = \liminf_{t \to s} g(t).$$

Then, by applying Remark 4.3, we get easily

$$\phi_f(s, z) = \begin{cases} \langle A(s)\, z, z \rangle + g(s), & \text{if } \langle A(s) z, z \rangle \geqslant g(s) - \bar{g}(s) \\ \bar{g}(s) + 2\sqrt{g(s) - \bar{g}(s)}\, \sqrt{\langle A(s) z, z \rangle}, & \text{if } \langle A(s) z, z \rangle < g(s) - \bar{g}(s). \end{cases}$$

EXAMPLE 4.5 (See [9, Example 3.9]). Let $n=1$ and let

$$f(s,z)=(1+|1+z|)^{|s|}.$$

Then, by applying Remark 4.3, we get easily

$$\phi_f(s,z)=\begin{cases} f(s,z) & \text{if} \quad |s|>1, \\ 1 & \text{if} \quad |s|\leqslant 1. \end{cases}$$

Remark 4.6. If Ω satisfies the cone property, then by using the Sobolev imbedding theorems, it is possible to prove Theorem 2.3 even if the function f verifies the following weaker estimates instead of the hypothesis (H_p):

$$\text{if} \quad p>n \qquad 0\leqslant f(s,z)\leqslant c(\psi(s)+|z|^p),$$

where $\psi\colon \mathbb{R}\to [0,+\infty[$ is a continuous function;

$$\text{if} \quad p=n, \qquad 0\leqslant f(s,z)\leqslant c(1+|s|^k+|z|^p) \qquad \text{where } k\in[1,+\infty[;$$

$$\text{if} \quad p<n, \qquad 0\leqslant f(s,z)\leqslant c(1+|s|^k+|z|^p) \qquad \text{where } k=\frac{np}{n-p}.$$

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