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SEPARATION THEOREMS FOR ORIENTED MATROIDS

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In this paper we show that Minty's lemma can be used to prove the Hahn–Banach theorem as well as other theorems in this class such as Radon's and Helly's theorem for oriented matroids having an intersection property which guarantees that every pair of flats intersects in some point extension $\mathcal{O}\cup p$ of the oriented matroid $\mathcal{O}$.

1. Introduction

In [12] Las Vergnas introduced the notion of convexity for oriented matroids in order to study analogues of the Hahn–Banach theorem. Cordovil [5, 6] proved versions of the Hahn–Banach theorem for oriented matroids of rank three. As Mandel [13] pointed out this theorem is no longer true for oriented matroids of rank greater than three. In this paper we show that Minty's lemma can be used to prove a slightly stronger form of the Hahn–Banach theorem as well as other theorems in this class such as Radon's and Helly's theorem for oriented matroids of any rank provided the oriented matroids have a so-called intersection property. This intersection property (IP) came across when investigating polars of oriented matroids. It defines a class of oriented matroids which is contained in the more general class of Euclidean matroids and which includes those oriented matroids having an oriented adjoint (i.e. allowing the construction of polars).

Loosely speaking the intersection property guarantees that every pair of flats intersects in some point extension $\mathcal{O}\cup p$ of $\mathcal{O}$. In case of unoriented matroids of rank 4 this is equivalent to what geometers call the bundle condition (cf. Kern [10]).

Let $\mathcal{O}$ be an oriented matroid. We consider the circuits of $\mathcal{O}$ as vectors X of $2^{\pm E}:=\{+,-,0\}^E$ and as usual write $\underline{X}(X^+, X^-, X^0)$ for the set of all $e\in E$ for which $X_e\neq 0$ ($X_e=+$, $X_e=-$, $X_e=0$) and also use the notation $X_A\geqslant 0$ ($\leqslant 0$) if $X_e\in\{+,0\}\forall e\in A$ ($X_e\in\{-,0\}\forall e\in A$) respectively $X\geqslant 0(\leqslant 0)$ if $A=E$. A vector $X\in 2^{\pm E}$ is called a cell of $\mathcal{O}$ (resp. cocell) if X is an element of the circuit span (resp. cocircuit span) of $\mathcal{O}$. Note that a vector of $X\in 2^{\pm E}$ is a cocell of $\mathcal{O}$ if and only if it is orthogonal to all circuits C of $\mathcal{O}$. For the sake of simplicity we shall

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always assume that $\mathcal{O}$ is acyclic, i.e. it has no circuit $X \geq 0$. Clearly any oriented matroid can be reoriented to an acyclic one.

2. Hahn–Banach theorems

To state the Hahn–Banach theorem we need the notation of convexity and separability. Two sets $A, B \subseteq E$ are called separable if $Y_A \geq 0$ and $Y_B \leq 0$ for some cocircuit Y of $\mathcal{O}$. In other words A and B are separable if A and B lie on opposite sides of the hyperplane Y^0 of $\mathcal{O}$. The convex hull of $A \subseteq E$ is defined as

$$\mathrm{conv}_{\mathcal{O}}(A) := A \cup \{e \in E \setminus A \mid \exists \text{ circuit } X \text{ of } \mathcal{O} \text{ with } X^- = \{e\}, X^+ \subseteq A\}$$

(cf. Las Vergnas [12]).

Theorem 2.1 (*Cordovil* [5]). *Let $\mathcal{O}$ be an acyclic oriented matroid of rank at most three and let $A, B \subseteq E$. If for all point extensions $\mathcal{O}' := \mathcal{O} \cup p$ $\mathrm{conv}_{\mathcal{O}'}(A) \cap \mathrm{conv}_{\mathcal{O}'}(B) = \emptyset$ then A and B are separable.*

Mandel [13] and Fukuda [8] gave examples of oriented matroids of rank greater than three which disprove this form of the Hahn–Banach theorem for arbitrary oriented matroids. Interesting enough these examples were also used to construct oriented matroid programming problems where the simplex algorithm cycles through nondegenerate pivots. Edmonds and Fukuda invented the class of BOM's (Bland-oriented-matroids) to exclude those possible cyclings. It turned out that the Bland-oriented-matroids are in fact euclidean oriented matroids introduced by Edmonds and Mandel (cf [13]), i.e. oriented matroids which do have hyperplanes through a given point and 'parallel' to a given hyperplane.

Hence the Hahn–Banach Theorem cannot be proved for all non-euclidean oriented matroids. Here we shall use the intersection property (which implies euclidean) to prove a slightly stronger form of Theorem 2.1 for oriented matroids of arbitrary rank.

We call two sets $A, B \subseteq E$ strongly separable if $Y_A \geq 0$, $Y_B \leq 0$ and $Y^0 = \emptyset$ for a cocell Y of $\mathcal{O}$. Since every cocell of an oriented matroid is the conformal sum of cocircuits (cf. [4]) strongly separable implies separable. In [6] Cordovil and Duchet proved that Theorem 2.1 is still valid if 'separable' is replaced by 'strongly separable'.

We say $\mathcal{O}$ has the intersection property (IP) if for every nonmodular pair of flats F, G (i.e. $r(F \vee G) + r(F \wedge G) < r(F) + r(G)$ where '$\vee$' stands for join and '$\wedge$' for meet) of rank at least 2 there exists a point extension $\mathcal{O}' := \mathcal{O} \cup p$ with $r_{\mathcal{O}}(F) = r_{\mathcal{O}'}(F \vee p)$ and $r_{\mathcal{O}}(G) = r_{\mathcal{O}'}(G \vee p)$ such that $r_{\mathcal{O}}(F \wedge G) < r_{\mathcal{O}'}$ $([F \vee p] \wedge [G \vee p])$. Informally the intersection property allows any two flats of $\mathcal{O}$ to intersect (either in a given point $e \in E$ or in a point extension p of $\mathcal{O}$.)

The intersection property as defined above is one of several intersection properties including Levi's and Euclidean intersection property. In [1] we discuss the classes of matroids having these intersection properties. There it is shown that representable and all rank three matroids do have the intersection properties. Graphic matroids and other important classes of matroids do satisfy the intersection property as was shown in [7]. However, in [1] we gave also infinite classes of nonisomorphic matroids not fulfilling the different intersection properties.

Theorem 2.2. (*Hahn-Banach*). *Let $\mathcal{O}$ be an acyclic oriented matroid with the intersection property. Then $A, B \subseteq E$ are strongly separable if and only if for every point extension $\mathcal{O}' := \mathcal{O} \cup p$ $\mathrm{conv}_{\mathcal{O}'}(A) \cap \mathrm{conv}_{\mathcal{O}'}(B) = \emptyset$.*

Proof. Let $A, B \subseteq E$ be strongly separable, i.e. $Y_A \geq 0$, $Y_B \leq 0$ and $Y^0 = \emptyset$ for some vector Y of the cocircuit span. Assume $\mathrm{conv}_{\mathcal{O}'}(A) \cap \mathrm{conv}_{\mathcal{O}'}(B) \neq \emptyset$ for some point extension $\mathcal{O}' := \mathcal{O} \cup p$ of $\mathcal{O}$. Clearly, since $Y^0 = \emptyset$ $A \cap B = \emptyset$, by definition of the convex hull operator either there exists a circuit X of $\mathcal{O}$ such that $X^- = \{e\} \subseteq A$ and $X^+ \subseteq B$ for some $e \in E$ (interchange A and B if necessary) or there exist circuits W and Z of $\mathcal{O}'$ such that $W^- = \{e\}$, $W^+ \subseteq A$ and $Z^+ = \{e\}$, $Z^- \subseteq B$. In the latter case we can use the circuit elimination axiom to construct a circuit X with $X^+ \subseteq (W^+ \cup Z^+) \backslash e = W^+ \subseteq A$ and $X^- \subset (W^- \cup Z^-)/e \subseteq B$. Hence in both cases there is a circuit X of $\mathcal{O}$ with $X^+ \subseteq A$ and $X^- \subseteq B$. Thus X is not orthogonal to Y, a contradiction.

For the converse assume A and B are not strongly separable, i.e. there is no conformal sum of cocircuits Y all of which fulfill $Y_B \leq 0$ and $Y_A \geq 0$, moreover there exists $e \in A \cup B$ (w.l.o.g. $e \in B$) such that $Y_B \leq 0$, $Y_A^- \geq 0$ holds for no cocircuit Y of $\mathcal{O}$. In other words there is no cocircuit Y with $e \in \underline{Y} \subseteq A \cup B$ and $Y^+ \cap B = Y^- \cap A = \emptyset$. Using the partition $e \in B \dot{\cup} A \dot{\cup} G \dot{\cup} R = E (G := E \backslash (A \cup B)$, $R := \emptyset)$ we can apply Minty's painting lemma to conclude the existence of a circuit X of $\mathcal{O}$ with $e \in \underline{X} \subseteq A \cup B$ and $X^+ \cap B = X^- \cap A = \emptyset$, i.e. $X_A \geq 0$ and $X_B \leq 0$. Since $\mathcal{O}$ is acyclic A and B both must have elements with $\underline{X}$ in common. If $A \cap \underline{X} = \{e\}$ for some $e \in A$ then the circuit $-X$ with $(-X)^- = \{e\}$ and $(-X)^+ \subseteq B$ shows $e \in \mathrm{conv}_{\mathcal{O}}(B) \cap A$, a contradiction. Thus we may assume $|A \cap \underline{X}| \geq 2$ and $|B \cap \underline{X}| \geq 2$ and clearly the flats $F = \mathrm{cl}(A \cap \underline{X})$, $G = \mathrm{cl}(B \cap \underline{X})$ have rank of at least 2. Since $A \cap \underline{X}$ and $B \cap \underline{X}$ are independent we have $r(F) + r(G) = |A \cap \underline{X}| + |B \cap \underline{X}| = |\underline{X}|$ and obviously $F \vee G = \mathrm{cl}(\underline{X})$, i.e. $r(F \vee G) = |\underline{X}| - 1$. Hence if F and G is a modular pair of flats then $r(F \wedge G) = 1$, i.e. $p \in F \cap G$ for some $p \in E$ and there are circuits S and T of $\mathcal{O}$ with $p \in \underline{S} \subseteq (A \cap \underline{X}) \cup p$ and $p \in \underline{T} \subseteq (B \cap \underline{X}) \cup p$. On the other side if F and G are non-modular there exists (by assumption) a point extension $\mathcal{O}' = \mathcal{O} \cup p$ and with the same arguments as above we obtain circuits S and T of $\mathcal{O}'$ with the above properties. Hence in either case we can use the circuit elimination axiom (applied to S and T or $-S$, T) to construct a new circuit R with $\underline{R} \subseteq \underline{X}$, i.e. $R = X$ or $R = -X$. Moreover since $R^+ \subseteq S^+ \cup T^+$ and

$R^- \subseteq S^- \cup T^-$ respec. $X_A \geqslant 0$ and $X_B \leqslant 0$ we have $S_A \geqslant 0$ and $T_B \leqslant 0$. The oriented matroid $\mathcal{O}$ is acyclic, i.e. $\mathcal{O}' = \mathcal{O} \cup p$ is acyclic (or may be chosen acyclic) and thus $S^- \neq \emptyset$ and $T^+ \neq \emptyset$. Hence $p \in S^- \cap T^+$, which proves $p \in \mathrm{conv}_{\mathcal{O}'}(A) \cap \mathrm{conv}_{\mathcal{O}'}(B)$. □

Since all oriented matroids of rank three do have the intersection property, Theorem 2.2 implies Theorem 2.1.

If $A, B \subseteq E$ are strongly separable there exists a cocell Z of $\mathcal{O}$ represented by a conformal sum of some cocircuits $Y^1, \ldots, Y^k$ such that $Z_A \geqslant 0$, $Z_B \leqslant 0$ and $Z^0 = \emptyset$. Thus for any given point $p \in A \cup B$ there exists always a hyperplane $H := E \backslash \underline{Y}^i$ (for some $i = 1, \ldots, k$) such that $p \in \underline{Y}^i$ and $Y_A \geqslant 0$ and $Y_B \leqslant 0$. Moreover the following stronger proposition holds.

Proposition 2.3. *If Y is a nonzero cocell of $\mathcal{O}$, then there is an extension $\mathcal{O}'$ of $\mathcal{O}$ in which Y is a cocircuit.*

Proof. A cocell Y of $\mathcal{O}$ is a cocircuit of $\mathcal{O}$ iff $d(Y) := r(\mathcal{O}) - r(E \backslash \underline{Y}) - 1$ is zero. Let Y be any cocell of $\mathcal{O}$ with $d(Y) > 0$. For the proof it suffices to construct a point extension $\mathcal{O}' := \mathcal{O} \cup p$ which contains Y as a cocell with $d'(Y) = d(Y) - 1$.

This can easily be done by using Las Vergnas's [11] method of lexicographic point extensions. Let x be a cocircuit of $\mathcal{O}$ which conforms to Y and choose a base $B := B_1 \cup B_2 \cup \{e\}$ of E such that B_1 is a base of $E \backslash \underline{Y}$ and $B_1 \cup B_2$ is a base of $E \backslash \underline{X}$. Let $e_1, \ldots, e_n$ be the elements of B with $B_1 \cup B_2 = \{e_1, \ldots, e_{n-1}\}$ and $e = e_n$. The lexicographic point extension lex $(e_1, \ldots, e_{n-1}, -e_n)$ now yields the desired extension $\mathcal{O}'$. □

Corollary 2.4. *If $A, B \subseteq E$ are strongly separable then there exists an extension $\mathcal{O}'$ of the oriented matroid $\mathcal{O}$ such that $A \subseteq \underline{Y}^+$ and $B \subseteq \underline{Y}^-$ for some cocircuit Y of $\mathcal{O}'$, i.e. A and B can be strictly separated by a hyperplane $H = E' \backslash \underline{Y}$.*

Theorem 2.2 shows that the intersection property is sufficient for the validity of the Hahn–Banach theorem. On the other side however the class of oriented matroids with the Hahn–Banach property properly includes the class of oriented matroids having the intersection property.

Proposition 2.5. *The Vamos matroid (oriented as usual) satisfies the statement of the Hahn–Banach Theorem but not the intersection property.*

Proof. Clearly, the Vamos matroid does not satisfy the intersection property. Assume that the Vamos matroid does not satisfy the Hahn–Banach property, i.e. there exist not strongly separable sets $A, B \subseteq E$ such that $\mathrm{conv}_{\mathcal{O}'}(A) \cap \mathrm{conv}_{\mathcal{O}'}(B) = \emptyset$ for every extension $\mathcal{O}'$ of $\mathcal{O}$. As shown in the proof of Theorem 2.2

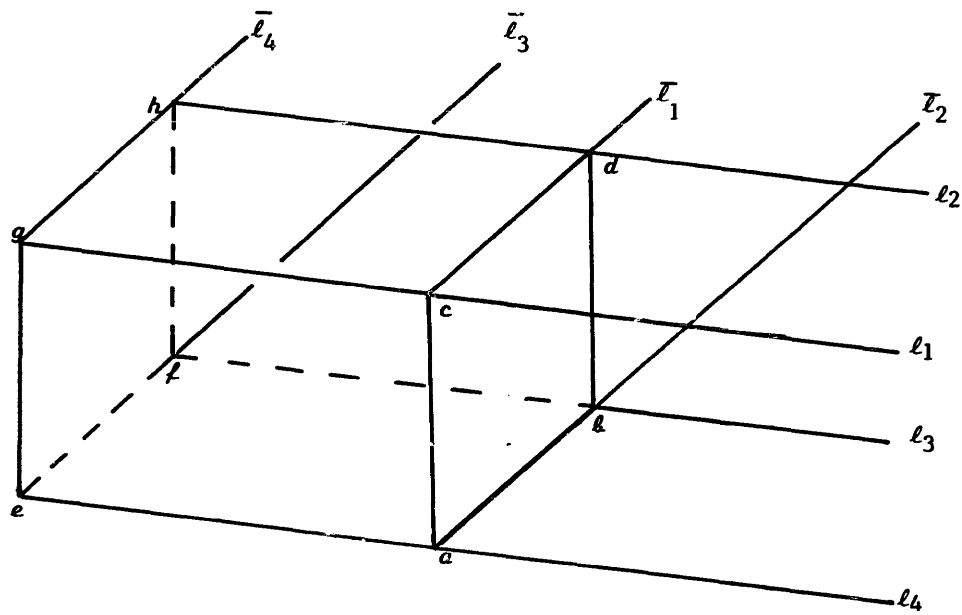

Fig. 1. Vamos matroid.

this proves the existence of two sets $A' \subseteq A$ and $B' \subseteq B$ such that the corresponding flats $F = \text{cl}(A')$ and $G = \text{cl}(B')$ are nonmodular and cannot be intersected (using any point extension). Hence both sets A and B must contain (at least) one of the lines l_i ($i = 1, \ldots, 4$) resp. $\bar{l}_j$ ($j = 1, \ldots, 4$) (cf. Fig. 1).

The Vamos matroid is a well-known example of a bad behaved matroid. Here we look at the Vamos matroid as a perturbed cube which misses exactly one 4-point hyperplane g, h, c, d. We can now enumerate all such possible combinations of A and B and prove that in any circumstance either A or B can be strongly separated or there exist flats (cf. the proof of Theorem 2.2) such that there exists a point p in the intersection of the convex hulls of A and B. Instead of proving every (similar) instance we look at a representable instance in a number of different cases. (This will prove the other instances, by symmetry).

(a) If $A = l_i$ or $A = \bar{l}_j$ for some $i,j \in \{1, \ldots, 4\}$, then A and $E \setminus A$ can easily be strongly separated. Let $A = \text{cl}(a, b)$ and B arbitrary. The hyperplanes $\text{cl}(c, d, e, f)$, $\text{cl}(a, b, e, f)$ and $\text{cl}(a, b, c, d)$ show that A, B can be strongly separated. Similar if $A = \text{cl}(c, d)$, we choose $\text{cl}(a, b, g, h)$, $\text{cl}(a, b, c, d)$ and $\text{cl}(c, d, g)$ resp. $\text{cl}(c, d, h)$ as separating hyperplanes.

(b) If A is a proper 3-element subset of one of the five four-point facets then A and $E \setminus A$ can be strongly separated as the following example $A = \text{cl}(a, c, d)$ shows. The separating hyperplanes are $\text{cl}(a, b, c, d)$, $\text{cl}(b, d, e, g)$ and $\text{cl}(a, d, g)$. In case A is a proper 3-element subset of the non facet $\{c, d, g, h\}$ (e.g. $A = \text{cl}(c, d, g)$) A can e.g. be strongly separated by $\text{cl}(a, d, e, h)$, $\text{cl}(a, b, g, h)$, $\text{cl}(a, d, g)$, $\text{cl}(a, d, g)$ and $\text{cl}(c, d, h)$ or $\text{cl}(d, h, g)$.

(c) If $|A| = 4$ and A is one of the four-point facets, then corresponding four-point hyperplanes (not facets) show the strong separation.

(d) In all other instances A and B include diagonal lines lying on a four-point hyperplane. Of course the corresponding circuit C satisfies $C_A \leq 0$ and $C_B \geq 0$. Hence as in the proof of Theorem 2.2, we can find a point extension p in the convex hulls of A and B. □

Usually an element x of a polytope $P \subseteq \mathbb{R}^n$ is called a vertex if and only if $P \setminus \{x\}$ is convex. This is a special case of the following more general definition of faces of a polytope. Let $P = \mathrm{conv}(v_1, \ldots, v_n)$ be a polytope and $F \subseteq E := \{v_1, \ldots, v_n\}$. Then $\mathrm{conv}(F)$ is a face of P if and only if $\mathrm{aff}(F) \cap \mathrm{conv}(E \setminus F) = \emptyset$. In oriented matroid language this translates to the following statement.

Let $\mathcal{O}$ be an acyclic oriented matroid on E and V the vertices of $\mathcal{O}$. Let $F \subseteq V$, then $\mathrm{conv}\, F$ is a face of $\mathcal{O}$ (in the notation of Las Vergnas) if and only if $\mathrm{conv}_{\mathcal{O}'}(E \setminus F) \cap \mathrm{cl}_{\mathcal{O}'}(F) = \emptyset$ in every point extensions $\mathcal{O}'$ of $\mathcal{O}$. (2.5)

Munson [14] proved the necessity of Statement (2.5) for general oriented matroids and Kern [10] gave an example of Vamos-type-matroid which shows that the condition in (2.5) does not suffice in general. Hence this characterization of facets does not carry over to oriented matroids.

Theorem 2.6. *For acyclic oriented matroids with the intersection property, let $F \subseteq V$. Then F is a face of the oriented matroid $\mathcal{O}$ if and only if $\mathrm{conv}_{\mathcal{O}'}(E \setminus F) \cap \mathrm{cl}_{\mathcal{O}'}(F) = \emptyset$ in every point extension $\mathcal{O}'$ of $\mathcal{O}$.*

Proof. Suppose that $F \subseteq E$ and $\mathrm{conv}(E \setminus F) \cap \mathrm{cl}(F) = \emptyset$ in every point extension of $\mathcal{O}$. Hence F is closed. By Theorem 2.2 $Y \in 2^{\pm E}$ with $Y^+ = E \setminus F$, $Y^- = F$ is a cocell. Consider the vector $W \in 2^{\pm E}$ with $W^+ = E \setminus F$, $W^- = \emptyset$. F is a face iff W is a cocell. So, if F is not a face, there is a circuit X not orthogonal to W. That means that $\emptyset \neq E \setminus F \supseteq X^+ \setminus F$. Since F is closed, $|X \setminus F| \geq 2$. Since $\mathcal{O}$ is acyclic, $X^- \neq \emptyset$ and since Y is orthogonal to X, $X^+ \cap F \neq \emptyset$, hence $|X \cap F| \geq 2$. From the intersection property, there is a point extension of $\mathcal{O}$ in which there is a $p \in \mathrm{conv}(X \setminus F) \cap \mathrm{cl}(E \setminus F)$, and arguing as in 2.2, $V \in 2^{\pm E}$ with $V^+ = X \setminus F$, $V^- = \{p\}$) is a circuit. Whence $p \in \mathrm{conv}(X \setminus F) \subseteq \mathrm{conv}(E \setminus F)$. Since $p \in \mathrm{cl}(X \cap F) \subseteq \mathrm{cl}(F)$, we have a contradiction. □

Corollary 2.7. *Let $\mathcal{O}$ be an oriented acyclic matroid with vertices V and F a face of $\mathcal{O}$. Then there exists an extension $\mathcal{O}'$ of $\mathcal{O}$ and a hyperplane H' of $\mathcal{O}'$ such that $F = \mathrm{conv}(V \cap H')$ and $V \subseteq C^+ \cup C^0$ for the cocircuit C (corresponding to H') of $\mathcal{O}'$.*

3. Caratheodory-, Radon- and Helly-type theorems

Once having the tool of a convexity operator one can easily prove a Caratheodory-type theorem for oriented matroids.

Proposition 3.1 (*Caratheodory*). *Let $\mathcal{O}$ be an acyclic oriented matroid of rank r on E. Let $A \subseteq E$ and $e \in \text{conv}_{\mathcal{O}}(A)$ for $e \in E$. Then there exists a subset $B \subseteq A$ of A such that $e \in \text{conv}(B)$ and $|B| \leq r$.*

Proof. Since $e \in \text{conv}_{\mathcal{O}}(A)$, there exists a circuit C of $\mathcal{O}$ with $C^- = \{e\}$ and $C^+ \subseteq A$. Since $|\underline{C}| \leq r+1$ we can use $B := \underline{C} \cap A$ to prove the proposition. □

Radon's and Helly's theorem are not valid in general oriented matroids. Here we are going to apply the proof technique of Theorem 2.2 to prove Radon's theorem (for oriented matroids having the intersection property) and then use Radon's theorem for a proof of Helly's theorem.

Corollary 3.2 (*Radon's theorem*). *Let $\mathcal{O}$ be an acyclic matroid of rank r with the intersection property and let $A \subseteq E$ with $|A| \geq r+1$. Then there exists a partition $A' \cup A'' = A$ of A such that $\text{conv}_{\mathcal{O}'}(A') \cap \text{conv}_{\mathcal{O}'}(A'') \neq \emptyset$ for some point extension $\mathcal{O}' = \mathcal{O} \cup p$ of $\mathcal{O}$.*

Proof. Since $|A| \geq r+1$, there exists a circuit C of $\mathcal{O}$ with $C \subseteq A$. Since $\mathcal{O}$ is acyclic neither $\tilde{A}' := A \cap C^+$ nor $\tilde{A}'' := A \cap C^-$ are empty. Hence $\tilde{A}'$ and $\tilde{A}''$ are not strongly separated, and the result follows from Theorem 2.2. □

Corollary 3.3 (*Helly's theorem*). *Let $\mathcal{O}$ be an acyclic oriented matroid of rank r with the intersection property and let $(K_i)_{i=1,\ldots,n}$ $n \geq r+1$ be a family of convex sets such that every intersection of $(n-1)$ of the sets K_i is nonempty. Then there exists an extension $\mathcal{O}'$ such that $\bigcap_{i=1,\ldots,n} \text{conv}_{\mathcal{O}'}(K_i) \neq \emptyset$.*

Proof. Due to the assumption there exists for every $i = 1, \ldots, n$ a point $x_i \in \text{conv}_{\mathcal{O}}(K_1) \cap \cdots \cap \text{conv}_{\mathcal{O}}(K_{i-1}) \cap \text{conv}_{\mathcal{O}}(K_{i+1}) \cap \cdots \cap \text{conv}_{\mathcal{O}}(K_n)$. Corollary 3.2 (Radon's theorem) gives us now a point extension $\mathcal{O}' = \mathcal{O} \cup x$ such that $x \in \text{conv}(\{x_j \mid j \in J\}) \cap \text{conv}(\{x_j \mid j \notin J\})$ for some $J \subseteq \{1, \ldots, n\}$. Since $x \in \text{conv}(\{x_j \mid j \in J\})$ we have $x \in \bigcap_{j \notin J} \text{conv}_{\mathcal{O}'}(K_j)$ and similar since $x \in \text{conv}(\{x_j \mid j \notin J\})$ we have $x \in \bigcap_{j \in J} \text{conv}_{\mathcal{O}'}(K_j)$. Hence $x \in \bigcap_{j=1}^{n} \text{conv}_{\mathcal{O}'}(K_j)$. □

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References

[1] A. Bachem and A. Wanka, Euclidean intersection properties. To appear in: J. Combinat. Theory B. Working Paper WP 86.27, Mathematisches Institut, University of Cologne (1985).

[2] A. Bachem and A. Wanka, Matroids without adjoints. Working Paper WP 86.29, Mathematisches Institut, University of Cologne (1986).
[3] L.S. Billera and B.S. Munson, Triangulations of oriented matroids and convex polytopes, SIAM T. Alg. Discr. Math. 5 (1985) 515–525.
[4] R.G. Bland and M. Las Vergnas, Orientability of matroids. J. of Comb. Theory B 24, (1978) 94–123.
[5] R. Cordovil, Sur un theoreme de separation des matroides orientes de rang trois. Discrete Math. 40 (1982) 163–169.
[6] R. Cordovil and P. Duchet, Séparations par une droite dans les matroides orientés de rang 3. Working paper, Lisboa (1985).
[7] R. Cordovil and I.P. Silva, Determining a matroid by non-radon partitions. Working paper, Lisboa (1986).
[8] K. Fukuda, Oriented matroid programming. Thesis, Waterloo, Canada (1982).
[9] W. Kern, Adjoints und Polare von orientierten Matroiden. Diploma Thesis, Erlangen (1982).
[10] W. Kern, Verbandstheoretische Dualität in kombinatorischen Geometrien und orientierten Matroiden. Dissertation, University of Cologne 1985 (also available as Working Paper WP 85.14, Mathematisches Institut, Universität Köln 1985).
[11] M. Las Vergnas, Extensions ponctuelles d'une géometrié combinatoire, pp. 265–270, in: Problémes combinatoires et theorie des graphes (Actes Coll. Intern. C.N.R.S., No. 260, Orsay 1976), Ed. C.N.R.S., Paris (1978).
[12] M. Las Vergnas, Convexity in Oriented matroids, J. Combin. Theory, Ser. B 29 (1980) 231–243.
[13] A. Mandel, Topology of oriented matroids, Thesis, Waterloo, Canada (1982).
[14] B.S. Munson, Face lattices of oriented matroids, Ph.D Thesis, Cornell University, Ithaca, USA (1981).