1. INTRODUCTION

Consider the linear system
\[ dX + AX \, dt = B(X) \, dw, \]
where \( A \) is a linear operator generating the strongly continuous semigroup \( U_t \) on the separable Hilbert space \( H \), and where \( w \) is a Wiener process on the separable Hilbert space \( K \) with covariance operator \( W \), a positive nuclear operator in \( \mathcal{L}(K, K) \), the space of continuous linear mappings of \( K \) into itself. \( B(\cdot) \) is an element of \( \mathcal{L}(H, \mathcal{L}(K, H)) \). We shall also be concerned with mild solutions, i.e., solutions of the equation
\[ X_t = U_t X_0 + \int_0^t U_{t-s} B(X_s) \, dw_s. \]

For details on Wiener processes and stochastic integrals in Hilbert space, see [1].

The question of the asymptotic stability of the second moment of \( X_t \) has received considerable attention in the literature. Willems [2], Brockett and Willems [3], and Brockett [4] have given sufficient (and in some cases necessary) conditions which guarantee asymptotic stability when the spaces are finite dimensional. Wonham [5], Willems and Willems [6], and the author [7, 8] have considered a related problem, the stabilization problem, again in finite dimension. Recently Ichikawa [9] and Zabczyk [20] have extended these results to infinite dimensions. Triggiani and Pritchard [10] and Datko [11, 12] have some results for the infinite-dimensional nonstochastic problem.

In Section 2 we study mild solutions. We give conditions for the second moment of \( X_t \) to decay exponentially. Next we attempt to follow Kozin [13, 14], who treats the finite-dimensional case, to deduce sample asymptotic stability.

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Under either a certain boundedness condition on $A$, or a diagonalizability condition on $A$ plus a restriction on the range of $B$, we show that the sample paths are ultimately exponentially bounded. Then we conclude that the zero solution is pathwise asymptotically stable relative to finite-dimensional initial conditions. The point is, of course, that we can give conditions for exponential stability of the second moment (Theorem 1) so that we can establish asymptotic stability almost surely, which is the kind of stability that one likes to have in a physical situation.

In Section 3 we carry out a similar program for strong solutions in the sense of Pardoux [15]. Now we no longer require the restrictions on $A$ and $B$ mentioned above, and in fact $B$ need no longer be bounded (e.g., $B(X) = \text{grad } X$) but a coercivity condition is added. In Section 4 we give three examples.

We write $|x|$ for the Hilbert space norm of $x$; $|B(x)|$ for the norm of $B(x)$ in $\mathcal{L}(H, \mathcal{L}(K, H))$. We write $D^*$ for the adjoint of $D$ and $\text{tr } D$ for the trace of $D$. The separable process $\omega$ is defined on a probability space $(\Omega, F, P)$. $E$ denotes expectation. The theory of stochastic integrals in Hilbert space can be found in [1, 15, 16].

2. MILD SOLUTIONS

We show now that the zero solution of (2) is exponentially asymptotically stable in the mean square, and from this fact we deduce that the sample paths converge to zero as $t \to \infty$. If $\{U_t\}$ is a strongly continuous semigroup and if $B \in \mathcal{L}(H, \mathcal{L}(K, H))$, it follows by a standard argument (cf. [17, p. 395]) that for any initial condition $X_0$, independent of $\omega_t - \omega_s$, $t > s > 0$, with $E |X_0|^2 < \infty$, and for any $T < \infty$ there exists a unique $\omega$-adapted solution $X_t$ of (2) in the space $C(0, T; L_2(\Omega, H))$, where $C(0, T; S)$ denotes the Banach space of continuous functions mapping $[0, T]$ into the Banach space $S$. Hence a unique solution is defined on $[0, \infty)$, such that $E |X_t|^2$ is continuous and for any $T < \infty$,

$$\sup_{0 \leq t \leq T} E |X_t|^2 < \infty.$$ 

To state the first theorem we need two conditions:

**H$_1$:** $\exists \epsilon > 0, \gamma > 0$ such that $|U_t| \leq C e^{-\gamma t}$ $\forall t > 0$.

This exponential stability of the semigroup is equivalent to the requirement (cf. [18, p. 99]) that for all $\lambda > -\gamma$

$$|(\lambda I + A)^{-1}| \leq C(\lambda + \gamma)^{-1}.$$ 

**H$_2$:** $|\int_0^\infty U^*_t A(I) U_t dt| < 1$, where, for $P \in \mathcal{L}(H, H)$,

$$(A(P) x, y) = \text{tr}[B(x)^* PB(y) W].$$
Theorem 1. Assume $H_1$ and $H_2$. There exist positive constants $a, b$, such that for any solution $X_t$ of (2),

$$E |X_t|^2 \leq a E |X_0|^2 e^{-bt}, \quad t \geq 0. \tag{3}$$

Proof. As in [9, Sect. 3], there exists a unique positive operator $P$ in $L(H, H)$ such that

$$P = \int_0^\infty U_t^* [(I + \Delta(P))] U_t \, dt,$$

$$E(PX_0, X_0) = E \int_0^\infty |X_t|^2 \, dt.$$ 

Although in [9] $H$ is assumed to be real, the complex case follows similarly. The semigroup property now implies that

$$E(PX_t, X_t) = E \int_t^\infty |X_s|^2 \, ds.$$ 

It follows that

$$\frac{d}{dt} E(PX_t, X_t) = -E |X_t|^2 \leq -|P|^{-1} E(PX_t, X_t) \tag{4}$$

and hence

$$E(PX_t, X_t) \leq E(PX_0, X_0) e^{-t/|P|}. \tag{5}$$

But from (2)

$$E |X_t|^2 \leq 2E |U_tX_0|^2 + 2E \left| \int_0^t U_{t-s}B(X_s) \, dw_s \right|^2 \tag{6}$$

$$\leq 2C^2 e^{-2yt} E |X_0|^2 + 2\text{tr}[W] \int_0^t E |U_{t-s}B(X_s)|^2 \, ds.$$ 

Now from (4) and (5)

$$\int_0^t E |U_{t-s}B(X_s)|^2 \, ds \leq \int_0^t \frac{C^2 e^{-2y(t-s)} |B|^2 E |X_s|^2 \, ds}{2} \tag{7}$$

$$= -C^2 |B|^2 e^{-2y(t-s)} \frac{d}{ds} E(PX_s, X_s) \, ds$$

$$\leq kE(PX_0, X_0) (e^{-vt} + e^{-t/|P|}),$$

where we integrated by parts and then considered the three cases $2\gamma |P| > 1$, $2\gamma |P| = 1$, and $2\gamma |P| < 1$. Combining this inequality with (6) yields (3).

Theorem 1 is identical to the finite-dimensional result in [6, 7]. It is also related to the work of Zabczyk, i.e., [20, Theorem 2]: both theorems begin with
the necessary and sufficient Liapunov function condition given in [20, Theorem 1] and $H_1$ is assumed in both cases. To construct the Liapunov function we impose the sufficient condition $H_2$. Zabczyk, who is only working with systems of Lurie type, i.e., $B(x) \phi_j = b_j \langle c_j, x \rangle$, where $\{\phi_j\}$ is a complete set of orthonormal eigenvectors of the covariance operator $W$, is able to give a necessary and sufficient condition such that

$$E \int_0^\infty |X_t|^2 \, dt < \infty.$$ 

However, the method does not extend beyond Lurie systems. Work similar to Zabczyk's but in finite dimensions can be found in [2].

Next we deduce that the sample paths converge to zero almost surely if we assume the following condition.

$H_3$: (i) $\{U_t\}$ is an analytic semigroup.

(ii) There exists a function $f > 0$ such that for all $t < \infty$

$$\int_0^t f(s)^2 \, ds < \infty$$

and for all $i, t > 0, x \in H$,

$$|AU_t B(x) \phi_i| \leq f(t) |x|.$$ 

We observe that $H_3(i)$ implies $|AU_t| \leq c/t$, but this is not quite $H_3(ii)$. If however $A$ has a complete set of orthonormal eigenvectors, $\{\phi_i\}$, and $B(x)$ maps onto span $\{\phi_{i-1}^{m-1}, m < \infty$, for all $x$, then $H_3(ii)$ holds. The point is that to apply Kozin's method we must somehow put (2) into the form (1) where the stochastic integral is a martingale in $t$. Hence the assumptions $H_3$ and later $H_4$ are made.

**Theorem 21.** Assume $H_3$. If $X_t$ is a solution of (2) satisfying (3), then there exist $\alpha, \beta > 0$, $T(\omega) < \infty$, such that for $t > T(\omega)$

$$|X_t|^2 \leq \alpha E |X_0|^2 e^{-\alpha t}, \quad \text{w.p. 1.} \quad (7)$$

**Proof.** Set $\tilde{X}_t = X_t + U_t Y_0$, where $Y_0$ is chosen such that $X_0 + Y_0 \in D(A)$. From $H_3$ and [16, Lemma 2.23] it follows that $\int_0^t U_{t-s} B(X_s) \, dw_s \in D(A)$, hence $\tilde{X}_t \in D(A)$. Again from $H_3$ and [16, Proposition 2.13, Definition 2.18, and Lemma 2.23], we obtain

$$\tilde{X}_t = \tilde{X}_0 - \int_0^t AX_s \, ds + \int_0^t B(X_s) \, dw_s.$$ 

$^1$ If $\{U_t\}$ is a group satisfying $H_2$ and $H_4$ for all real $t$, then the theorem is true without $H_3$. 

Moreover from $H_3$ and (2) it follows for $t \geq \delta$ that

$$|A\tilde{X}_t| \leq K |\tilde{X}_{t-\delta}| + \left| \sum_{t}^{t} \int_{t-\delta}^{t} AU_{t-\tau} B(X_{\tau}) \phi_{\tau} \, d\beta_{\tau} \right|, \quad (8)$$

where $\{\beta_{\tau}\}$ is a family of independent Wiener processes such that

$$E((\beta_{\tau}(s) - \beta_{\tau}(t))^2) = \mu_{\tau} \, |t - s|$$

if $\mu_{\tau}$ is the eigenvalue corresponding to $\phi_{\tau}$.

Now

$$\tilde{X}_t = \tilde{X}_N + (\tilde{X}_t - \tilde{X}_N) \quad \text{for} \ t \geq N > \delta,$$

so that

$$\Pr \left\{ \sup_{N \leq t \leq N + 1} |\tilde{X}_t| \geq \epsilon_N \right\}$$

$$\leq \Pr \{ |\tilde{X}_N| \geq \epsilon_N/3 \} + \Pr \left\{ \sup_{N \leq t \leq N + 1} \left| \int_{N}^{t} AX_{s} \, ds \right| \geq \epsilon_N/3 \right\}$$

$$+ \Pr \left\{ \sup_{N \leq t \leq N + 1} \left| \int_{N}^{t} B(X_{s}) \, dw_{s} \right| \geq \epsilon_N/3 \right\}.$$

But from (8), (3), and $H_3$

$$\Pr \left\{ \sup_{N \leq t \leq N + 1} \left| \int_{N}^{t} AX_{s} \, ds \right| \geq \epsilon_N/3 \right\}$$

$$\leq \Pr \left\{ \int_{N}^{N + 1} |\tilde{X}_{s-\delta}| \, ds \geq \epsilon_N/6K \right\}$$

$$+ \Pr \left\{ \int_{N}^{N + 1} \left| \sum_{s-\delta}^{s} AU_{s-\tau} B(X_{\tau}) \phi_{\tau} \, d\beta_{\tau}(\tau) \right| \, ds \geq \epsilon_N/6 \right\}$$

$$\leq \frac{6K}{\epsilon_N} \left\{ \int_{N}^{N + 1} (aE |X_{0}|^2)^{1/2} e^{-b_3t/2} \, ds \epsilon_{b_8t/2} + \int_{N}^{N + 1} CE |Y_0| e^{-\nu s} \, ds \right\}$$

$$+ \frac{6 (\text{tr } W)^{1/2}}{\epsilon_N} \left\{ \int_{N}^{N + 1} (aE |X_{0}|^2)^{1/2} \left\{ \int_{s-\delta}^{s} f^2(s - \tau) e^{-bt} \, d\tau \right\}^{1/2} \, ds \right\}$$

$$\leq k_1 ([E |X_0|^2]^{1/2} e^{-bN/2} |E |Y_0| e^{-\nu N})/\epsilon_N.$$

Also from [15, p. 6],

$$\Pr \left\{ \sup_{N \leq t \leq N + 1} \left| \int_{N}^{t} B(X_{s}) \, dw_{s} \right| \geq \epsilon_N/3 \right\} \leq \frac{18 \text{tr } W}{\epsilon_N^2} |B|^{2} \int_{N}^{N + 1} aE |X_{0}|^2 e^{-bs} \, ds$$

$$\leq k_2 E |X_0|^2 e^{-bN}/\epsilon_N^2.$$
It now follows that

\[
\begin{align*}
\Pr \left\{ \sup_{N \leq t \leq N+1} |\bar{X}_t| \geq \varepsilon_N \right\} \\
\leq k_2 E |X_0|^2 e^{-bN/\varepsilon_N^2} + k_3 ((E |X_0|^2)^{1/2} e^{-bN/2} + E |Y_0| e^{-\gamma N})/\varepsilon_N \\
\leq k_4 e^{-bN/4}
\end{align*}
\]

if \( \varepsilon_N = \{E |Y_0| + (E |X_0|^2)^{1/2}\} e^{-bN/4} \) where we have assumed without loss of generality that \( 2\gamma \geq b \). The Borel–Cantelli lemma now implies that there is \( N(\omega) \) such that if \( N > N(\omega) \), then

\[
\sup_{N \leq t \leq N+1} |\bar{X}_t| < e^{-bN/2}(E |X_0|^2)^{1/2} + E |Y_0|)^2;
\]

hence

\[
\sup_{N \leq t \leq N+1} |X_t|^2 < 2e^{-bN/2}(E |X_0|^2)^{1/2} + E |Y_0|)^2 + C^2E |Y_0|^2
\]

and (7) follows since \( D(A) \) is dense in \( H \).

We give now a weaker result which however replaces \( H_3 \) by another condition.

\( H_4: \) \( A \) has a complete orthonormal set of eigenvectors \( \{\psi_j\} \) with corresponding eigenvalues \( \lambda_j \) where

\[
0 \leq \Re \lambda_1 \leq \Re \lambda_2 \leq \cdots.
\]

We observe that \( H_4 \) implies \( H_3 \) with \( \gamma = \Re \lambda_1 \). Let \( \Pi_m \) be the projection of \( H \) onto the span of \( \{\psi_1, \psi_2, \ldots, \psi_m\} \).

**Theorem 3.** Assume \( H_4 \). If \( X_t \) is a solution of (2) satisfying (3), then there exist \( \alpha, \beta > 0 \), \( T_m(\omega) < \infty \), such that if \( t > T_m(\omega) \) then for all \( m \)

\[
|\Pi_m X_t|^2 \leq \alpha E |X_0|^2 e^{-\beta t}, \quad \text{w.p. 1.} \quad (9)
\]

Moreover if \( \bigcup_{t>0} X_t(\omega) \) lies in a compact set \( K(\omega) \), then

\[
\lim_{t \to \infty} X_t(\omega) = 0 \quad \text{w.p.1.} \quad (10)
\]

**Proof.** The proof of (9) is similar to that of (7) but one begins with

\[
\begin{align*}
\Pi_m X_t &= U_{t-N} \Pi_m X_N + \sum_{i=1}^m \int_N^t U_{t-s} \Pi_m B(X_s) \, dw_s. \quad \text{But} \quad |\int_N^t U_{t-s} \Pi_m B(X_s) \, dw_s|^2 \\
&= \sum_{i=1}^m \exp[-2\Re \lambda_i t] |\sum_j \int_N^t e^{\lambda_i s} \langle B(X_s) \psi_j, \psi_i \rangle \, d\beta_j|^2 \leq \exp[2\Re \lambda_m] |\int_N^t U_{N+1-s} \Pi_m B(X_s) \, dw_s|^2.
\end{align*}
\]
Then one estimates

\[
\Pr \left\{ \sup_{N < t \leq N+1} \left| \int_0^t U_{t-s} \Pi_m B(X_s) \, dw_s \right| \geq \epsilon_N \right\} \leq \Pr \left\{ \sup_{N < t \leq N+1} \left| \int_0^t U_{N+1-s} \Pi_m B(X_s) \, dw_s \right|^2 \geq \epsilon_N^2/4 \exp(2 \Re \lambda^m) \right\}
\]

\[
\leq \frac{8 \tr W \epsilon N^2 - e^{2 \Re \lambda_m} \int_0^{N+1} E \left| U_{N+1-s} \Pi_m B(X_s) \right|^2 \, ds}
\]

\[
\leq \frac{8 \tr W \epsilon N^2 - e^{2 \Re \lambda_m} \left| B \right|^2 \alpha E \left| X_0 \right|^2 \epsilon^{-bN} C \right|^2.
\]

It follows that

\[
\Pr \left\{ \sup_{N < t \leq N+1} \left| \Pi_m X_t \right| \geq \epsilon_N \right\} \leq k_m E \left| X_0 \right|^2 \epsilon^{-bN}/\epsilon N^2.
\]

and the proof is completed as before with an application of the Borel–Cantelli lemma.

Suppose now that (10) fails. Then there is a sequence of times \( t_m \uparrow \infty \) and \( \delta > 0 \) such that \( \left| X_{t_m}(\omega) \right| \geq \delta \). Since \( X_{t_m}(\omega) \in K(\omega) \), then there is a convergent subsequence again called \( X_{t_m} \) such that \( \lim_{m \to \infty} X_{t_m}(\omega) = X_\omega(\omega) \) exists with \( \left| X_\omega(\omega) \right| > \delta \). However, from (9) we have that \( \langle X_\omega(\omega), \psi_m \rangle \leq \alpha E \left| X_0 \right|^2 \epsilon^{-b t} \) for \( t > T_m(\omega) \), any \( m < \infty \). Hence \( \langle X_\omega(\omega), \psi_m \rangle = 0 \), i.e., \( X_\omega(\omega) = 0 \) and a contradiction is reached.

**Corollary.** Assume \( H_4 \) and (3). If there exists \( m \) such that \( \Pi_m B(x) y = B(x) y \) for all \( x \in H, y \in K \), then (7) holds.

**Proof.** This result follows from the above proof since

\[
X_t = U_{t-N} X_N + \int_0^t U_{t-s} \Pi_m B(X_s) \, dw_s.
\]

**Remark.** To complete the pathwise stability analysis one would like to deduce that the zero solution is pathwise stable w.p. 1 (hence pathwise asymptotically stable if (7) holds), i.e., for \( X(0) = x_0 \) nonrandom

\[
\Pr \{ \lim \sup_{n \to 0} \sup_{|z_n| < n, t > 0} |X_t| = 0 \} = 1.
\]

Unfortunately we cannot quite do this. However, in most physical examples one treats finite dimensional initial conditions, so let \( \{ \varphi_i \} \) be any orthonormal basis.
of $H$ and write $\Pi_m$ for the projection of $H$ onto the span of $\{\bar{\psi}_1, \ldots, \bar{\psi}_m\}$. We let $X_t^i$ be the solution of (2) with $X_0^i = \bar{\psi}_i$. If

$$x = \sum_{i=1}^{n} x_i \bar{\psi}_i \in H$$

and if

$$X_t = U_t x + \int_0^t U_{t-s} B(X_s) \, dw_s,$$

then

$$X_t = \sum_{i=1}^{n} x_i X_t^i.$$ 

Assuming the conditions of the corollary to Theorem 3, one can estimate as in the proof that

$$\Pr\left\{\sup_{|x_0| < \eta} \sup_{t > 0} \left| \sum_{i=1}^{n} x_i X_t^i \right| > \epsilon \right\} \leq \Pr\left\{\sup_{t > 0} \left( \sum_{i=1}^{n} |X_t^i|^2 \right)^{1/2} > \epsilon \right\} \leq \sum_{i=1}^{n} \Pr\left\{\sup_{t > 0} |X_t^i| > \frac{\epsilon}{n^{1/2} \eta} \right\} \leq nk \left( \frac{n \eta^2}{\epsilon^2} + \frac{n^{1/2} \eta}{\epsilon} \right).$$

Since the events are monotonically decreasing in $\eta$ we have shown that for any $n < \infty$

$$\Pr\{\lim_{n \to 0} \sup_{|x_0| < \eta} \sup_{t > 0} |X_t|: |x_0| < \eta, x_0 \in \Pi_m H\} = 0\} = 1.$$

If $H_3$ holds, this equality is true with $t > 0$ replaced by $t > \delta > 0$, but the appropriate sets are monotone in $\delta$, and $\lim_{n \to 0} \sup_{|x_0| < \eta} \sup_{t > \delta} |X_t| = 0$ a.e. uniformly in $\delta$. Hence stability w.p. 1 again holds.

3. STRONG SOLUTIONS

We now develop a similar theory for Eq. (1). To obtain existence of strong solutions we restrict ourselves to the case studied in [15]. Hence let $V$ be a dense subspace of the real Hilbert space $H$, and assume that it is a Banach space under the norm $\| \cdot \|$. Let $V^*$ be its dual with norm denoted by $\| \cdot \|_*$. Then $V \subset H \subset V^*$, and we assume that the injection is continuous, i.e., $|x| \leq c \| x \|$ for $x \in V$. We write $\langle x, y \rangle$ for $x(y)$ if $x \in V^*$, $y \in V$. Now we assume that
$A: V \to V^*$ is a bounded map, with norm $\|A\|$, which is coercive, i.e., $\exists \alpha > 0$, $\lambda$ such that $\forall x \in V$,\[ \langle Ax, x \rangle + \lambda \| x \|^2 \geq \alpha \| x \|^2. \quad (11) \]

$B$ is, as usual, an element of $\mathcal{L}(H, \mathcal{L}(K, H))$. Then [15, p. 83, Theorem 1.1] tells us that for any $T < \infty$ there is a unique solution \[ X \in L_2(\Omega \times (0, T); V) \cap L_2(\Omega; C(0, T; H)). \]

Hence we have a solution defined on $[0, \infty)$. We observe that in [15], $B$ is required to be Hilbert–Schmidt but this is because the $B$ there is really $BW^{1,2}$, which is Hilbert–Schmidt if $W$ is nuclear.

Under the above conditions, $-A$ generates a strongly continuous semigroup and the strong solution is also a mild solution. Hence if we assume $H_1$ and $H_2$ then according to Theorem 1 \[ E \| X_t \|^2 \leq aE \| X_0 \|^2 e^{-bt}. \quad (12) \]

There is one condition, stronger than $H_1$ and $H_2$, which together with the stochastic energy equality implies (12) directly. This condition is \[ H_{12}: \exists \nu > 0 \text{ s.t. } \forall x \in V \]
\[ 2\langle Ax, x \rangle \geq \nu \| x \|^2 + (\Delta(I), x x). \]

Lemma 1 tells us that in fact the solution lies in $L_2(\Omega \times (0, \infty); V) \cap L_2(\Omega; C(0, \infty; H))$.

**Lemma 1.** Assume (12). There exists $K_0 < \infty$ such that \[ E(\sup_{0 \leq t < \infty} \| X_t \|^2) \leq K_0 E \| X_0 \|^2. \quad (13) \]

**Proof.** From the energy equality [15, p. 57, Theorem 3.1],
\[ \| X_t \|^2 = \| X_0 \|^2 - 2 \int_0^t \langle AX_s, X_s \rangle \, ds \]
\[ + 2 \int_0^t (X_s, B(X_s) \, dw_s) + \int_0^t (\Delta(I) X_s, X_s) \, ds \]
\[ \leq \| X_0 \|^2 + (2\lambda + |\Delta(I)|) \int_0^t \| X_s \|^2 \, ds + 2 \int_0^t (X_s, B(X_s) \, dw_s). \quad (14) \]

Hence
\[ E \sup_{0 \leq t < T} \| X_t \|^2 \leq E \| X_0 \|^2 \cdot (2\lambda + |\Delta(I)|) \int_0^T E \| X_s \|^2 \, ds \]
\[ + 2E \sup_{0 \leq t < T} \left| \int_0^t (X_s, B(X_s) \, dw_s) \right|. \quad (15) \]
But according to [15, p. 6, Theorem 1.3],

\[
2E \sup_{0 < t < T} \left| \int_0^t (X_s, B(X_s) \, dw_s) \right| \\
\leq 6E \left\{ \int_0^T |X_t|^2 (\Delta(I) X_s, X_s) \, ds \right\}^{1/2} \\
\leq 3E \left\{ 2 \sup_{0 < t < T} |X_t| \left[ \int_0^T (\Delta(I) X_s, X_s) \, ds \right]^{1/2} \right\} \\
\leq 3IE \{ \sup_{0 < t < T} |X_t|^2 \} + 3I^{-1} \int_0^T E(\Delta(I) X_s, X_s) \, ds.
\]

(16)

If we take \( I = \frac{1}{2} \) and substitute into (15) we obtain after using (12),

\[
E \sup_{0 < t < T} |X_t|^2 \leq 2E |X_0|^2 + (4\lambda + 38 |\Delta(I)|) ab^{-1}E |X_0|^2 (1 - e^{-bt}) \\
\leq \{ 2 + ab^{-1}(4\lambda + 38 |\Delta(I)|) \} E |X_0|^2.
\]

We can now obtain the pathwise exponential bound for strong solutions without further assumptions.

**Theorem 4.** Assume (11). If \( X_t \) is a solution of (1) satisfying (12), then there exist constants \( \alpha, \beta > 0, T(\omega) < \infty \) such that for \( t > T(\omega) \)

\[
|X_t|^2 \leq \alpha E |X_0|^2 e^{-\beta t} \quad \text{w.p. 1.}
\]

**Proof.** From (14) it follows that for \( t \geq N \)

\[
|X_t|^2 = |X_N|^2 - 2 \int_N^t \langle AX_s, X_s \rangle \, ds \\
+ 2 \int_N^t (X_s, B(X_s) \, dw_s) + \int_N^t (\Delta(I) X_s, X_s) \, ds \\
\leq |X_N|^2 + (2\lambda + |\Delta(I)|) \int_N^t |X_s|^2 \, ds + 2 \int_N^t (X_s, B(X_s) \, dw_s).
\]

Hence

\[
\Pr \left\{ \sup_{N \leq t < N+1} |X_t| \geq \epsilon_N \right\} \\
\leq \Pr( |X_N|^2 \geq \epsilon_N^2/3) + \Pr \left\{ \int_N^{N+1} |X_t|^2 \, dt \geq \epsilon_N^2/3(2\lambda + |\Delta(I)|) \right\} \\
+ \Pr \left\{ \sup_{N \leq t < N+1} \int_N^t (X_s, B(X_s) \, dw_s) \geq \epsilon_N^2/6 \right\}.
\]

(17)
Now from (16), (13), and (12),

\[
\Pr \left\{ \sup_{N \leq t \leq N+1} \left| \int_0^t (X_s, B(X_s) \, dw_s) \right| \geq \epsilon_N^2/6 \right\} \\
\leq 6\epsilon_N^{-2} \left\{ E \sup_{N \leq t \leq N+1} \left| \int_0^t (X_s, B(X_s) \, dw_s) \right| \right\} \\
\leq 18\epsilon_N^{-2}E \left\{ \sup_{N \leq t \leq N+1} \left| X_t \right| \left( \int_0^{N+1} E(\Delta(I) + X_s, X_s) \, ds \right)^{1/2} \right\} \\
\leq 18\epsilon_N^{-2}K^{1/2} E \left| X_0 \right|^{1/2} \left| \Delta(I) \right|^{1/2} d^{1/2} e^{-bN^2/2} \\
\leq k_1E \left| X_0 \right|^2 e^{-bN^2/\epsilon_N^2}.
\]

The proof can be completed as for Theorem 2. The remark at the end of Section 2 also applies here so that we have asymptotic stability relative to finite-dimensional initial conditions.

Let us now relax our conditions on \( B \), i.e., \( B \) will only be required to lie in \( \mathcal{L}(V, \mathcal{L}(K, H)) \), but the coercivity (11) is strengthened to

\[
2\langle Ax, x \rangle + \lambda \left| x \right|^2 \geq \alpha \left| x \right|^2 + \langle \Delta(I) x, x \rangle. \tag{18}
\]

Moreover for \( P \in \mathcal{L}(H, H) \), \( \Delta(P) \in \mathcal{L}(V, V^*) \) is defined by

\[
\langle \Delta(P) x, y \rangle = \text{tr} [B(x)^* PB(y) W], \quad x, y \in V.
\]

According to [15, p. 105, Theorem 3.1], a unique solution exists in \( L_2(\Omega \times (0, T); V) \cap L_2(\Omega; C(0, T; H)) \). We write \( \| B \| \) for the norm of \( B \) in \( \mathcal{L}(V, \mathcal{L}(K, H)) \). Next we must ensure that \( H_2 \) has a meaning. Under (11) (or (18)) \( U_t \) maps \( H \) into \( V \) such that there is a constant \( C_0 \) satisfying

\[
\int_0^\infty \| U_t x \|^2 e^{-2\lambda t} \, dt \leq C_0 \left| x \right|^2 \quad \forall x \in H;
\]

cf. [19, Chap. IV, Theorem 1.1]. Hence for \( T < \infty \)

\[
\left| \int_0^T U^*_t \Delta(I) U_t \, dt \right| \leq \sup_{|x|<1} \| \Delta(I) \| \left| \int_0^T \| U_t x \|^2 \, dt \right| < \infty.
\]

Hence \( H_2 \) holds for example if \( \lambda = 0 \) and if \( \text{tr} [W] \| B \|^2 C_0 < 1 \). In fact if \( \lambda = 0 \), then (18) implies \( H_{18} \) and hence (12) holds directly again.

Theorem 5 gives the bound (12) for the case \( \lambda > 0 \).

**Theorem 5.** Assume \( H_1 \) and \( H_2 \). There exist positive constants \( a, b \) such that for any solution \( X_t \) of (1)

\[
E \left| X_t \right|^2 \leq aE \left| X_0 \right|^2 e^{-bt}, \quad t \geq 0.
\]
Proof. Our proof parallels that of Theorem 1. \( P \) can be defined as 
\[ P = \lim_{n \to \infty} P^n, \]
where
\[ 0 < P^1 = \int_0^\infty U^*_t U_t \, dt \in \mathcal{L}(H, H) \]
and
\[ P^{n+1} = P^1 + T(P^n) \]
with
\[ T(Q) = \int_0^\infty U^*_t \Delta(Q) \, U_t \, dt \]
for \( Q \in \mathcal{L}(H, H) \). Moreover, \( T(Q) \geq 0 \) if \( Q \geq 0 \) so that \( P^n \neq 0 \) in addition for \( Q \geq 0 \)
\[ |T(Q)| = \sup_{|x|=1} \int_0^\infty \langle U^*_t \Delta(Q) \, U_t x, x \rangle \, dt \]
\[ = \sup_{|x|=1} \int_0^\infty \text{tr}[B(U_t x)^* Q B(U_t x) \, W] \, dt \]
\[ \leq |Q| \cdot |T(I)|. \]
According to \( H_2 \), \( |T(I)| < 1 \) so that \( P = \lim_{n \to \infty} P^n \) exists, and
\[ P = \int_0^\infty U^*_t [I + \Delta(P)] \, U_t \, dt \]
or
\[ -2(P A x, x) + \langle \Delta(P) \, x, x \rangle + |x|^2 = 0 \]
for all \( x \in D(A) = A^{-1}H \), a subset of \( V \) dense in \( H \).
As before it follows for \( x_0 \in D(A) \), that
\[ E(PX_t, X_s) = E(PX_s, X_s) - \int_s^t E |X_r|^2 \, dr, \quad t \geq s \geq 0. \quad (19) \]
Moreover the energy equality, (14), and (18) yield
\[ E |X_t|^2 \leq E |X_s|^2 + \lambda \int_s^t E |X_r|^2 \, dr - \alpha \int_s^t E \| X_r \|^2 \, dr. \quad (20) \]
Set \( \eta_t = \lambda E(PX_t, X_t) + E |X_t|^2 \). From (19) and (20) we obtain
\[ \eta_t \leq \eta_s - \alpha \int_s^t E \| X_r \|^2 \, dr \]
\[ \leq \eta_s - \alpha k_0 \int_s^t \eta_r \, dr, \quad t \geq s \geq 0, \]
because \( E \| X_t \|^2 \geq c^{-2} E \| X_t \|^2 \geq c^{-2} \eta_t (1 + \lambda \| P \|)^{-1} \). It follows that

\[
\frac{d\eta_t}{dt} \leq -\alpha k_0 \eta_t = -\nu \eta_t, \tag{21}
\]

and

\[
E \| X_t \|^2 \leq -\alpha^{-1} \frac{d\eta_t}{dt}. \tag{22}
\]

Now using the fact that \( X_t \) also satisfies (2), we deduce

\[
E \| X_t \|^2 \leq 2e^{-\nu t} E \| X_0 \|^2 + 2E \left| \int_0^t U_{t-s} B(X_s) \, dw_s \right|^2
\]

and

\[
E \left| \int_0^t U_{t-s} B(X_s) \, dw_s \right|^2 \leq \text{tr}[W] \int_0^t E \| U_{t-s} B(X_s) \|^2 \, ds \\
 \leq \text{tr}[W] \| B \|^2 \int_0^t e^{-2\nu(t-s)} E \| X_s \|^2 \, ds \\
 \leq k_4 e^{-k_2 t}
\]

if (21) and (22) are applied; cf. Theorem 1. Hence

\[
E \| X_t \|^2 \leq a E \| X_0 \|^2 e^{-bt}
\]

if \( X_0 \in D(A) \) w.p. 1. But since \( D(A) \) is dense in \( H \) the result holds for all \( X_0 \).

Now the proofs of Lemma 1 and Theorem 4 go through with (14) and (16) changed to

\[
\| X_t \|^2 \leq \| X_0 \|^2 + \lambda \int_0^t \| X_s \|^2 \, ds + 2 \int_0^t (X_s, B(X_s) \, dw_s), \tag{14'}
\]

\[
2E \sup_{0 \leq t \leq T} \left| \int_0^t (X_s, B \, dw_s) \right| \leq \frac{1}{2} E \left( \sup_{0 \leq t \leq T} \| X_t \|^2 \right) + 18 \int_0^t E\langle \Delta(I) X_s, X_s \rangle \, ds \\
 \leq \frac{1}{2} E \left( \sup_{0 \leq t \leq T} \| X_t \|^2 \right) + kE \| X_0 \|^2, \tag{16'}
\]

where we used (21) together with

\[
E\langle \Delta(I) X_s, X_s \rangle = E \text{tr}[B(X_s)^* B(X_s) W] \\
 \leq \text{tr}[W] \| B \|^2 E \| X_s \|^2 \\
 \leq -\alpha^{-1} \text{tr}[W] \| B \|^2 \frac{d\eta_t}{ds}
\]

by (22). Hence we have
Theorem 6. Assume (18) with $B \in \mathcal{L}(V; \mathcal{L}(K, H))$. If $X_t$ is a solution of (1) satisfying (12), then there exist constants $\alpha, \beta > 0$, $T(\omega) < \infty$, such that for $t > T(\omega)$,

$$|X_t|^2 \leq \alpha E |X_0|^2 e^{-\beta t} \quad \text{w.p. 1.}$$

Again we remark that we now have pathwise asymptotic stability relative to finite dimensional initial conditions.

4. Examples

Consider a one-dimensional rod of length $\pi$ whose ends are maintained at $0^\circ$ and whose sides are insulated. Assume that there is an exothermic reaction taking place inside the rod with heat being produced proportionally to the temperature. The temperature in the rod may be modeled to satisfy

$$\frac{\partial X}{\partial t} = \frac{\partial^2 X}{\partial y^2} + rX, \quad T > 0, \quad 0 < y < \pi,$$

$$X(t, 0) = X(t, \pi) = 0,$$

$$X(0, y) = x_0(y),$$

where $r$ depends on the rate of reaction. If we assume $r = r_0$, a constant, then we can solve

$$X(t, y) = \sum_{n=1}^{\infty} a_n e^{-(n^2 - r_0)t} \sin ny,$$

where $x_0(y) = \sum_{n=1}^{\infty} a_n \sin ny$. Hence we obtain asymptotic stability if $n^2 > r_0$ for all $n$, i.e., if $r_0 < 1$. This is condition $H_1$.

Suppose now that $r$ is random, and assume it is modeled as $r = r_0 + r_1 \omega$, so that (23) becomes

$$dX = \left(\frac{\partial^2}{\partial y^2} + r_0\right)X dt + r_1 X \, d\omega, \quad (24)$$

where $\omega$ is a one-dimensional Wiener process. We put this into our formulation by setting $K = R^1$, $H = L_2[0, \pi]$, $-A = \frac{\partial^2}{\partial y^2} + r_0$, $B(X) = r_1 X$. Now $\Delta(P)$ can be shown to be $\left(r_1^2 + r_0\right)$ so that $H_2$ becomes $r_1^2 < 2(1 - r_0)$. This is exactly $H_{12}$. Hence if the unperturbed system is very stable, i.e., $r_0 \ll 1$, then the perturbations (i.e., $r_1$) can be fairly large and according to Theorem 1 we still have $E |X_t|^2 \leq aE |X_0|^2 e^{-bt}$.

Condition $H_3$ is not met (but $H_4$ is); however, we can apply the theory of strong solutions. We set

$$V = W_0^{1,2} = H_0^1, \quad \langle Au, v \rangle = \int_0^\pi \left(\frac{\partial u}{\partial y} \frac{\partial v}{\partial y} - r_0 uv\right) dy.$$
Then $\langle Au, u \rangle = \| u \|^2 - r_0 \| u \|^2$ so $\alpha = 1$ and $\lambda = r_0$ in (11). The conclusion is that if $t > T(\omega)$ then

$$\| X_t \|^2 \leq \alpha E \| X_0 \|^2 e^{-\lambda t} \quad \text{w.p. 1.}$$

For the next example we suppose that Eq. (24) is replaced by

$$dX = \left( \frac{\partial^2}{\partial y^2} + r_a \frac{\partial}{\partial y} \right) X \, dt + r_1 \frac{\partial X}{\partial y} \, dw,$$

(25)
i.e., we are observing heat diffusion in a rod relative to an origin moving with velocity $r_0 + r_1 \tilde{v}$. Take $K, H,$ and $V$ as before. Then

$$\langle Au, v \rangle = \int_0^\pi \left( \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} - r_0 \frac{\partial u}{\partial y} v \right) \, dy,$$

$$\langle A(I) u, v \rangle = r_1^2 \int_0^\pi \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \, dy,$$

so that (18) becomes

$$2 \| u \|^2 + \lambda \| u \|^2 \geq \alpha \| u \|^2 + r_1^2 \| u \|^2.$$

This inequality is satisfied if $\lambda = 0$ and $\alpha = 2 - r_1^2, r_1^2 < 2$. Since $\lambda = 0$ then $H_{12}$ holds, and consequently for arbitrary $r_0$ and for $r_1^2 < 2$, pathwise asymptotic stability relative to finite-dimensional initial conditions follows with

$$\| X_t \|^2 \leq \alpha E \| X_0 \|^2 e^{-\lambda t}, \quad \text{w.p. 1.}$$

The last example (also treated in [20]) deals with delay systems of the form

$$dx_t + \int_{-h}^0 dN(s) x_{t+s} \, dt = B(x_t) \, dw_t, \quad x_t = 0, \ t < 0,$$

(26)
where $x_t \in \mathbb{R}^n$ and $N(\cdot)$ is a left-continuous function of bounded variation defined on $[-h, 0]$ into the space of $n \times n$ matrices. Let $H = \mathbb{R}^n \times L_2(-h, 0; \mathbb{R}^n)$ and let $D(A) = W^{1,2}(-h, 0; \mathbb{R}^n)$. Then $D(A)$ can be embedded in $H$ as $D(A) \sim \{ (f(0), f(\cdot)) \in H : f \in W^{1,2}(-h, 0; \mathbb{R}^n) \}$ and moreover $D(A)$ is dense in $H$. Note of course that $f \in D(A)$ is continuous. We write $f$ for its generalized derivative, and define

$$Af = \left( \int_{-h}^0 dN(s) f_s, -f(\cdot) \right).$$

We shall assume that $-A$ generates a strongly continuous semigroup $\{ U_t \}$ in
$H$; cf. [21] for examples. Now if we set $x^t = (x(t), x(t + \cdot)) \in H$, $B(x^t) w = (B(x_t) w, 0)$, then (26) implies

$$dx^t + A x^t \, dt = B(x^t) \, dw_i.$$ 

Now $H_3$ becomes

$$\sup \left\{ \Re \lambda : \det \left( \int_{-h}^{0} e^{\lambda s} \, dN_s + \lambda I \right) = 0 \right\} < 0. \quad (27)$$

To compute $H_2$ let us be more specific and assume $K = \mathbb{R}^n$, $B(x) w = \sum_{j=1}^{p} B_j x^j w_j$ where $B_j$ is an $n \times n$ matrix. Then

$$\langle \Delta(I) x, x \rangle = \left( \sum_{j=1}^{p} B^*_j B_j x, x \right)$$

and $H_2$ becomes

$$\sup_{|f| = 1} \int_{0}^{\infty} \sum_{j=1}^{p} |B_j y(t; f)|^2 \, dt < 1, \quad (28)$$

where $y(t; f)$ is the solution at time $t$ of

$$\dot{x}_t + \int_{-h}^{0} dN_s x_{t+s} = 0, \quad t \geq 0$$

with $y(t; f) = f(t)$ for $t \leq 0, f \in D(A)$.

Continuing now we note that $A$ is not usually coercive, and moreover $H_3$ and $H_4$ are not satisfied. However, Theorem 1 gives the exponential decay of the second moment, and (26) is actually a finite dimensional equation, i.e., $x(t) \in \mathbb{R}^n$. Hence we can proceed as in the proof of Theorem 2 to estimate

$$\Pr \left\{ \sup_{N \leq t \leq N+1} \int_{-h}^{t} \left| \int_{-h}^{0} dN_s x_{t+s} \right| \, dt > \epsilon \right\} \leq \Pr \left\{ \int_{N}^{N+1} \left| \int_{-h}^{0} dN_s x_{t+s} \right| \, dt > \epsilon \right\} \leq \int_{N}^{N+1} E \left( \int_{-h}^{0} dN_s x_{t+s} \right)^2 \, dt / \epsilon^2.$$

Now we assume that $H = \mathbb{R}^n \times L_2$, where either $L_2$ is taken under the measure $d | N_1 |$, or $L_2$ is taken under the usual Lebesgue measure and

$$dN_s = \left[ \sum_{i=1}^{m} c_i \delta_{s_i}(s) + n(s) \right] ds,$$

where $\delta_{s_i}(s)$ is the delta function at $s_i$, and $\int_{-h}^{0} |n(s)|^2 \, ds < \infty$. In either case

$$\int_{N}^{N+1} E \left( \int_{-h}^{0} dN_s x_{t+s} \right)^2 \, dt \leq K \sup_{N-h \leq t \leq N+1} E \left| x^t \right|^2 \leq K \epsilon E \left| x^0 \right|^2 e^{-\beta N}.$$
so that sample paths are asymptotic to 0, w.p. 1. Moreover since $x \in \mathbb{R}^n$, asymptotic stability w.p. 1 follows as in Section 2.

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