Linear Algebra and its Applications 435 (2011) 330–341

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Hamming graphs in Nomura algebras

Ada Chan [a,∗,1], Akihiro Munemasa [b]

[a] *Department of Mathematics and Statistics, York University, Toronto, Ontario, Canada M3J 1P3*
[b] *Graduate School of Information Sciences, Tohoku University, Sendai 980-8579, Japan*

ARTICLE INFO

Article history:
Received 29 November 2010
Accepted 18 January 2011
Available online 12 February 2011

Submitted by R.A. Brualdi

AMS classification:
05E30

Keywords:
Type II matrix
Nomura algebra
Hamming scheme
Duality of association scheme

ABSTRACT

Let $\mathcal{A}$ be an association scheme on $q \geqslant 3$ vertices. We show that the Bose–Mesner algebra of the generalized Hamming scheme $\mathcal{H}(n, \mathcal{A})$, for $n \geqslant 2$, is not the Nomura algebra of any type II matrix.

This result gives examples of formally self-dual Bose–Mesner algebras that are not the Nomura algebras of type II matrices.

1. Introduction

A $v \times v$ matrix W is a *type II matrix* if

$$\sum_{x=1}^{v} \frac{W(x,a)}{W(x,b)} = \delta_{a,b} v \quad \text{for } a, b = 1, \ldots, v. \tag{1}$$

Hadamard matrices and the character tables of finite abelian groups satisfy this condition. Type II matrices also arise from combinatorial objects such as symmetric designs, tight sets of equiangular lines and strongly regular graphs [2].

In [8], Nomura constructed the Bose–Menser algebra of an association scheme from each type II matrix W, hence another connection to combinatorics. We call this algebra the *Nomura algebra of* W, and denote it by $\mathcal{N}_W$. Jaeger et al. [8] showed that a type II matrix W belongs to $\mathcal{N}_W$ if and only if cW is

∗ Corresponding author.
E-mail addresses: ssachan@yorku.ca (A. Chan), munemasa@math.is.tohoku.ac.jp (A. Munemasa).

1 The first author gratefully acknowledges the support from a JSPS postdoctoral fellowship and a NSERC discovery grant.

0024-3795/$ - see front matter
doi:10.1016/j.laa.2011.01.020

a spin model for some non-zero scalar c. Spin models give link invariants and they are difficult to find. The ability to identify the Bose–Mesner algebras that are the Nomura algebras of type II matrices is a step towards the search of spin models.

We say two type II matrices W_1 and W_2 are *type II equivalent* if

$$W_1 = P_1 D_1 W_2 D_2 P_2$$

for some invertible diagonal matrices D_1 and D_2 and permutation matrices P_1 and P_2. Suzuki [11] showed that if $\mathcal{N}_W$ contains the Hamming scheme $\mathcal{H}(n, 3)$, then W is type II equivalent to the character table of the group $\mathbb{Z}_3^n$.

It is straightforward to show that W_1 and W_2 are type II matrices if and only if their Kronecker product $W_1 \otimes W_2$ is a type II matrix. We generalize Suzuki's result and show that if $\mathcal{N}_W$ contains the adjacency matrix of the Hamming graph $H(n, q)$, $n \geqslant 2$ and $q \geqslant 3$, then W is type II equivalent to

$$W_1 \otimes W_2 \otimes \cdots \otimes W_n$$

where $W_1, W_2, \ldots, W_n$ are $q \times q$ type II matrices. In this case, $\mathcal{N}_W$ is isomorphic to

$$\mathcal{N}_{W_1} \otimes \mathcal{N}_{W_2} \otimes \cdots \otimes \mathcal{N}_{W_n}.$$

As a consequence, when $n \geqslant 2$ and $q \geqslant 3$, the Bose–Mesner algebra of the Hamming scheme $\mathcal{H}(n, q)$ and the Bose–Mesner algebra of the generalized Hamming scheme $\mathcal{H}(n, \mathcal{A})$, for any association scheme $\mathcal{A}$ on q vertices, cannot be the Nomura algebras of type II matrices.

If W is type II, then so is W^T. In [8], Jaeger et al. showed that $\mathcal{N}_W$ and $\mathcal{N}_{W^T}$ are formally dual. If $\mathcal{N}_W = \mathcal{N}_{W^T}$, which includes the case where W is symmetric, then $\mathcal{N}_W$ is formally self-dual. The Hamming scheme $\mathcal{H}(n, q)$ is a formally self-dual association scheme. It also satisfies Delsarte's notion [4] of self-duality and the notion of hyper self-duality defined by Curtin and Nomura [3]. We now have a family of association schemes satisfying all three notions of duality but are not the Nomura algebras of type II matrices.

In this paper, we use I_v and J_v to denote the $v \times v$ identity matrix and the matrix of all ones, respectively. We use $\mathbf{1}_v$ to denote the column vector of all ones of length v. We omit the subscript if the size or the length is clear.

2. Nomura algebras

We recall type II matrices and their Nomura algebras, see [8,10] for details.

Given matrices M and N of the same order, their *Schur product*, $M \circ N$, is defined by

$$(M \circ N)(i, j) = M(i, j)N(i, j) \qquad \text{for all } i \text{ and } j.$$

If M has no zero entry, we use $M^{(-)}$ to denote the matrix whose (i, j)-entry is $M(i, j)^{-1}$.

A $v \times v$ matrix W is type II if $W^{(-)T}W = vI$. The $v \times v$ matrix

$$W = (t - 1)I + J$$

is type II if and only if t is a root of the quadratic $t^2 + (v - 2)t + 1 = 0$. This is the simplest spin model (after an appropriate scaling), called the Potts model, and the associated link invariant is the Jones polynomial [9].

Let the vectors $e_1, \ldots, e_v$ be the standard basis of $\mathbb{C}^v$. Given a $v \times v$ type II matrix W, we define v^2 vectors

$$Y_{a,b} = We_a \circ W^{(-)}e_b \qquad \text{for } a, b = 1, \ldots, v.$$

It follows from (1) that W is invertible and has no zero entries. Hence, for all a, the set $\{Y_{a,b} : b = 1, \ldots, v\}$ is a basis for $\mathbb{C}^v$.

The *Nomura algebra*, $\mathcal{N}_W$, of W is the set of matrices that have $Y_{a,b}$ as eigenvectors for all a, b. It follows immediately that $\mathcal{N}_W$ contains I and it is closed under matrix multiplication. By (1), $JY_{a,b} = \delta_{a,b}vY_{a,b}$. Hence $J \in \mathcal{N}_W$ and $\dim \mathcal{N}_W \geqslant 2$.

For each matrix $M \in \mathcal{N}_W$, let $\Theta_W(M)$ be the $v \times v$ matrix whose (a, b)-entry satisfies

$$MY_{a,b} = \Theta_W(M)(a, b)Y_{a,b} \quad \text{for } a, b = 1, \ldots, v.$$

The following results, obtained from Theorems 1 to 3 of [8], are useful in subsequent sections.

Theorem 2.1. *Let W be a $v \times v$ type II matrix. Then $\mathcal{N}_W$ is closed under matrix multiplication, Schur product and transpose. It is commutative with respect to matrix multiplication.*

Moreover $\Theta_W(\mathcal{N}_W) = \mathcal{N}_{W^T}$, and

1. $\Theta_W(M_1M_2) = \Theta_W(M_1) \circ \Theta_W(M_2)$,
2. $\Theta_W(M_1 \circ M_2) = \frac{1}{v}\Theta_W(M_1)\Theta_W(M_2)$,
3. $\Theta_W(M_1^T) = \Theta_W(M_1)^T$,

for $M_1, M_2 \in \mathcal{N}_W$.

Lemma 2.2. *Let W be a type II matrix. For any invertible diagonal matrices D_1 and D_2 and for any permutation matrices P_1 and P_2,*

$$\mathcal{N}_{P_1D_1WD_2P_2} = P_1\mathcal{N}_WP_1^T.$$

An *association scheme* on v elements with n classes is a set of $v \times v$ 01-matrices

$$\mathcal{A} = \{A_0, A_1, \ldots, A_n\}$$

satisfying

1. $A_0 = I$.
2. $\sum_{i=0}^n A_i = J$.
3. $A_i^T \in \mathcal{A}$ for $i = 0, \ldots, n$.
4. A_iA_j lies in the span of $\mathcal{A}$ for $i, j = 0, \ldots, n$.
5. $A_iA_j = A_jA_i$ for $i, j = 0, \ldots, n$.

The simplest association scheme is $\{I, J - I\}$, called the *trivial association scheme*.

The span of an association scheme over $\mathbb{C}$ is called its *Bose–Mesner algebra*. The Bose–Mesner algebra of $\mathcal{A}$ is closed under matrix multiplication, Schur product and transpose. It is commutative with respect to matrix multiplication, and it contains I and J. Conversely, any algebra satisfying these conditions is the Bose–Mesner algebra of an association scheme [1].

A *formal duality* between two Bose–Mesner algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ is an invertible linear map $\Theta : \mathcal{B}_1 \to \mathcal{B}_2$ satisfying $\Theta(MN) = \Theta(M) \circ \Theta(N)$ and $\Theta(M \circ N) = \frac{1}{v}\Theta(M)\Theta(N)$. When $\mathcal{B}_1 = \mathcal{B}_2$ and $\Theta^2(M) = vM^T$, we say $\mathcal{B}_1$ is *formally self-dual*.

Theorem 2.3. *If W is a type II matrix, then $\mathcal{N}_W$ and $\mathcal{N}_{W^T}$ are a formally dual pair of Bose–Mesner algebras.*

3. Products

Suppose W_1 and W_2 are type II matrices then so is their Kronecker product $W_1 \otimes W_2$. Proposition 7 of [8] determines the Nomura algebra of $W_1 \otimes W_2$.

Lemma 3.1. *If W_1 and W_2 are type II matrices, then*

$$\mathcal{N}_{W_1 \otimes W_2} = \mathcal{N}_{W_1} \otimes \mathcal{N}_{W_2}.$$

Hosoya and Suzuki [7] studied the structure of W if $J \otimes I$ belongs to $\mathcal{N}_W$. In [11], Suzuki showed that if $I \otimes J$ and $J \otimes I$ belong to $\mathcal{N}_W$, then W is type II equivalent to the Kronecker product of two type II matrices. We tailor Theorem 1.2 of [11] to Lemma 3.4 and Theorem 3.5 by showing the type II equivalence explicitly, for later use.

For the remainder of this section, we assume that W is an $mn \times mn$ type II matrix and

$$I_n \otimes J_m, J_n \otimes I_m \in \mathcal{N}_W.$$

For a vector u of length mn, we use $u[i]$ to denote its ith block of length m for $i = 1, \ldots, n$. We also denote by $W[i, j]$ the $m \times m$ submatrix located at the (i, j)-block of W.

Lemma 3.2. *For $1 \leqslant a, b \leqslant mn$, the following statements hold.*

1. $\Theta_W(I_n \otimes J_m)(a, b) = m$ *if and only if*

$$Y_{a,b}[i] \in \text{span}(\mathbf{1}_m) \quad \textit{for all } i = 1, \ldots, n,$$

or equivalently, $We_a[i] \in \text{span}(We_b[i])$ *for all* $i = 1, \ldots, n$.

2. $\Theta_W(J_n \otimes I_m)(a, b) = n$ *if and only if*

$$Y_{a,b}[i] = Y_{a,b}[1] \quad \textit{for all } i = 1, \ldots, n,$$

or equivalently,

$$(We_a[i]) \circ (W^{(-)}e_a[1]) = (We_b[i]) \circ (W^{(-)}e_b[1]) \quad \textit{for all } i = 1, \ldots, n.$$

Proof. Straightforward. □

Lemma 3.3. *There exists a permutation matrix P such that*

$$\Theta_{WP}(I_n \otimes J_m) = m(J_n \otimes I_m) \quad \textit{and} \quad \Theta_{WP}(J_n \otimes I_m) = n(I_n \otimes J_m). \tag{2}$$

Proof. For $1 \leqslant a, b \leqslant mn$, we write $a \sim_1 b$ when $\Theta_W(I_n \otimes J_m)(a, b) = m$, and $a \sim_2 b$ when $\Theta_W(J_n \otimes I_m)(a, b) = n$. Since $\{Y_{a,1}, \ldots, Y_{a,mn}\}$ is a basis of $\mathbb{C}^{mn}$ consisting of eigenvectors of $I_n \otimes J_m$ which has eigenvalue m with multiplicity n, $|\{b \mid a \sim_1 b\}| = n$, for each given a. It follows from Lemma 3.2 that $\sim_1$ is an equivalence relation with m equivalence classes of size n. Similarly, for each given a, $|\{b \mid a \sim_2 b\}| = m$, and Lemma 3.2 implies that $\sim_2$ is an equivalence relation with n equivalence classes of size m.

Furthermore, Lemma 3.2 implies that $a \sim_1 b$ and $a \sim_2 b$ occur simultaneously only when $a = b$. Hence an equivalence class of $\sim_1$ meets every equivalence class of $\sim_2$ in exactly one element, and there exists a permutation matrix P so that the equivalence classes of $\sim_1$ and $\sim_2$ defined for WP are

$$\{h, m + h, 2m + h, \ldots, (n - 1)m + h\} \quad \text{for } h = 1, \ldots, m$$

and

$$\{rm + 1, rm + 2, \ldots, rm + m\} \qquad \text{for } r = 0, \ldots, n - 1,$$

respectively. By Lemma 2.2, $\mathcal{N}_W = \mathcal{N}_{WP}$. We conclude that $\mathcal{N}_{WP}$ contains $I_n \otimes J_m$ and $J_n \otimes I_m$, and (2) holds. □

We say a type II matrix is *normalized* if all entries in its first row and its first column are 1. Given any type II matrix W, there exists invertible diagonal matrices D and D' such that $W' = DWD'$ is normalized. By Lemma 2.2, we have $\mathcal{N}_{W'} = \mathcal{N}_W$. Note that the eigenvector $W'e_a \circ W'^{(-)}e_b$ is a scalar multiple of $We_a \circ W^{(-)}e_b$. We conclude that $\Theta_{W'}(M) = \Theta_W(M)$ for all $M \in \mathcal{N}_W$.

Lemma 3.4. *Suppose W is normalized and*

$$\Theta_W(I_n \otimes J_m) = m(J_n \otimes I_m) \quad \text{and} \quad \Theta_W(J_n \otimes I_m) = n(I_n \otimes J_m).$$

Then $W = U \otimes V$ for some $n \times n$ type II matrix U and some $m \times m$ type II matrix V.

Proof. By Lemma 3.2, we have $a \equiv b \pmod{m}$ if and only if

$$We_a[i] \in \text{span}(We_b[i]) \quad \text{for } i = 1, \ldots, n.$$

Since the first row of W is $\mathbf{1}_{mn}^T$, setting $i = 1$ gives

$$W[1,1] = W[1,2] = \ldots = W[1,n]. \tag{3}$$

Let $V = W[1,1]$. Further, the first column of W is $\mathbf{1}_{mn}$, so there exists a non-zero scalar $U(i,j)$ such that

$$We_{(j-1)m+1}[i] = U(i,j)\mathbf{1}_m \quad \text{for } i,j = 1, \ldots, n. \tag{4}$$

Suppose $a = (j-1)m + h$ and $c = (j-1)m + 1$ for $j = 1, \ldots, n$ and $h = 1, \ldots, m$. As $\Theta_W(J_n \otimes I_m) = n(I_n \otimes J_m)$, Lemma 3.2 implies that

$$\begin{aligned} We_a[i] &= (Ve_h) \circ (U(i,j)\mathbf{1}_m) \circ (W^{(-)}[1,1]e_1) \qquad \text{(by (3), (4))} \\ &= U(i,j)(Ve_h). \end{aligned}$$

Hence $W[i,j] = U(i,j)V$ for $i,j = 1, \ldots, n$. It is straightforward to check that both U and V are type II. □

Theorem 3.5. *Suppose W is a type II matrix and*

$$I_n \otimes J_m, J_n \otimes I_m \in \mathcal{N}_W.$$

Then

$$W = D_1(U \otimes V)D_2P$$

for some $n \times n$ type II matrix U, $m \times m$ type II matrix V, permutation matrix P, and invertible diagonal matrices D_1 and D_2. In this case,

$$\mathcal{N}_W = \mathcal{N}_U \otimes \mathcal{N}_V.$$

Proof. By Lemma 3.3, there exists a permutation matrix P such that (2) holds. There exist invertible diagonal matrices D and D' such that $W' = DWPD'$ is normalized. Then by Lemma 3.4, $W' = U \otimes V$ for some $n \times n$ type II matrix U and $m \times m$ type II matrix V. The rest of the proof is immediate from Lemmas 2.2 and 3.1. □

4. Generalized Hamming schemes

We recall from [5] the definition of and some facts concerning the generalized Hamming scheme $\mathcal{H}(n, \mathcal{A})$. Let $\mathcal{A} = \{A_0, A_1, \ldots, A_d\}$ be an association scheme on q vertices. Consider the product association scheme $\mathcal{A}^{\otimes n}$ and the symmetric group $\mathcal{S}_n$ acting on $\{1, \ldots, n\}$. For each element $\sigma \in \mathcal{S}_n$, define

$$(A_{i_1} \otimes A_{i_2} \otimes \cdots \otimes A_{i_n})^{\sigma} = A_{i_{1\sigma^{-1}}} \otimes A_{i_{2\sigma^{-1}}} \otimes \cdots \otimes A_{i_{n\sigma^{-1}}}.$$

Then $\mathcal{S}_n$ is a group of algebra automorphism of the span of $\mathcal{A}^{\otimes n}$. The set of matrices in the span of $\mathcal{A}^{\otimes n}$ fixed by every element of $\mathcal{S}_n$ is closed under matrix multiplication, Schur product and transpose, and this set contains I_{q^n} and J_{q^n}. It is the Bose–Mesner algebra of a subscheme of $\mathcal{A}^{\otimes n}$ [5]. This subscheme is called the *generalized Hamming scheme* $\mathcal{H}(n, \mathcal{A})$. In particular, for $i = 1, \ldots, d$, the matrix

$$(A_i \otimes I_q \otimes \cdots \otimes I_q) + (I_q \otimes A_i \otimes \cdots \otimes I_q) + \cdots + (I_q \otimes I_q \otimes \cdots \otimes A_i)$$

lies in $\mathcal{H}(n, \mathcal{A})$. The *Hamming scheme* $\mathcal{H}(n, q)$ is $\mathcal{H}(n, \mathcal{A})$ when $\mathcal{A}$ is the trivial association scheme on q vertices.

Let Ω be the set of words of length n over an alphabet of size q. The Hamming graph $H(n, q)$ has vertex set Ω, and two words are adjacent if and only if they differ in exactly one position. We use $A(n)$ to denote the adjacency matrix of $H(n, q)$. Up to permutation of the vertices, we can write $A(n)$ as

$$\begin{aligned}&[(J_q - I_q) \otimes I_q \otimes \cdots \otimes I_q] + [I_q \otimes (J_q - I_q) \otimes \cdots \otimes I_q] + \cdots + [I_q \otimes \cdots \otimes I_q \otimes (J_q - I_q)] \\ &= \sum_{i=1}^{d} \left[(A_i \otimes I_q \otimes \cdots \otimes I_q) + (I_q \otimes A_i \otimes \cdots \otimes I_q) + \cdots + (I_q \otimes I_q \otimes \cdots \otimes A_i)\right].\end{aligned} \quad (5)$$

Therefore $A(n)$ lies in the Bose–Mesner algebra of $\mathcal{H}(n, \mathcal{A})$ for any association scheme $\mathcal{A}$ on q vertices. More importantly, $A(n)$ satisfies the recursion

$$\begin{aligned}A(n) &= (J_q - I_q) \otimes I_{q^{n-1}} + I_q \otimes A(n-1) \\ &= \begin{pmatrix} A(n-1) & I_{q^{n-1}} & \cdots & I_{q^{n-1}} \\ I_{q^{n-1}} & A(n-1) & \cdots & I_{q^{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ I_{q^{n-1}} & I_{q^{n-1}} & \cdots & A(n-1) \end{pmatrix}.\end{aligned}$$

Here are some facts about $A(n)$ that are useful in the next section, see [1,6] for details. The matrix $A(n)$ has $n + 1$ eigenvalues

$$\theta_h(n) = (q-1)(n-h) - h \quad \text{for } h = 0, \ldots, n.$$

The eigenspace of $\theta_h(n)$, denoted by $V_h(n)$, has dimension $(q-1)^h \binom{n}{h}$. Note that $\theta_0(n) = (q-1)n$ is the valency of the vertices in the Hamming graph $H(n, q)$, so $\mathbf{1}_{q^n}$ is an eigenvector of $A(n)$ belonging to the eigenvalue $\theta_0(n)$. Since $V_0(n)$ has dimension one, $V_0(n) = \text{span}(\mathbf{1}_{q^n})$.

The next lemma exhibits the recursive nature of the eigenvectors of $A(n)$ in $V_h(n)$ when $h \geqslant 1$. Given a column vector u of length q^n, we use $u[i]$ to denote the ith block of u of length q^{n-1}.

Lemma 4.1. *Let $1 \leqslant h \leqslant n$. Then $u \in V_h(n)$ if and only if*

$$u[i] - u[j] \in V_{h-1}(n-1) \quad \text{for } i, j = 1, \ldots, q, \quad (6)$$

and

$$\sum_{i=1}^{q} u[i] \in V_h(n-1). \quad (7)$$

In particular, $u \in V_1(n)$ if and only if there exist a vector $w \in V_1(n-1)$ and scalars $a_1, \ldots, a_q$ such that $a_1 + \cdots + a_q = 0$ and

$$u[i] = w + a_i \mathbf{1}_{q^{n-1}} \quad \text{for } i = 1, \ldots, q.$$

Proof. From

$$\begin{pmatrix} A(n-1) & I_{q^{n-1}} & \cdots & I_{q^{n-1}} \\ I_{q^{n-1}} & A(n-1) & \cdots & I_{q^{n-1}} \\ \vdots & \ddots & \vdots & \vdots \\ I_{q^{n-1}} & I_{q^{n-1}} & \cdots & A(n-1) \end{pmatrix} \begin{pmatrix} u[1] \\ u[2] \\ \vdots \\ u[q] \end{pmatrix} = \theta_h(n) \begin{pmatrix} u[1] \\ u[2] \\ \vdots \\ u[q] \end{pmatrix},$$

we get

$$\left(A(n-1) - I_{q^{n-1}}\right) u[i] + \sum_{l=1}^{q} u[l] = \theta_h(n) u[i] \tag{8}$$

for $i = 1, \ldots, q$. It follows that

$$\left(A(n-1) - I_{q^{n-1}}\right) (u[i] - u[j]) = \theta_h(n)(u[i] - u[j])$$

or

$$A(n-1)(u[i] - u[j]) = \theta_{h-1}(n-1)(u[i] - u[j])$$

for $i, j = 1, \ldots, q$ and (6) follows.

We also get from (8) that

$$\sum_{i=1}^{q} \left((A(n-1) - I_{q^{n-1}}) u[i] + \sum_{l=1}^{q} u[l] \right) = \theta_h(n) \sum_{i=1}^{q} u[i]$$

which leads to

$$A(n-1) \sum_{i=1}^{q} u[i] = \theta_h(n-1) \sum_{i=1}^{q} u[i].$$

Hence (7) is true. The converse is straightforward.

Suppose $u \in V_1(n)$. From (6), there exist scalars $a_1, \ldots, a_q$ such that

$$\frac{1}{q} \sum_{j=1}^{q} (u[i] - u[j]) = a_i \mathbf{1}_{q^{n-1}} \quad \text{for } i = 1, \ldots, q.$$

Set

$$w = \frac{1}{q} \sum_{j=1}^{q} u[j].$$

Then by (7), $w \in V_1(n-1)$ and $w + a_i \mathbf{1}_{q^{n-1}} = u[i]$ holds for $i = 1, \dots, q$. Since

$$\sum_{i=1}^{q} a_i \mathbf{1}_{q^{n-1}} = \frac{1}{q} \sum_{i,j=1}^{q} (u[i] - u[j]) = 0,$$

we see that $a_1 + \cdots + a_q = 0$. The converse is again straightforward. □

5. When $\mathcal{N}_W$ contains the Hamming graph

In this section, we assume W is a type II matrix and $A(n)$ is the adjacency matrix of the Hamming graph $H(n, q)$ given in (5) for some $n \geqslant 2$ and $q \geqslant 3$.

Lemma 5.1. *Suppose $A(n) \in \mathcal{N}_W$. If $Y_{a,b} \in V_1(n)$, then either*

$$Y_{a,b} = \begin{pmatrix} a_1 \mathbf{1}_{q^{n-1}} & a_2 \mathbf{1}_{q^{n-1}} & \cdots & a_q \mathbf{1}_{q^{n-1}} \end{pmatrix}^T$$

where $a_1 + a_2 + \cdots + a_q = 0$, or

$$Y_{a,b} = \begin{pmatrix} w & w & \cdots & w \end{pmatrix}^T$$

for some non-zero vector $w \in V_1(n-1)$.

Proof. From Lemma 4.1, there exist $w \in V_1(n-1)$ and scalars $a_1, \dots, a_q$ satisfying $a_1 + \cdots + a_q = 0$ such that

$$Y_{a,b}[i] = w + a_i \mathbf{1}_{q^{n-1}} \quad \text{for } i = 1, \dots, q. \tag{9}$$

Now suppose w is not the zero vector and not all a_i's are zero, and we shall derive a contradiction. By Theorem 2.1 and the symmetry of $A(n)$,

$$\Theta_W(A(n))(a, b) = \Theta_W(A(n))(b, a)$$

so $Y_{b,a} \in V_1(n)$. Similar to $Y_{a,b}$, it follows from (6) that there exist scalars c_{ij} such that

$$Y_{b,a}[i] - Y_{b,a}[j] = Y_{a,b}[i]^{(-)} - Y_{a,b}[j]^{(-)} = c_{ij} \mathbf{1}_{q^{n-1}} \tag{10}$$

for all $i, j = 1, \dots, q$.

Applying (10) to the rth and the sth blocks gives

$$\frac{1}{w(l) + a_r} - \frac{1}{w(l) + a_s} = c_{rs} \quad \text{for } l = 1, \dots, q^{n-1}. \tag{11}$$

There exists $r \in \{1, \dots, q\}$ such that $a_r \neq 0$, and since $a_1 + \cdots + a_q = 0$, there exists $s \in \{1, \dots, q\}$ such that $a_s \neq a_r$. Then $c_{rs} \neq 0$ by (11). Hence, for $l = 1, \dots, q^{n-1}$, $w(l)$ is a root of the quadratic

$$x^2 + (a_r + a_s)x + a_r a_s + \frac{a_r - a_s}{c_{rs}} = 0. \tag{12}$$

Since $w \in V_1(n-1)$ is orthogonal to $\mathbf{1}_{q^{n-1}}$ and $w \neq 0$, there exist l and l' such that $w(l) \neq w(l')$. Then $w(l)$ and $w(l')$ are the roots of the quadratic (12). This implies that w has two distinct entries, the sum of which is $-(a_r+a_s)$. Also, since s was arbitrary subject to $a_s \neq a_r$, we see that the a_i's take exactly two distinct values.

Let w have x entries equal $\frac{-(a_r+a_s)}{2}+\alpha$ and $(q^{n-1}-x)$ entries equal $\frac{-(a_r+a_s)}{2}-\alpha$ where $\alpha \neq 0$. Let $Y_{a,b}$ have y blocks equal $w+a_r\mathbf{1}_{q^{n-1}}$, $q-y$ blocks equal $w+a_s\mathbf{1}_{q^{n-1}}$. Since $Y_{a,b}, Y_{b,a} \in V_1(n)$, we have

$$\mathbf{1}_{q^n}^T Y_{a,b} = (2y-q)q^{n-1}\frac{(a_r-a_s)}{2} + (2x-q^{n-1})q\alpha = 0,$$
$$\mathbf{1}_{q^n}^T Y_{b,a} = \frac{1}{(\frac{a_r-a_s}{2})^2-\alpha^2}\left((2y-q)q^{n-1}\frac{(a_r-a_s)}{2} - (2x-q^{n-1})q\alpha\right) = 0$$

These two equations give

$$x = \frac{q^{n-1}}{2} \quad \text{and} \quad y = \frac{q}{2}.$$

This is a contradiction if q is odd.

Now assume q is even. Then $\sum_{i=1}^{q} a_i = \frac{q}{2}a_r + \frac{q}{2}a_s = 0$, so $a_s = -a_r$. Assume, without loss of generality, that the first $\frac{q}{2}$ blocks of $Y_{a,b}$ are $w+a_r\mathbf{1}_{q^{n-1}}$ and the last $\frac{q}{2}$ blocks are $w-a_r\mathbf{1}_{q^{n-1}}$.

Since $\dim V_1(n) = (q-1)n > 1$ and $\{Y_{b,c} : c \in \Omega\}$ is a basis of $\mathbb{C}^{q^n}$, there exists $c \neq a$ such that $Y_{b,c} \in V_1(n)$. From (6), there exist scalars b_{ij} such that

$$Y_{b,c}[i] - Y_{b,c}[j] = b_{ij}\mathbf{1}_{q^{n-1}} \qquad \text{for } i,j = 1,\ldots,q.$$

There exists $k \in \{0, 1 \ldots, n\}$ such that $Y_{a,c} \in V_k(n)$. Then by (6), we have

$$Y_{a,c}[i] - Y_{a,c}[j] \in V_{k-1}(n-1).$$

On the other hand,

$$Y_{a,c}[i] - Y_{a,c}[j] = \begin{cases} b_{ij}(w+a_r\mathbf{1}_{q^{n-1}}) & \text{if } 1 \leqslant i,j \leqslant \frac{q}{2}, \\ b_{ij}(w-a_r\mathbf{1}_{q^{n-1}}) & \text{if } \frac{q}{2}+1 \leqslant i,j \leqslant q. \end{cases}$$

Since w and $a_r\mathbf{1}_{q^{n-1}}$ are non-zero vectors in distinct eigenspaces of $A(n-1)$, we have $b_{ij} = 0$ for $1 \leqslant i,j \leqslant \frac{q}{2}$ and for $\frac{q}{2}+1 \leqslant i,j \leqslant q$. Therefore the first $\frac{q}{2}$ blocks of $Y_{b,c}$ are identical and the last $\frac{q}{2}$ blocks of $Y_{b,c}$ are identical. If we let $u = Y_{b,c}[1] + Y_{b,c}[q]$, then

$$Y_{b,c}[i] = \begin{cases} \frac{b_{1q}}{2}\mathbf{1}_{q^{n-1}} + \frac{1}{2}u & \text{if } i = 1,\ldots,\frac{q}{2}, \\ -\frac{b_{1q}}{2}\mathbf{1}_{q^{n-1}} + \frac{1}{2}u & \text{if } i = \frac{q}{2}+1,\ldots,q. \end{cases}$$

By (7),

$$\sum_{i=1}^{q} Y_{b,c}[i] = \frac{q}{2}u \in V_1(n-1).$$

So if $Y_{b,c} \in V_1(n)$, then it lies in the span of

$$\left\{ \begin{pmatrix} \mathbf{1}_{q^{n-1}} \\ \vdots \\ \mathbf{1}_{q^{n-1}} \\ -\mathbf{1}_{q^{n-1}} \\ \vdots \\ -\mathbf{1}_{q^{n-1}} \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} u \\ \vdots \\ u \\ u \\ \vdots \\ u \end{pmatrix} : u \in V_1(n-1) \right\},$$

which has dimension at most $1 + (q-1)(n-1)$. The set $\{Y_{b,c} : c \in \Omega\}$ is a basis of $\mathbb{C}^{q^n}$, so there should be $\dim V_1(n) = (q-1)n$ eigenvectors of the form $Y_{b,c}$ in $V_1(n)$. But when $q \geqslant 3$, $(q-1)n > 1 + (q-1)(n-1)$. This is a contradiction. □

Lemma 5.2. *If* $A(n) \in \mathcal{N}_W$, *then*

$$I_q \otimes J_{q^{n-1}} \in \mathcal{N}_W.$$

Proof. Suppose $Y_{a,b} \in V_0(n)$. Then $Y_{a,b}[i] \in \operatorname{span}(\mathbf{1}_{q^{n-1}})$ for $i = 1, \ldots, q$, and $Y_{a,b}$ is an eigenvector of $I_q \otimes J_{q^{n-1}}$ belonging to the eigenvalue q^{n-1}.

Suppose $Y_{a,b} \in V_1(n)$. By the previous lemma, either all of $Y_{a,b}[1], \ldots, Y_{a,b}[q]$ lie in $\operatorname{span}(\mathbf{1}_{q^{n-1}})$ or they all lie in $V_1(n-1)$. In the former case, $Y_{a,b}$ is an eigenvector of $I_q \otimes J_{q^{n-1}}$ belonging to the eigenvalue q^{n-1}. In the latter case, $Y_{a,b}$ is an eigenvector of $I_q \otimes J_{q^{n-1}}$ belonging to the eigenvalue 0.

Suppose $Y_{a,b} \in V_h(n)$ for some $h > 1$. It follows from (6) that

$$J_{q^{n-1}}(Y_{a,b}[i] - Y_{a,b}[j]) = \mathbf{0}$$

for all $1 \leqslant i, j \leqslant q$. From (7), we have

$$\sum_{i=1}^{q} J_{q^{n-1}} Y_{a,b}[i] = \mathbf{0}.$$

These two equations give

$$J_{q^{n-1}} Y_{a,b}[i] = \mathbf{0} \qquad \text{for } i = 1, \ldots, q.$$

Therefore $Y_{a,b}$ is an eigenvector of $I_q \otimes J_{q^{n-1}}$ belonging to the eigenvalue 0. □

Theorem 5.3. *If* $A(n) \in \mathcal{N}_W$, *then* W *is type II equivalent to* $W_1 \otimes \cdots \otimes W_n$ *and*

$$\mathcal{N}_W = \mathcal{N}_{W_1} \otimes \cdots \otimes \mathcal{N}_{W_n},$$

where $W_1, \ldots, W_n$ *are* $q \times q$ *type II matrices.*

Proof. By Lemma 5.2, we have $I_q \otimes J_{q^{n-1}} \in \mathcal{N}_W$. Then

$$J_q \otimes I_{q^{n-1}} = A(n) - \left(A(n) \circ (I_q \otimes J_{q^{n-1}})\right) + I_{q^n}$$

also belongs to $\mathcal{N}_W$.

Theorem 3.5 tells us that W is type II equivalent to $W_1 \otimes V$ for some $q \times q$ type II matrix W_1 and $q^{n-1} \times q^{n-1}$ type II matrix V, and

$$\mathcal{N}_W = \mathcal{N}_{W_1} \otimes \mathcal{N}_V.$$

Observe that

$$A(n) \circ \left(I_q \otimes J_{q^{n-1}}\right) = I_q \otimes A(n-1) \in \mathcal{N}_W,$$

so $A(n-1) \in \mathcal{N}_V$. The theorem follows by induction. □

We now give Theorem 1.3 of [11] as an immediate consequence of this theorem and the fact that the unique 3×3 type II matrix, up to type II equivalence, is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \tag{13}$$

where ω is a primitive cube root of unity.

Corollary 5.4. *If $\mathcal{H}(n, 3) \subseteq \mathcal{N}_W$, then W is type II equivalent to a character table of $\mathbb{Z}_3^n$.*

Theorem 5.5. *Let $\mathcal{A}$ be an association scheme on $q \geqslant 3$ vertices. Then, for $n \geqslant 2$, the Bose–Mesner algebra of $\mathcal{H}(n, \mathcal{A})$ is not the Nomura algebra of a type II matrix.*

Proof. Suppose that the Bose–Mesner algebra of $\mathcal{H}(n, \mathcal{A})$ coincides $\mathcal{N}_W$ for some type II matrix W. Since $A(n)$ belongs to the span of $\mathcal{H}(n, \mathcal{A})$, it follows from Theorem 5.3 that

$$\mathcal{N}_W = \mathcal{N}_{W_1} \otimes \cdots \otimes \mathcal{N}_{W_n},$$

where $W_1, \ldots, W_n$ are $q \times q$ type II matrices. There exists a Schur idempotent $A_1 \neq I$ of $\mathcal{N}_{W_1}$, and $A_1 \otimes I \otimes \cdots \otimes I$ belongs to the association scheme defined by $\mathcal{N}_W$. This forces $A_1 \otimes I \otimes \cdots \otimes I \in \mathcal{H}(n, \mathcal{A})$ which is absurd. □

It is known that if $\mathcal{A}$ is formally self-dual, then so is $\mathcal{H}(n, \mathcal{A})$ [5]. The corollary gives plenty of examples of formally self-dual association schemes that are not the Nomura algebras of type II matrices.

Corollary 5.6. *The Bose–Mesner algebra of $\mathcal{H}(n, q)$, $n \geqslant 2$ and $q \geqslant 3$, is not the Nomura algebra of a type II matrix.*

When $n = 1$, $\mathcal{H}(1, q)$ is the trivial scheme on q vertices. It follows from Theorem 6.4 of [2] that the Nomura algebra of the Potts model of size q, for $q \geqslant 5$, is trivial. The Nomura algebra of the 3×3 type II matrix in (13) has dimension three. The Nomura algebra of a 4×4 type II matrix has dimension at least three [8]. So the Bose–Mesner algebra of $\mathcal{H}(1, q)$ is the Nomura algebra of a type II matrix exactly when $q \geqslant 5$.

The Bose–Mesner algebra of $\mathcal{H}(2, 2)$ is the Nomura algebra of

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & \alpha & -\alpha \\ 1 & -1 & -\alpha & \alpha \end{pmatrix}$$

when α is not a fourth root of unity [8].

Acknowledgement

The authors would like to thank Hiroshi Suzuki for bringing our attention to this problem and his result for $q = 3$ in [11], and many useful discussions.

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