Boundedness and Compactness of Integral Operators on Spaces of Homogeneous Type and Applications, I

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Submitted by William F. Ames

Received November 2, 2000

1. INTRODUCTION

Let $X$ be a space of homogeneous type. For a function $f$ on $X$, let $M_f$ be the corresponding multiplication operator on some function space over $X$. Let $T_K$ be a standard Calderón–Zygmund (C-Z) operator that is bounded on $L^2(X)$. Let $C_f = [M_f, T_K]$ be the commutator of $M_f$ and $T_K$. When $X = \mathbb{R}^N$, a fundamental theorem of Coifman et al. [CRW] gives a characterization of boundedness of $C_f$ when the $T_K$ are the Riesz transforms $R_j$ ($j = 1, 2, \ldots, n$). The characterization of compactness was given by Uchiyama [UCH] and Janson [JAN]. The boundedness result was generalized to other contexts and important applications to some non-linear PDEs were given by Coifman et al. [CLMS]. Boundedness and compactness for $C_f$ on $L^p$ space over $X \times (0, \infty)$ was proved by Beatrous and Li [BEL], and applications to Hankel-type operators on Bergman spaces were given there. More general operators—multilinear operator or bilinear forms rather than commutators on $L^p(X)$—were studied by several authors (see examples, [COG, GLY, CLMS]), and characterizations of boundedness were given in that context when $X$ is $\mathbb{R}^N$ or the Heisenberg group. A sufficient condition on $f$ so that $C_f$ is bounded on $L^p(X)$ ($1 < p < \infty$) was given in [BC].

\textsuperscript{1} Author partially supported by NSF Grants DMS-9631359 and DMS-9531967.
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The main purpose of the present paper, the first one in a series of two papers on integral operators on function space over $X$, is to study the boundedness of generalized Toeplitz operators (including the commutator of a singular integral operator and a multiplication operator) on function spaces on a space of homogeneous type (Theorem 3.1). In particular, we shall generalize the results in [COG, CRW, JAN, LI] from $\mathbb{R}^n$ or $S^{2n-1}$ to much more general settings.

The second paper [KRL2] in this two-paper series has as its purpose to develop compactness theorems for operators on $L^p(X)$. Moreover, as applications, we formulate and prove characterization theorems (Theorems 2.2 and 2.4 in [KRL2]) for the boundedness and compactness of Hankel operators or commutators $[M_f, S]$ on holomorphic Hardy spaces $H^2(D)$, where $D$ is a bounded, strictly pseudoconvex domain in $\mathbb{C}^n$ and $S$ is the Szegö projection.

This paper is organized as follows: In Section 2, we recall some preliminary results, some notation, and some definitions. In Section 3, we study the boundedness of some generalized Toeplitz operators on $L^p$.

2. PRELIMINARIES

Let $X$ be a locally compact Hausdorff space. A **homogeneous structure** on $X$ consists of a positive regular Borel measure $\mu$ on $X$ and a family $\{B(x, r) : x \in X, r > 0\}$ of basic open subsets of $X$ such that, for some constants $c > 1$ and $K > 1$, we have

1. $x \in B(x, r)$ for all $x \in X$ and every $r > 0$;
2. If $x \in X$ and $0 < r_1 \leq r_2$, then $B(x, r_1) \subset B(x, r_2)$;
3. $0 < \mu(B(x, r)) < \infty$ for all $x \in X$ and all $r > 0$;
4. $X = \cup_{r>0} B(x, r)$ for some (and hence every) $x \in X$;
5. $\mu(B(x, cr)) \leq K\mu(B(x, r))$ for all $x \in X$ and all $r > 0$;
6. If $B(x_1, r_1) \cap B(x_2, r_2) \neq \emptyset$ and $r_1 \geq r_2$, then $B(x_2, r_2) \subset B(x_1, cr_1)$;
7. For $x \in X$, $\cap_{r>0} B(x, r) = \{x\}$.

We say that $X$ is a **space of homogeneous type** if $X$ is a locally compact Hausdorff space having a homogeneous structure. We usually denote our space of homogeneous type by $(X, \mu)$. Following Christ [CHR1], we assume from now on that $\mu(\{x\}) = 0$ for all $x \in X$.

If $X$ is a space of homogeneous type, then one may define a quasi-distance on $X$ as follows: If $x, y \in X$, then we let

\[d(x, y) = \inf\{t : y \in B(x, t), \text{ and } x \in B(y, t)\}.\]

It is clear that $d(x, y) = d(y, x)$. Since $X$ is Hausdorff and the sets $\{B(x, r) : x \in X, r > 0\}$ are basic open subsets of $X$, we may conclude from (3), (6),

\[\text{(2.1)}\]
and (7) that \( d(x, y) = 0 \) if and only if \( x = y \). From the so-called “doubling property” (5), we have that \( d(x, z) \leq C(c, K)[d(x, y) + d(y, z)] \). The doubling property is of course a substitute for the classical triangle inequality. Therefore \( d(\cdot, \cdot) \) is called a quasi-metric on \( X \).

Yet another approach is to define a “distance” by

\[
d(x, y) = \inf \{ \mu(B(z, t)) : y \in B(z, t), \ x \in B(z, t) \}.
\]

Coifman and Weiss refer to this quasi-metric as the measure distance.

For certain purposes, it is useful to choose a quasi-metric such that the measure of a ball \( B(x, r) \) associated to the quasi-metric is comparable to a fixed power \( r^\gamma \) of its radius. Following Theorem 3 in [MS], we have the following result:

**Lemma 2.1.** Let \( (X, \mu) \) be a space of homogeneous type. For any positive number \( \gamma \), there is a quasi-metric \( d_\gamma \) on \( X \) such that if

\[
B_\gamma(x_0, r) = \{ x \in X : d_\gamma(x, x_0) < r \},
\]

then

\[
\mu(B_\gamma(x_0, r)) \approx r^\gamma,
\]

where \( r > 0 \) and small when \( X \) is compact. Furthermore, with this quasi-metric we have

\[
\int_{X \setminus B_\gamma(x_0, t)} \mu(B_\gamma(x, t))^{-1+s} d\mu(x) \leq C_{\gamma} \mu(B_\gamma(x_0, t))^{-s+1}
\]

for all \( s > 1 \).

An examination of [MS] reveals that \( d_\gamma \) is simply a suitable power of the measure distance. Of course \( d_\gamma \) will still have the doubling property and will satisfy a quasi-triangle inequality.

Now we define the maximum mean oscillation on balls with fixed radius \( r \) as

\[
M(r, f) = \sup_{x \in X} \left\{ \mu(B(x, r)) \int_{B(x, r)} |f - m_B(f)| \, d\mu \right\}.
\]

(Here \( m_B(f) \) is the mean value of \( f \) on the ball \( B = B(x, r) \). In some contexts we will also use \( f_S \) to denote the mean of \( f \) over \( S \), where \( S \) is a more general set than a ball.)

**Definition 2.2.** Let \( (X, \mu) \) be a space of homogeneous type. Let \( f \in L^1_{\text{loc}}(X) \). We say that \( f \in BMO(X) \) if

\[
\|f\|_* = \sup_{0 \leq r < \infty} M(r, f) < \infty.
\]
We say that \( f \in VMO(X) \) if \( f \in BMO(X) \) and
\[
\lim_{r \to 0^+} M(r, f) = 0.
\]

Observe that this definition of \( BMO \) depends on the balls \( B(x, r) \) (see [KRA2] for more on this matter). Our definition of \( VMO \) is formulated for convenience; it suits the applications that we will discuss later. It is the same as other traditional definitions when \( X \) is compact or \( \mu(X) < \infty \).

Now we may define the atomic \( H^1 \) space as follows:

**Definition 2.3.** Let \((X, \mu)\) be a space of homogeneous type. Let \( a \in L^\infty(X) \). We say that \( a \) is an atom (or a 1-atom) if there is a ball \( B \) such that \( \text{supp}(a) \subseteq B \) and
\[
\begin{align*}
(i) & \quad |a(x)| \leq 1/\mu(B) \\
(ii) & \quad \int_B a(x) d\mu = 0.
\end{align*}
\]
Then
\[
H^1(X) = \left\{ u = \sum_{j=1}^\infty \lambda_j a_j : a_j \text{ are atoms and } \{\lambda_j\}_{j=1}^\infty \in \ell^1, \lambda_j \geq 0 \right\}
\]
with norm
\[
\|u\|_{H^1} = \inf \left\{ \sum_{j=1}^\infty |\lambda_j| : u = \sum_{j=1}^\infty \lambda_j a_j \right\}.
\]

The following result was proved in [COW2] (a version also appeared in [MS]):

**Theorem 2.4.** Let \( X \) be a space of homogeneous type. Then
\[
\begin{align*}
(i) & \quad [H^1(X)]^* = BMO(X); \\
(ii) & \quad [VMO(X)]^* = H^1(X).
\end{align*}
\]

The question of how to define atomic \( H^p(X) \) spaces for the full range \( 0 < p < 1 \) is more complex. The case for \( p \) very close to 1 (for example, if \( X = \mathbb{R}^N \) we take \( n/p \leq p < 1 \)) was treated by Coifman and Weiss [COW1, COW2] (also see [MS]). Let \( a \) be a bounded function on \( X \) with support in some ball \( B = B(x_0, r) \). We say that \( a \) is a \( p \)-atom in the sense of Coifman and Weiss if
\[
\begin{align*}
(i) & \quad |a(x)| \leq \mu(B)^{-1/p}; \\
(ii) & \quad \int_X a(x) d\mu(x) = 0.
\end{align*}
\]
The atoms of Coifman and Weiss are natural for values of $p$ that are close to 1 (in which context the elementary mean-value-zero property (ii) suffices for the purpose of studying singular integrals); when the value of $p$ is small, then the definition of Hardy space requires a higher order moment condition and is unworkable on an arbitrary space of homogeneous type. We have in fact developed a way to define a class of polynomials on a space of homogeneous type which helps us to distinguish the different degrees of smoothness and, therefore, the different moment conditions for $H^p$ spaces and Zygmund spaces. As a result we may formulate and prove the duality theorems on $H^p$ which cover all known duality theorems on $\mathbb{R}^N$, the Heisenberg group, nilpotent Lie groups, and the boundaries of certain classes of pseudoconvex domains. We will provide the details of these constructions in a future paper. The consideration of $H^p$, for $p$ small, on a space of homogeneous type will play no role in the remainder of the present paper.

Next we need some definitions (from Christ [CHR2]) for singular integrals on a space of homogeneous type.

**Definition 2.5.** Let $(X, \mu)$ be a space of homogeneous type. A **standard kernel** is a function $K : X \times X \setminus \{x = y\} \to \mathbb{C}$ such that there exist $\epsilon > 0$, and $0 < C < \infty$ satisfying

$$|K(x, y)| \leq \frac{C}{\lambda(x, y)}$$

for all distinct $x, y \in X$;

(2.7)

here

$$\lambda(x, y) = \mu(B(x, d(x, y)))$$

(2.8)

and

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq \left( \frac{d(x, x')}{d(x, y)} \right)^\epsilon \cdot \left( \frac{C}{\lambda(x, y)} \right)$$

(2.9)

whenever $d(x, y) \geq cd(x, x')$, where $c > 1$ is given in the doubling property (5) of this section.

**Remark.** On Euclidean space $\mathbb{R}^N$, the most important standard kernel is the classical Calderón–Zygmund kernel (see [CHR2])

$$K(x, y) = \frac{\Omega(x - y)}{|x - y|^N}.$$  

Here the function $\Omega(x)$ on $\mathbb{R}^N \setminus \{0\}$ is assumed to be homogeneous of degree 0, continuously differentiable, and $\int_{|x|=1} \Omega(x) \sigma(x) = 0$. Then (2.7) and (2.9) are easy to verify.
The idea of focusing on an estimate like (2.9), rather than homogeneity and the mean-value-zero property, goes back to Hörmander (see [HOR]). The forms of these estimates have been standardized in [DAJ] (see also [CHR2, KRA1]). In [KRA1], an explicit computation is performed to relate (2.9) to the classical Calderón–Zygmund conditions on the kernel.

**Definition 2.6.** A continuous linear operator $T : \Lambda_\delta \to \Lambda_\delta'$ is said to be associated to $K$ if $K$ is locally integrable away from the diagonal and

$$\langle Tf, g \rangle = \int \int K(x, y) f(y) g(x) \, d\mu(y) \, d\mu(x)$$

for all $f, g \in \Lambda_\delta$ whose supports are separated by a positive distance. Here $\Lambda''_\delta$ denotes the dual space of $\Lambda_\delta$.

**Definition 2.7.** A singular integral operator $T$ is an integral operator associated to a standard kernel $K$ so that $T$ is a continuous linear operator from $\Lambda_\delta$ to $\Lambda''_\delta$ for some $\delta \in (0, \delta_0]$.

The last two definitions are a standard part of the theory of the David–Journé $T(1)$ theorem. See [CHR2] for a nice discussion of these ideas.

The following theorem is due to Coifman and Weiss [COW1]; the basic arguments appear in [CAZ, STE].

**Theorem 2.8.** Let $T$ be any singular integral operator which is bounded on $L^2(X, \mu)$. In addition, if $X$ is non-compact then we assume that $T(1) = 0$. Then $T$ is bounded on $L^p(X, \mu)$ for all $p \in (1, \infty)$, is weak type $(1,1)$, and is bounded on BMO($X$).

The celebrated $T(1)$ theorem of David and Journé gives necessary and sufficient conditions to test when a singular integral operator is bounded on $L^2$. Let $T$ be a singular integral operator. We say that $T$ is weakly bounded on $L^2(X)$ if

$$|\langle T\phi, \psi \rangle| \leq C\mu(B(x_0, r))$$

for all $\phi, \psi \in \Lambda_\delta(B(x_0, r))$ supported in $B$ and

$$\|\phi\|_{L^\infty} + \|\psi\|_{L^\infty} \leq 1$$

for all $x_0 \in X$ and $r > 0$. It is easy to show that if $T$ is bounded on $L^2(X)$ then $T$ is weakly bounded on $L^2$. Conversely, we have the following $T(1)$ theorem of David and Journé (see [DAJ]).

**Theorem 2.9.** Any singular integral operator $T$ is bounded on $L^2$ if and only if the following holds:

(i) $T$ is weakly bounded on $L^2(X)$;

(ii) $T(1) \in BMO(X)$ and $T^*(1) \in BMO(X)$.
3. GENERALIZED TOEPLITZ OPERATORS

In this section, we shall study the boundedness of some generalized Toeplitz operators which come from a family of singular integral operators (see (3.2)) in the context of the Lebesgue spaces $L^p(X)$ ($1 < p < \infty$) on a space of homogeneous type.

Let $f \in L^2(X, \mu)$. Then we define the multiplication operator $M_f$ with symbol $f$ as follows: For any function $g$ on $X$, let 

$$M_f(g)(x) = f(x) \cdot g(x), \quad \text{all } x \in X.$$ 

Let $T_K$ be a singular integral operator with standard kernel $K(x, y)$. Then we denote by $C_f = M_f T_K - T_K M_f$ the commutator of $M_f$ and $T_K$. In order to obtain better control over the singular integrals to be defined later, we let 

$$K^n(x, y) = K(x, y) \text{ if } d(x, y) \geq \eta; \quad K^n(x, y) = 0 \text{ if } d(x, y) < \eta.$$ 

We also set 

$$T^n(g)(x) = \int_X K^n(x, y) g(y) d\mu(y)$$ 

and 

$$\tilde{T}(f)(x) = \sup_{0 < \eta < 1} |T^n(f)(x)|.$$ 

Then, by Theorem 12 in [CHR1], we have 

$T$ is bounded on $L^2(X) \iff \tilde{T}$ is bounded on $L^p(X)$ for $1 < p < \infty$.

We assume that the balls $B$ and measure $\mu$ satisfy the following condition: There is an $\epsilon > 0$ (depending only on $X$) such that 

$$c^\epsilon \mu(B(x, t)) \leq \mu(B(x, c^\epsilon t)) \leq C(c, K) \mu(B(x, t))$$

for all $t > 0$ and $x \in X$. Here $c > 1$ is given by the doubling property (5) in Section 2. This condition is sufficient to guarantee that (2.4) holds. Notice that (3.1) holds if $\frac{1}{C} \leq \mu(B(x, t)) \leq C t^{1/C}$ for some $C > 1$ and all $0 < t < 1$. Therefore, by Lemma 2.1, we can always choose a family of balls so that the above is true. However, most known examples of space of homogeneous type, especially the ones we consider in [KRL2], satisfy this condition.

Let $T_{j,1}, T_{j,2}$ ($j = 1, \ldots, m$) be a finite sequence of C-Z type operators. We shall consider a generalized Toeplitz operator 

$$T_b = \sum_{j=1}^{m} T_{j,1} M_b T_{j,2},$$

We shall always assume that $T_{j,1}, T_{j,2}$ are bounded on $L^2(X)$. 

Then, by Theorem 12 in [CHR1], we have 

$T_b$ is bounded on $L^2(X) \iff \tilde{T}_b$ is bounded on $L^p(X)$ for $1 < p < \infty$.
The main purpose of this section is to prove the following theorem.

**Theorem 3.1.** Let \((X, \mu)\) be a space of homogeneous type on which (3.1) holds. Let \(T_{j,i}\) be a sequence of C-Z operators which are bounded on \(L^2(X)\). If \(g \in L^p(X)\) and \(\overline{T}_1(g) = 0\) (here \(\overline{T}_1\) is the operator of type (3.2) that is associated to the symbol \(b \equiv 1\)) then, for any \(b \in \text{BMO}(X)\), we know that \(\overline{T}_b(g) \in L^p(X)\). Moreover,

\[
\|\overline{T}_b(g)\|_{L^p(X)} \leq C_p \left( \sum_{j=1}^{m} \|T_{j,1}\| \right) \left( \sum_{j=1}^{m} \|T_{j,2}\| \right) \|g\|_{L^p} \|b\|_* \quad \text{for all } 1 < p < \infty,
\]

where \(\|T_{j,i}\|\) denotes the operator norm of \(T_{j,i}\) on \(L^2(X)\).

When \(X\) is the Heisenberg group, this last result was proved by Grafakos et al. [GLY].

If we choose \(m = 2\) and \(T_{1,1} = T_{2,2} = I\) and \(T_{1,2} = T_{2,1} = T_K\), then \(\overline{T}_b = C_b\). Thus we have the following corollary which was independently proved in [BC].

**Corollary 3.2.** Let \((X, \mu)\) be a space of homogeneous type. Let \(T_K\) be a C-Z operator which is bounded on \(L^2(X)\). If \(b \in \text{BMO}(X)\), then \([M_b, T_K]\) is bounded on \(L^p(X)\) for all \(1 < p < \infty\).

Let \(f \in L^1_{\text{loc}}(X, \mu)\). Then we define the sharp maximal function of \(f\) on \(X\) as follows: For a.e. \(x \in X\),

\[
(3.3) \quad f^#(x) = \sup \left\{ \frac{1}{\mu(B)} \int_B |f(y) - f_B| \, d\mu(y) : r > 0 \right\}.
\]

Also the \(q\)-maximal function of \(f\) is

\[
(3.4) \quad M_q(f)(x) = \sup \left\{ \left( \frac{1}{\mu(B)} \int_B |f(y)|^q \, d\mu(y) \right)^{1/q} : r > 0 \right\}.
\]

Then we have Lemmas 3.3 and 3.4.

**Lemma 3.3.** Let \((X, \mu)\) be a space of homogeneous type. Let \(f \in L^{p_0}(X, \mu)\) for some \(1 \leq p_0 \) and \(p > p_0\). Then \(f \in L^p(X, \mu)\) if and only if \(f^#(x) \in L^p(X, \mu)\); and \(f \in L^p(X, \mu)\) if and only if \(M(f) = M_1(f) \in L^p(X, \mu)\) for all \(1 < p < \infty\).

The proof of Lemma 3.3 can be found in Christ and Fefferman [CHF]; see also Calderón [CAL] for \(X = \mathbb{R}^N\); Aimar and Macià [AIM] contains results on the Hardy–Littlewood maximal function on spaces of homogeneous type. Also see Strömberg and Torchinski [STT] for sharp maximal functions on spaces of homogeneous type.
LEMMA 3.4. Let $T$ be a Calderón–Zygmund operator.

(a) $T$ is bounded on $L^2(X) \iff \tilde{T}$ is bounded on $L^p(X)$ for $1 < p < \infty$.

(b) Let $1 < p < \infty$. Then $M_q(f) \in L^p(X, \mu)$ for all $1 < q < p$.

(c) If $f \in \text{BMO}(X)$, then we have $|f|_c B - |f|_B| \leq C(c, K)k\|f\|_\infty$, and

$$\sup \left\{ \frac{1}{|B|} \int_B |f(y) - f_B|^p d\mu(y) : B = B(x_0, r) \subset X \right\} \leq C_p\|f\|_p^p,$$

where $B = B(x_0, r) \subset X$.

Now we are ready to begin the proof of Theorem 3.1.

Proof. Without loss of generality, we may assume that

$$\|T_B\| \leq 1, \quad \text{for all } 1 \leq i \leq m, \ i = 1, 2.$$ 

Let $b \in \text{BMO}(X)$ have compact support, and $g \in L^p(X)$ with $\mathcal{T}_1(g) = 0$. Then $\mathcal{T}_B(g) \in L^p(X, \mu)$ for some $1 \leq p_0 < p$. By Lemma 6.4, it suffices to prove that $\mathcal{T}_B(g) \in L^p(X, \mu)$ and $\|\mathcal{T}_B(g)\|_p \leq C_p\||b||g||_p$.

Let $B = B(x, r)$ be any ball in $X$, and $cB = B(x, cr)$. We let

$$\mathcal{X}^1 = \mathcal{X}_{cB}; \quad \mathcal{X}^2 = 1 - \mathcal{X}_{cB}.$$ 

We have that $\mathcal{T}_1(g) = 0$, and so $\mathcal{T}_B(g) = b_B\mathcal{T}_1(g) = 0$. Thus

$$\mathcal{T}_B(g) = \mathcal{T}_{(b-b_B)x^1}(g) + \mathcal{T}_{(b-b_B)x^2}(g) = g_1 + g_2.$$ 

Note that

$$g_1(y) = \mathcal{T}_{(b-b_B)x^1}(g)(y) = \sum_{j=1}^m T_{j,1}[(b - b_B)b^1 T_{j,2}(g)](y).$$

Thus, for each $1 < q < p$, we can choose $1 < \gamma < \infty$ such that $q\gamma < p$. Now

$$\left( \int_B |g_1(y)|^q d\mu \right)^{1/q} \leq \sum_{j=1}^m C_{q,1} \left( \int_{cB} |b - b_B|^{q\gamma} |T_{j,2}(g)(y)|^q d\mu(y) \right)^{1/q}$$

$$\leq \sum_{j=1}^m C_{q,\gamma,1} \left( \int_{cB} |b - b_B|^{q\gamma} d\mu \right)^{1/(q\gamma)} \left( \int_{cB} |T_{j,2}(g)|^{q\gamma} d\mu \right)^{1/(q\gamma)}$$

$$\leq \sum_{j=1}^m C_{q,\gamma,1} ||b||_\gamma \mu(B)^{1/(q\gamma)} M_q(g, T_{j,2}(g)(x)) \mu(2B)^{1/q\gamma}.$$
Now we consider $g$ and where

$$y \leq 1 < \gamma < p$$

We shall present the proof of (3.7) here. Let $b \in B = B(x, r)$, we have

\begin{align}
|T_K(g\varphi^2)(y) − T_K(g\varphi^2)(x)| & ≤ C_{\epsilon, \gamma}M(g)(x) \\
|T_K((b - b_B)\varphi^2)(y) − T_K((b - b_B)\varphi^2)(x)| & ≤ C_{\epsilon, \gamma}M_\gamma(g)(x),
\end{align}

where $1 < \gamma < p$.

**Proof.** The proof of (3.6) is similar to and easier than the proof of (3.7). We shall present the proof of (3.7) here. Let $y \in B(x, r)$. Then

\begin{align}
|T_K((b - b_B)g\varphi^2)(y) − T_K((b - b_B)g\varphi^2)(x)| &= \left| \int_X(b - b_B)g(z)\varphi^2(z)(K(y, z) - K(x, z))d\mu(z) \right| \\
&= \left| \int_{X\setminus B}b - b_B||g(z)||K(y, z) - K(x, z)|d\mu(z) \right| \\
&≤ \sum_{k=2}^\infty \int_{c^kB \cup (c^{k-1}B)}|b - b_B||g(z)||K(y, z) - K(x, z)|d\mu(z) \\
&≤ C \sum_{k=2}^\infty \left( \int_{c^kB} \frac{|b - b_B|^\gamma d\mu}{\mu(2^kB)} \right)^{1/\gamma} \left( \int_{c^kB} \frac{|g(z)|^\gamma d\mu(z)}{\mu(c^kB)} \right)^{1/\gamma} c^{-(k-1)\epsilon} \\
&≤ C \sum_{k=2}^\infty C_{\gamma} \|b\|^\gamma kM_\gamma(g)(x)c^{-(k-1)\epsilon}
\end{align}
\[
\leq C_\gamma \|b\| \|M_\gamma(g)(x)\| \sum_{k=2}^{\infty} c^{-(k-1)\epsilon} k
\]

\[
\leq C C \epsilon \gamma \|b\| \|M_\gamma(g)(x)\|
\]

and the proof of the lemma is complete. ■

Observe that

\[
\frac{1}{\mu(B)} \int_B |g_2(y) - (g_2)_B| d\mu(y) \leq \frac{2}{\mu(B)} \int_B \int_B |g_2(y) - g_2(x)| d\mu(x) d\mu(y),
\]

where

\[
g_2(y) = \sum_{j=1}^{m} T_{j,i,1}(b - b_B) x^2 T_{j,2}(g))(y).
\]

Then, for each \( y \in B(x, r) \), we have

\[
|g_2(y) - g_2(x)| \leq \sum_{j=1}^{m} C_{\epsilon, \epsilon_0, \gamma, j} \|b\| \|M_\gamma(T_{j,2}(g))(x)\|
\]

Thus

\[
\frac{1}{\mu(B)} \int_B |g_2(y) - (g_2)_B| d\mu(y) \leq \sum_{j=1}^{m} C_{\epsilon, \epsilon_0, \gamma, j} \|b\| \|M_\gamma(T_{j,2}(g))(x)\|
\]

We know that the \( T_{j,i} \) are bounded on \( L^p(X) \) and

\[
T_\delta(g)^\#(x) = g_1^\#(x) + g_2^\#(x).
\]

Combining this result and the estimation of \( g^\#_1(x) \) and \( g^\#_2(x) \), we see that

\[
(3.8) \quad \|T_\delta(g)^\#\|_{L^p} \leq C \sum_{j=1}^{m} \|T_{j,1}\| \|T_{j,2}\| \|b\| \|g\|_{L^p(X)}.
\]

By Lemma 3.4, we have that \( T_\delta(g) \in L^p(X, \mu) \) for all \( 1 < p < \infty \) when \( b \) has compact support. For general \( b \), we may assume that \( g \) is smooth and compactly supported; then the above arguments show that (3.8) holds. The fact that \( C_\gamma(X) \) is dense in \( L^p(X, \mu) \) implies that the proof of Theorem 3.1 is complete. ■

As a direct consequence of Theorem 3.1, we have the following theorem (see [COG, GLY] for the special cases when \( X \) is \( \mathbb{R}^n \) or a homogeneous group).

**Theorem 3.6.** Let \( (X, \mu) \) be a space of homogeneous type. If \( f \in L^p(X) \) is such that \( T_\delta(f) = 0 \), then the linear operator \( B_\delta(g) = \sum_{j=1}^{m} (T_{j,1}(g), T_{j,2}(f)) \) is bounded from \( L^q(X) \to H^1(X) \), where \( p, q > 1 \) and \( 1/p + 1/q = 1 \) and \( (\cdot, \cdot) \) denotes the inner product in \( \mathbb{C}^n \).
Proof. Since

\[ \langle \mathcal{J}_b(f), g \rangle = \sum_{j=1}^{m} \langle bT_j, [b, T_j](f) \rangle, g \rangle \]

\[ = \sum_{j=1}^{m} \langle [b, T_j], T_j^*(g) \rangle \]

\[ = \sum_{j=1}^{m} \langle b, T_j T_j^*(g) \rangle \]

\[ = \langle b, B_j(g) \rangle \]

for all \( b \in BMO(X) \) with compact support, and by Theorem 3.1, we have

\[ |\langle b, B_j(g) \rangle| \leq C_{p,q} \left( \sum_{j=1}^{m} \|T_j\| \right) \left( \sum_{j=1}^{n} \|T_j^*\| \right) \|b\|_{L_p} \|f\|_{L_p} \|g\|_{L_q}. \]

Now the set of VMO functions with compact support is dense in VMO(X); using Theorem 2.4, we have that \( B_j(g) \in H^1(X) \), and the proof is complete.

REFERENCES


