Stability of shunting inhibitory cellular neural networks with unbounded time-varying delays

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Abstract

In this work the stability of shunting inhibitory cellular neural networks with unbounded time-varying delays is considered. Some new sufficient conditions for the existence and local stability of equilibrium points of the networks are established.

Keywords: Shunting inhibitory cellular neural networks; Stability; Existence; Equilibrium point; Unbounded time-varying delays

1. Introduction

Consider the shunting inhibitory cellular neural networks (SICNNs) with time-varying delays given by

$$x'_{ij}(t) = -a_{ij}x_{ij}(t) - \sum_{C_{kl} \in N_r(i,j)} C_{kl} f(x_{kl}(t - \tau(t)))x_{ij}(t) + L_{ij},$$

where $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, $C_{ij}$ denotes the cell at the $(i, j)$ position of the lattice, the $r$-neighborhood $N_r(i, j)$ of $C_{ij}$ is

$$N_r(i, j) = \{ C_{kl} : \max(|k - i|, |l - j|) \leq r, 1 \leq k \leq m, 1 \leq l \leq n \},$$

$x_{ij}$ is the activity of the cell $C_{ij}$, $L_{ij}$ is the external input to $C_{ij}$, the constants $a_{ij} > 0$ represent the passive decay rate of the cell activity, $C_{kl} \geq 0$ is the connection or coupling strength of postsynaptic activity of the cell transmitted to the cell $C_{ij}$, and the activity function $f(\cdot)$ is a continuous function representing the output or firing rate of the cell $C_{kl}$, and $\tau(t) \geq 0$ corresponds to the transmission delay.

Since Bouzerdoum and Pinter in \cite{1–3} described SICNNs as new cellular neural networks (CNNs), SICNNs have been extensively applied in psychophysics, speech, perception, robotics, adaptive pattern recognition, vision, and image processing. Hence, they have been the object of intensive analysis by numerous authors in recent years. In particular, there have been extensive results on the problem of the existence and stability of the equilibrium points,

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and periodic and almost periodic solutions of SICNNs with time-varying delays in the literature. We refer the reader to [4–7] and the references cited therein. Suppose that there exists a constant \( \tau \) such that

\[
(H_0) \quad \tau = \sup_{t \in R} \tau (t)
\]

is satisfied. Most of the authors of the bibliographies listed above obtained some sufficient conditions for the existence and stability of an equilibrium point, and periodic and almost periodic solutions for system (1.1). However, to the best of our knowledge, few authors have considered dynamic behaviors of system (1.1) without the assumption \((H_0)\). Thus, it is worthwhile to continue to investigate the stability of system (1.1) in this case.

The main purpose of this work is to give the new criteria for the stability of the equilibria of system (1.1). By applying mathematical analysis techniques similar to those in [8], without assuming \((H_0)\), we derive some sufficient conditions ensuring that the equilibrium point of system (1.1) is locally stable, which are new and complement previously known results. Moreover, an example is also provided to illustrate the effectiveness of our results.

Throughout this work, it will be supposed that

\[
\sup_{t \in R} \tau (t) = + \infty, \quad \tau (t) < t (t > 0).
\]

We assume that \( L_{ij} \) is a constant, and set

\[
\{x_{ij}(t)\} = (x_{11}(t), \ldots, x_{ln}(t), \ldots, x_{i1}(t), \ldots, x_{in}(t), \ldots, x_{m1}(t), \ldots, x_{mn}(t)).
\]

For \( \forall x(t) = \{x_{ij}(t)\} \in R^{m \times n} \), we define the norm \( \|x(t)\| = \max_{i,j}\{|x_{ij}(t)|\} \).

We also assume that the following conditions (T₁) and (T₂) hold.

(T₁) There exist constants \( M_f \) and \( \mu_f \) such that

\[
|f(u) - f(v)| \leq \mu_f |u - v|, \quad |f(u)| \leq M_f, \quad \text{for all } u, v \in R.
\]

(T₂) There exist constants \( L, 1 > \delta > 0 \) and \( \eta > 0, i j = 11, 12, \ldots, 1n, m1, m2, \ldots, mn \), such that

\[
L = \max_{(i, j)} \left\{ \frac{|L_{ij}|}{a_{ij}} \right\}, \quad \delta = \max_{(i, j)} \left\{ \sum_{C_{kl} \in N_r(i, j)} C_{ij}^k \mu_f \right\},
\]

\[
-a_{ij} + \sum_{C_{ij} \in N_r(i, j)} C_{ij}^k \mu_f \left( \frac{L}{1 - \delta} + M_f \right) < -\eta < 0. \tag{1.4}
\]

The initial conditions associated with system (1.1) are of the form

\[
x_{ij}(s) = \varphi_{ij}(s), \quad s \in (-\infty, 0], \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n, \tag{1.5}
\]

where \( \varphi_{ij}(\cdot) \) denotes real-valued bounded continuous function defined on \((-\infty, 0] \).

Since the existence of the equilibrium point of (1.1) is independent of delays, from Theorem 1 in [5], we can easily show the following lemma.

**Lemma 1.1.** Let (T₁) and (T₂) hold. Then system (1.1) has exactly one equilibrium point \( Z^* = \{x_{ij}^*\} = (x_{11}^*, x_{12}^*, \ldots, x_{mn}^*)^T \), and \( |x_{ij}^*| \leq \frac{L}{1-\delta}, i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n \).

2. **Main results**

**Theorem 2.1.** Let (T₁) and (T₂) hold. Suppose that \( Z^* = \{x_{ij}^*\} = (x_{11}^*, x_{12}^*, \ldots, x_{mn}^*)^T \) is the equilibrium point of system (1.1). Then, \( Z^* \) is locally stable. Namely, for \( \forall \varepsilon > 0 \), there exists a constant \( N > 0 \) such that for every solution \( Z(t) = \{x_{ij}(t)\} = (x_{11}(t), x_{12}(t), \ldots, x_{mn}(t))^T \) of system (1.1) with any initial value \( \varphi = \{\varphi_{ij}(t)\} = (\varphi_{11}(t), \varphi_{12}(t), \ldots, \varphi_{mn}(t))^T \) and

\[
\|\varphi - Z^*\| = \sup_{i,j} \left\{ \sup_{-\infty \leq t \leq 0} |\varphi_{ij} - Z_{ij}^*| \right\} < N.
\]
it holds that
\[ |x_{ij}(t) - x^*_{ij}| < \varepsilon, \quad \text{for all } t \geq 0, \ i j = 11, 12, \ldots, m n. \]

**Proof.** Let \( Z(t) = [x_{ij}(t)] = (x_{11}(t), x_{12}(t), \ldots, x_{mn}(t))^T \) be a solution of system (1.1) with any initial value \( \varphi = \{\varphi_j(t)\} = (\varphi_{11}(t), \varphi_{12}(t), \ldots, \varphi_{mn}(t))^T \), and define

\[
u(t) = [u_{ij}(t)] = (u_{11}(t), u_{12}(t), \ldots, u_{mn}(t))^T = Z(t) - Z^*.
\]

Then
\[
u'_{ij}(t) = -a_{ij}\nu_{ij}(t) - \sum_{c_{kl} \in \mathbb{N}_{i,j}} C^k_{ij}[f(x_{kl}(t - \tau(t)))x_{ij}(t) - f(x^*_{kl}(t - \tau(t)))x^*_{ij}], \quad (2.1)
\]
where \( i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n. \)

Let \((ij)_r\) be an index such that
\[
u_{(ij)_r}(t) = \|\nu(t)\|. \quad (2.2)
\]
Calculating the upper right derivative of \( \|\nu_{(ij)_r}(s)\| \) along (2.1), in view of (2.1) and (T1), we have
\[
D^+ \{\nu_{(ij)_r}(s)\}_{s=t} = \sign(\nu_{(ij)_r}(t)) \left\{ -a_{(ij)_r}y_{(ij)_r}(t) - \sum_{c_{kl} \in \mathbb{N}_{i,j}} C^k_{(ij)_r}[f(x_{kl}(t - \tau(t)))x_{(ij)_r}(t) - f(x^*_{kl}(t - \tau(t)))x^*_{(ij)_r}] \right\}
\leq \left\{ -a_{(ij)_r}|\nu_{(ij)_r}(t)| + \sum_{c_{kl} \in \mathbb{N}_{i,j}} C^k_{(ij)_r}[|f(x_{kl}(t - \tau(t)))x_{(ij)_r}(t)| - |f(x^*_{kl}(t - \tau(t)))x^*_{(ij)_r}|] \right\}
\leq \left\{ -a_{(ij)_r}|\nu_{(ij)_r}(t)| + \sum_{c_{kl} \in \mathbb{N}_{i,j}} C^k_{(ij)_r}(|\nu_{(ij)_r} + M_f|\nu_{(ij)_r}(t)|] \right\}
\leq \left\{ -a_{(ij)_r}|\nu_{(ij)_r}(t)| + \sum_{c_{kl} \in \mathbb{N}_{i,j}} C^k_{(ij)_r} \left[ \mu_f \frac{L}{1 - \delta} |\nu_{(ij)_r}(t)| + M_f|\nu_{(ij)_r}(t)| \right] \right\}. \quad (2.3)
\]
Let
\[
M(t) = \max_{s \leq t} \{\|\nu(s)\|\}. \quad (2.4)
\]
It is obvious that \( \|\nu(t)\| \leq M(t) \), and \( M(t) \) is non-decreasing. Now, we consider two cases.

**Case (i).** Suppose
\[
M(t) > \|\nu(t)\| \quad \text{for all } t \geq 0. \quad (2.5)
\]
Then, we claim that
\[
M(t) \equiv M(0) \quad \text{is a constant for all } t \geq 0. \quad (2.6)
\]
By way of contradiction, assume that (2.6) does not hold. Consequently, there exists \( t_1 > 0 \) such that \( M(t_1) > M(0) \). We have
\[
\|\nu(t)\| \leq M(0) \quad \text{for all } t \leq 0.
\]
So there must exist \( \beta \in (0, t_1) \) such that
\[
\|\nu(\beta)\| = M(t_1) \geq M(\beta).
\]
which contradicts (2.5). This contradiction implies that (2.6) holds. It follows that
\[ \|u(t)\| < M(t) = M(0) \quad \text{for all } t \geq 0. \tag{2.7} \]

Case (ii). If there is such a point \( t_0 \geq 0 \) that \( M(t_0) = \|u(t_0)\| \), then, using Eqs. (2.1) and (2.3), for \( \forall \varepsilon > 0 \), we get
\[ D^+ \left( [[u(ij)uisine]_t]_{t=t_0} \right) \leq -a_{ij} |[u(ij)uisine]_{t=t_0}| + \sum_{C_{kl} \in N_{ij(i,j)uisine}} C_{kl}^{ij} \left[ \mu_f \frac{L}{1-\delta} |[u_{kl}(t_0 - \tau(t_0))| + M_f |[u_{ij}](t_0)| \right] \]
\[ \leq -a_{ij} + \sum_{C_{kl} \in N_{ij(i,j)uisine}} C_{kl}^{ij} \left( \mu_f \frac{L}{1-\delta} + M_f \right) M(t_0) \]
\[ < -\eta M(t_0) + \eta \varepsilon. \tag{2.8} \]

In addition, if \( M(t_0) \geq \varepsilon \), then \( M(t) \) is strictly decreasing in a small neighborhood \((t_0, t_0 + \delta_0)\). This contradicts that \( M(t) \) is non-decreasing. Hence,
\[ \|u(t_0)\| = M(t_0) < \varepsilon. \tag{2.9} \]

Furthermore, for any \( t > t_0 \), by the same approach as is used in the proof of (2.9), we have
\[ \|u(t)\| < \varepsilon, \quad \text{if } M(t) = \|u(t)\|. \tag{2.10} \]

On the other hand, if \( M(t) > \|u(t)\|, t > t_0 \), we can choose \( t_0 \leq t_3 < t \) such that
\[ M(t_3) = \|u(t_3)\| < \varepsilon \quad \text{and} \quad M(s) > \|u(s)\| \quad \text{for all } s \in (t_3, t]. \]

Using an argument similar to that in the proof of Case (i), we can show that
\[ M(s) \equiv M(t_3) \quad \text{is a constant for all } s \in (t_3, t], \tag{2.11} \]

which implies that
\[ \|u(t)\| < M(t) = M(t_3) = \|u(t_3)\| < \varepsilon. \]

In summary, for all \( t \geq 0 \), we obtain
\[ \|u(t)\| < \max \{M(0), \varepsilon\}. \tag{2.12} \]

Hence, for \( \forall \varepsilon > 0 \), set
\[ N = \varepsilon > 0. \]

Then, for every solution \( Z(t) = (x_{11}(t), x_{12}(t), \ldots, x_{mn}(t))^T \) of system (1.1) with any initial value \( \varphi = (\varphi_{11}(t), \varphi_{12}(t), \ldots, \varphi_{nn}(t))^T \) and \( \|\varphi - Z^*\| = \max_{i,j} \{\sup_{-\infty < t \leq 0} |\varphi_{ij} - Z_{ij}^*|\} < N \), in view of (2.12), we have
\[ |x_{ij}(t) - x_{ij}^*| < \varepsilon, \quad \text{for all } t \geq 0, i,j = 11, 12, \ldots, mn. \]

This implies that the proof of Theorem 2.1 is now complete. \( \square \)

3. An example

In this section, we give an example to demonstrate the results obtained in previous sections.

Example 3.1. Consider the following SICNNs with time-varying delays:
\[ x'_{ij}(t) = -a_{ij}x_{ij}(t) - \sum_{C_{kl} \in N_{ij(i,j)}} C_{kl}^{ij} f(x_{kl}(t - \tau(t)))x_{ij}(t) + L_{ij}, \quad i,j = 1, 2, 3 \tag{3.1} \]
\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{bmatrix} = \begin{bmatrix}
  1 & 1 & 3 \\
  3 & 1 & 3 \\
  3 & 1 & 3
\end{bmatrix}
\] (3.2)

\[
\begin{bmatrix}
  c_{11} & c_{12} & c_{13} \\
  c_{21} & c_{22} & c_{23} \\
  c_{31} & c_{32} & c_{33}
\end{bmatrix} = \begin{bmatrix}
  0.1 & 0.2 & 0.1 \\
  0.2 & 0.2 & 0.1 \\
  0.1 & 0.2 & 0.1
\end{bmatrix},
\begin{bmatrix}
  L_{11} & L_{12} & L_{13} \\
  L_{21} & L_{22} & L_{23} \\
  L_{31} & L_{32} & L_{33}
\end{bmatrix} = \begin{bmatrix}
  0.94 & 0.91 & 2.94 \\
  2.91 & 0.87 & 2.91 \\
  2.94 & 0.91 & 2.94
\end{bmatrix}.
\] (3.3)

Set \( r = 1, \tau(t) = \frac{1}{2} |t \sin t|, i = 1, 2, 3, j = 1, 2, 3, \) and \( f(x) = \frac{1}{20} (|x - 1| - |x + 1|); \) clearly, \( M_f = 0.1, \mu_f = 0.1, \sum_{C_{kl} \in N_1(1,1)} C_{kl}^{1} = 0.5, \sum_{C_{kl} \in N_1(1,2)} C_{kl}^{2} = 0.8, \sum_{C_{kl} \in N_1(1,3)} C_{kl}^{3} = 0.5, \sum_{C_{kl} \in N_1(2,1)} C_{kl}^{2} = 0.8, \sum_{C_{kl} \in N_1(2,2)} C_{kl}^{2} = 1.2, \sum_{C_{kl} \in N_1(2,3)} C_{kl}^{2} = 0.8, \sum_{C_{kl} \in N_1(3,1)} C_{kl}^{3} = 0.5, \sum_{C_{kl} \in N_1(3,2)} C_{kl}^{3} = 0.8, \sum_{C_{kl} \in N_1(3,3)} C_{kl}^{3} = 0.5. \)

Then
\[
L = \max_{(i,j)} \frac{|L_{ij}|}{a_{ij}} = 0.98, \quad \delta = \max_{(i,j)} \left\{ \frac{\sum_{C_{kl} \in N_i(i,j)} C_{kl}^{ij} M_f}{a_{ij}} \right\} = 0.12,
\]
\[
-ai_j + \sum_{C_{kl} \in N_i(i,j)} C_{kl}^{ij} \left( \mu_f \frac{L}{1 - \delta} + M_f \right) < -0.5 < 0, \quad i, j = 1, 2, 3,
\]
which implies that system (3.1) satisfies all the conditions in Theorem 2.1. Hence, system (3.1) has exactly one equilibrium point \( Z^* \), and \( Z^* \) is locally stable.

**Remark 3.1.** System (3.1) is a form of shunting inhibitory cellular neural network with unbounded time-varying delays. Therefore, all the results in [2–8] and the references therein are inapplicable for proving that the equilibrium point of system (3.1) is locally stable.

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**References**


