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Viscosity approximation methods for a finite family of nonexpansive mappings in Banach spaces ☆

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Abstract

By using viscosity approximation methods for a finite family of nonexpansive mappings in Banach spaces, some sufficient and necessary conditions for the iterative sequence to converging to a common fixed point are obtained. The results presented in the paper extend and improve some recent results in [H.K. Xu, Viscosity approximation methods for nonexpansive mappings, J. Math. Anal. Appl. 298 (2004) 279–291; H.K. Xu, Remark on an iterative method for nonexpansive mappings, Comm. Appl. Nonlinear Anal. 10 (2003) 67–75; H.H. Bauschke, The approximation of fixed points of compositions of nonexpansive mappings in Banach spaces, J. Math. Anal. Appl. 202 (1996) 150–159; B. Halpern, Fixed points of nonexpansive maps, Bull. Amer. Math. Soc. 73 (1967) 957–961; J.S. Jung, Iterative approaches to common fixed points of nonexpansive mappings in Banach spaces, J. Math. Anal. Appl. 302 (2005) 509–520; P.L. Lions, Approximation de points fixes de contractions', C. R. Acad. Sci. Paris Sér. A 284 (1977) 1357–1359; A. Moudafi, Viscosity approximation methods for fixed point problems, J. Math. Anal. Appl. 241 (2000) 46–55; S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces, J. Math. Anal. Appl. 75 (1980) 128–292; R. Wittmann, Approximation of fixed points of nonexpansive mappings, Arch. Math. 58 (1992) 486–491].

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1. Introduction and preliminaries

Throughout this paper, we assume that E is a real Banach space, E^* is the dual space of E, C is a nonempty closed convex subset of E, $\mathrm{Fix}(T)$ is the set of fixed points of mapping T and $J: E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{ f \in E^*,\ \langle x, f \rangle = \|x\| \|f\|,\ \|f\| = \|x\| \}, \quad x \in E. \tag{1.1}$$

Definition 1.

(1) A mapping $f: C \to C$ is said to be a *contraction on C with a contractive constant* $\alpha \in (0,1)$, if

$$\| f(x) - f(y) \| \leqslant \alpha \|x - y\|, \quad \forall x, y \in C.$$

In the sequel, we always use Π_C to denote the collection of all contractions on C with a suitable contractive constant $\alpha \in (0,1)$. That is

$$\Pi_C = \{ f\colon\ C \to C,\ \text{a contraction with a suitable contractive constant} \}. \tag{1.2}$$

(2) Let $T: C \to C$ be a mapping. T is said to be *nonexpansive* if

$$\|Tx - Ty\| \leqslant \|x - y\|, \quad \forall x, y \in C.$$

(3) Suppose that to each $x \in E$, there exists a unique $Px \in C$ such that $\|x - Px\| = d(x, C)$. Then C is said to be a *Chebyshev set* and the mapping $P: E \to C$ is called *the metric projection onto C*.

(4) Let K be a subset of C. A mapping P of C onto K is said to be *sunny*, if $P(Px + t(x - Px)) = Px$ for each $x \in C$ and $t \geqslant 0$ with $Px + t(x - Px) \in C$ (see, for example, [4] or [5]).

(5) A subset K of C is called a *nonexpansive retract of C*, if there exists a nonexpansive retraction of C onto K.

Definition 2. Let $U = \{x \in E\colon\ \|x\| = 1\}$. E is said to be *uniformly smooth*, if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists and is attained uniformly in $x, y \in U$.

It is well known that the following proposition is true:

Proposition 1. [6] *If E is a uniformly smooth Banach space, then the normalized duality mapping J defined by* (1.1) *is single-valued and uniformly continuous on each bounded subset of E from the norm topology of E to the norm topology of E^*.*

Let $T: C \to C$ be a nonexpansive mapping. For given $f \in \Pi_C$ and for given $t \in (0,1)$ define a contraction mapping $T_t: C \to C$ by

$$T_t x = t f(x) + (1 - t) T x, \quad x \in C. \tag{1.3}$$

By Banach's contraction principle it yields a unique fixed point $z_t \in C$ of T_t, i.e., z_t is the unique solution of the equation

$$z_t = tf(z_t) + (1-t)Tz_t. \tag{1.4}$$

Concerning the convergence problem of $\{z_t\}$, in 2000, Moudafi [10] by using the viscosity approximation method proposed by himself proved that if E is a real Hilbert spaces, then the sequence $\{z_t\}$ converges strongly to a fixed point x_* of T in C which is the unique solution to the following variational inequality:

$$\langle (I-f)x^*, x-x^* \rangle \geqslant 0, \quad \forall x \in \mathrm{Fix}(T). \tag{1.5}$$

It should be pointed out that Moudafi's above result is a generalization of the corresponding result in Browder [2].

In 2004, Xu [14] studied further the viscosity approximation method for a nonexpansive mapping and proved the following result:

Theorem. (Xu [14, Theorem 4.1]) *Let E be a uniformly smooth Banach space, C be a nonempty closed convex subset of E, $T : C \to C$ be a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$ *and $f \in \Pi_C$. Then the sequence $\{z_t\}$ defined by* (1.4) *converges strongly to a fixed point in* $\mathrm{Fix}(T)$. *If we define $P : \Pi_C \to \mathrm{Fix}(T)$ by*

$$P(f) := \lim_{t\to 0} x_t, \quad f \in \Pi_C, \tag{1.6}$$

then $P(f)$ solves the variational inequality

$$\big\langle (I-f)P(f), J\big(p - P(f)\big)\big\rangle \geqslant 0, \quad \forall p \in \mathrm{Fix}(T). \tag{1.7}$$

In particular, if $f = u \in C$ (where u is a given point in C), then (1.6) *is reduced to the sunny nonexpansive retraction of Reich* [11] *from C onto* $\mathrm{Fix}(T)$:

$$\big\langle P(u) - u, J\big(p - P(u)\big)\big\rangle \geqslant 0, \quad \forall p \in \mathrm{Fix}(T).$$

On the other hand, in 1996, Bauschke [1] introduced and studied the following iterative process for a finite family of nonexpansive mappings $T_1, T_2, \ldots, T_r$ in a Hilbert space:

$$x_{n+1} = \alpha_{n+1}u + (1-\alpha_{n+1})T_{n+1}x_n, \quad \forall n \geqslant 0, \tag{1.8}$$

where u and x_0 are any given two points in C, $\{\alpha_n\}$ is a real sequence in (0, 1) and $T_n = T_{n\,(\mathrm{mod}\, r)}$. Under suitable conditions he proved the convergence of the sequence $\{x_n\}$ to converge to a common fixed point $P_F u$ of $T_1, T_2, \ldots, T_r$ in C, where $P_F : H \to F = \bigcap_{i=1}^r \mathrm{Fix}(T_i)$ is the metric projection.

The purpose of this paper is by using the viscosity approximation method for a finite family of nonexpansive mappings $\{T_1, T_2, \ldots, T_N\}$ on C to obtain some sufficient and necessary criteria for the following iterative sequence:

$$x_{n+1} = \alpha_{n+1} f(x_n) + (1-\alpha_{n+1})T_{n+1}x_n, \quad \forall n \geqslant 0. \tag{1.9}$$

to converging to a common fixed point of $T_1, T_2, \ldots, T_N$ in Banach spaces, where $\{\alpha_n\}$ is real sequence in (0, 1), f is a given mapping in Π_C, $x_0 \in C$ is any given point and $T_n = T_{n\,(\mathrm{mod}\, N)}$.

Special cases. Now we consider some special cases of sequence (1.9):

(1) If E is a Hilbert space, $f = u \in C$ is a constant and $N = 1$, then (1.9) is reduced to the following iterative sequence:

$$x_{n+1} = \alpha_{n+1} u + (1 - \alpha_{n+1}) T x_n, \quad \forall n \geqslant 0, \tag{1.10}$$

which was studied in Halpern [7], Lions [9], Wittmann [13]. Under suitable conditions on the mapping T and the sequence $\{\alpha_n\}$, some strong convergence theorems for iterative sequence $\{x_n\}$ to converge to the nearest point projection of u onto $\mathrm{Fix}(T)$ are obtained.

(2) Let E be a Hilbert space, $f = u \in C$ be a constant and $T_1, T_2, \ldots, T_N : C \to C$ be a finite family of nonexpansive mappings with $\bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \neq \emptyset$. Then the iterative sequence $\{x_n\}$ defined by (1.9) is deduced to (1.8) which was considered by Bauschke [1].

(3) Let E be a uniformly smooth Banach space, C be a nonempty closed convex subset of E, $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$ and $f \in \Pi_C$. Then (1.9) is reduced to the following iterative sequence:

$$x_{n+1} = \alpha_{n+1} f(x_n) + (1 - \alpha_{n+1}) T x_n, \quad \forall n \geqslant 0, \tag{1.11}$$

which was considered in Xu [14].

Summing up the above arguments, we know that (1.9) is a more general sequence which contains (1.8), (1.10) and (1.11) as its special cases.

The following theorem is the main results in the paper.

Theorem 1. *Let E be a uniformly smooth Banach space, C be a nonempty closed convex subset of E, $f \in \Pi_C$, T_i, $i = 1, 2, \ldots, N$, be a finite family of nonexpansive mappings of C into itself such that the set $\bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$ of common fixed points of $T_1, T_2, \ldots, N$ is nonempty and satisfies the following condition*:

$$\begin{aligned} \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) &= \mathrm{Fix}(T_1 T_N \cdots T_3 T_2) \\ &= \cdots \\ &= \mathrm{Fix}(T_{N-1} T_{N-2} \cdots T_1 T_N) \\ &= \mathrm{Fix}(T_N T_{N-1} \cdots T_1) := F(S), \end{aligned}$$

where

$$S = T_N T_{N-1} \cdots T_1. \tag{1.12}$$

Suppose further that $f \in \Pi_C$ with $p \neq f(p)$, $\forall p \in \bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$, $x_0 \in C$ is a given point, $\{\alpha_n\}$ is a sequence in $[0, 1]$ *and $\{x_n\}$ is the iterative sequence defined by* (1.9)*, then the following conclusions hold*:

(1) *$\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, \ldots, T_N$ if and only if*
  (a) $\lim_{n\to\infty} \alpha_n = 0$;
  (b) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
  (c) $\|x_n - S x_n\| \to 0$ $(n \to \infty)$;

(2) *if* $\{x_n\}$ *converges strongly to some common fixed point* $z \in \bigcap_{i=1}^N F(T_i)$ *and if* $P(f) = z = \lim_{n\to\infty} x_n$ *for each* $f \in \Pi_C$, *then* $P(f)$ *solves the following variational inequality*:

$$\langle (f - I)P(f), J(P(f) - p)\rangle \geqslant 0, \quad \forall p \in \bigcap_{i=1}^N F(T_i).$$

In order to prove our results, we need the following lemmas.

Lemma 1. (Goebel and Reich [6, p. 48]) *Let* C *be a nonempty convex subset of a smooth Banach space* E. *If* $C_0 \subset C$ *and* P *is a retraction of* C *onto* C_0 *such that*

$$\langle x - Px, J(Px - y)\rangle \geqslant 0,$$

for all $x \in C$ *and* $y \in C_0$, *then* $P : C \to C_0$ *is sunny and nonexpansive.*

Lemma 2. (Wang [12]) *Let* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ *be three nonnegative real sequences satisfying the following conditions*:

$$a_{n+1} \leqslant (1 - \lambda_n)a_n + b_n + c_n, \quad \forall n \geqslant n_0,$$

where n_0 *is some nonnegative integer,* $\{\lambda_n\} \subset (0, 1)$ *with* $\sum_{n=0}^\infty \lambda_n = \infty$, $b_n = o(\lambda_n)$ *and* $\sum_{n=0}^\infty c_n < \infty$, *then* $a_n \to 0$ (*as* $n \to \infty$).

Lemma 3. [3] *Let* E *be a real Banach space and* $J : E \to 2^{E^*}$ *be the normalized duality mapping, then for any* $x, y \in E$ *the following holds*:

$$\|x + y\|^2 \leqslant \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall j(x + y) \in J(x + y).$$

2. Proof of Theorem 1

Proof of conclusion (1) *of Theorem* 1

Sufficiency

(I) Let S be the mapping defined by (1.12). It is easy to see that $S : C \to C$ is a nonexpansive mapping. For given $f \in \Pi_C$, $t \in (0, 1)$, we define a contraction mapping $T_t : C \to C$ by

$$T_t x = tf(x) + (1 - t)Sx, \quad x \in C.$$

By Banach's contraction mapping principle it yields a unique fixed point $z_t \in C$ of T_t which is a unique solution of the equation

$$z_t = tf(z_t) + (1 - t)Sz_t. \tag{2.1}$$

By Theorem 4.1 in Xu [14], $z_t \to z \in \mathrm{Fix}(S) := \bigcap_{i=1}^N \mathrm{Fix}(T_i)$ which is a solution of the following variational inequality:

$$\langle (I - f)P(f), J(p - P(f))\rangle \geqslant 0, \quad \forall p \in \bigcap_{i=1}^N \mathrm{Fix}(T_i), \tag{2.2}$$

therefore the sequence $\{z_t\}$ is bounded.

(II) Now we prove that the sequence $\{x_n\}$ defined by (1.9) is bounded. In fact, for any $p \in \bigcap_{i=1}^N \mathrm{Fix}(T_i)$ and for any $n \geqslant 0$ we have

$$\begin{aligned}\|x_n - p\| &= \big\|(1-\alpha_n)(T_n x_{n-1} - p) + \alpha_n\big(f(x_{n-1}) - p\big)\big\| \\ &\leqslant (1-\alpha_n)\|T_n x_{n-1} - p\| + \alpha_n\big\|f(x_{n-1}) - p\big\| \\ &\leqslant (1-\alpha_n)\|x_{n-1} - p\| + \alpha_n\big\{\big\|f(x_{n-1}) - f(p)\big\| + \big\|f(p) - p\big\|\big\} \\ &\leqslant (1-\alpha_n)\|x_{n-1} - p\| + \alpha_n\big\{\alpha\|x_{n-1} - p\| + \big\|f(p) - p\big\|\big\} \\ &= \big(1 - \alpha_n(1-\alpha)\big)\|x_{n-1} - p\| + \alpha_n\big\|f(p) - p\big\| \\ &\leqslant \max\left\{\|x_{n-1} - p\|, \frac{\|f(p) - p\|}{1-\alpha}\right\}.\end{aligned}$$

By induction, we can prove that

$$\|x_n - p\| \leqslant \max\left\{\|x_0 - p\|, \frac{\|f(p) - p\|}{1-\alpha}\right\}, \quad \forall n \geqslant 0. \tag{2.3}$$

This shows that $\{x_n\}$ is bounded, and so $f(x_n)$ and $\{T_{n+1}x_n\}$ both are bounded. Let

$$M = \sup_{t\geqslant 0}\sup_{n\geqslant 0}\big\{\|x_n - z\|^2 + \|x_n - z\| + \|z_t - x_n\| + \|z_t - x_n\|^2\big\} < \infty. \tag{2.4}$$

(III) Now we prove that

$$\limsup_{n\to\infty}\big\langle z - f(z), J(z - x_n)\big\rangle \leqslant 0, \tag{2.5}$$

where $z = P(f)$ is the strong limit of the sequence $\{z_t\}$ defined by (2.1). Indeed, it follows from (2.1), (1.9), (2.4) and Lemma 3 that

$$\begin{aligned}\|z_t - x_n\|^2 &= \big\|(1-t)(Sz_t - x_n) + t\big(f(z_t) - x_n\big)\big\|^2 \\ &\leqslant (1-t)^2\|Sz_t - x_n\|^2 + 2t\big\langle f(z_t) - x_n, J(z_t - x_n)\big\rangle \\ &\leqslant (1-t)^2\big\{\|Sz_t - Sx_n\| + \|Sx_n - x_n\|\big\}^2 \\ &\quad + 2t\big\langle f(z_t) - z_t + z_t - x_n, J(z_t - x_n)\big\rangle \\ &\leqslant (1-t)^2\big\{\|z_t - x_n\| + \|Sx_n - x_n\|\big\}^2 \\ &\quad + 2t\|z_t - x_n\|^2 + 2t\big\langle f(z_t) - z_t, J(z_t - x_n)\big\rangle \\ &= (1-t)^2\big\{\|z_t - x_n\|^2 + \sigma_n(t)\big\} \\ &\quad + 2t\|z_t - x_n\|^2 + 2t\big\langle f(z_t) - z_t, J(z_t - x_n)\big\rangle,\end{aligned} \tag{2.6}$$

where

$$\begin{aligned}\sigma_n(t) &:= 2\|z_t - x_n\| \cdot \|Sx_n - x_n\| + \|Sx_n - x_n\|^2 \\ &\leqslant 2M\|Sx_n - x_n\| + \|Sx_n - x_n\|^2, \quad \forall t \in (0,1).\end{aligned}$$

By condition (c),

$$\lim_{n\to\infty}\sigma_n(t) = 0 \tag{2.7}$$

uniformly in $t \in (0,1)$. Hence from (2.6) we have

$$\big\langle z_t - f(z_t), J(z_t - x_n)\big\rangle \leqslant \frac{t}{2}\|z_t - x_n\|^2 + \frac{1}{2t}\sigma_n(t) \leqslant \frac{t}{2}M + \frac{1}{2t}\sigma_n(t).$$

By using (2.7), we have

$$\limsup_{n\to\infty}\big\langle z_t - f(z_t), J(z_t - x_n)\big\rangle \leqslant \frac{t}{2}M, \quad \forall t \in (0,1),$$

where M is the constant defined by (2.4). And so we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\big\langle z_t - f(z_t), J(z_t - x_n)\big\rangle \leqslant 0. \tag{2.8}$$

On the other hand, from (2.4) we know that $\|z_t - x_n\| \leqslant M$ and $\|x_n - z\| \leqslant M$, $\forall t \in (0,1)$, $n \geqslant 0$. Take $r \geqslant 2M$ and denote $B_r = \{x \in X\colon \|x\| \leqslant r\}$. Since X is uniformly smooth, the normalized duality mapping J is uniformly continuous on the closed ball B_r from norm topology to norm topology. Therefore for any given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for any $x, y \in B_r$, if $\|x - y\| < \delta$, then

$$\big\|J(x) - J(y)\big\| < \varepsilon.$$

In particular, if we take $t_0 \in (0,1)$ such that $\|z_t - z\| < \delta$, $\forall t \in (0, t_0)$. From (2.4) we know that $(z_t - x_n)$ and $(z - x_n) \in B_r$, and so

$$\big\|J(z_t - x_n) - J(z - x_n)\big\| < \varepsilon, \quad \forall t \in (0, t_0),\ n \geqslant 0.$$

Therefore for all $t \in (0, t_0)$ and $n \geqslant 0$ we have

$$\begin{aligned}\big\langle z - f(z), J(z - x_n)\big\rangle &= \big\langle z - f(z), J(z - x_n) - J(z_t - x_n)\big\rangle \\ &\quad + \big\langle z - z_t - f(z) + f(z_t), J(z_t - x_n)\big\rangle \\ &\quad + \big\langle z_t - f(z_t), J(z_t - x_n)\big\rangle \\ &\leqslant \big\|z - f(z)\big\| \cdot \big\|J(z - x_n) - J(z_t - x_n)\big\| \\ &\quad + \big\{\|z - z_t\| + \big\|f(z) - f(z_t)\big\|\big\}\|z_t - x_n\| \\ &\quad + \big\langle z_t - f(z_t), J(z_t - x_n)\big\rangle \\ &\leqslant \varepsilon\big\|z - f(z)\big\| + 2r\|z - z_t\| + \big\langle z_t - f(z_t), J(z_t - x_n)\big\rangle. \end{aligned} \tag{2.9}$$

By taking lim sup first with respect to $n \to \infty$ and then to $t \to 0$ and noticing (2.8), we obtain

$$\limsup_{n\to\infty}\big\langle z - f(z), J(z - x_n)\big\rangle \leqslant \varepsilon\big\|z - f(z)\big\|.$$

By the arbitrariness of $\varepsilon > 0$, we have

$$\limsup_{n\to\infty}\big\langle z - f(z), J(z - x_n)\big\rangle \leqslant 0. \tag{2.10}$$

The conclusion (2.5) is proved.

Letting $\gamma_n = \max\{\langle z - f(z), J(z - x_n)\rangle, 0\} \geqslant 0$, $\forall n \geqslant 0$, now we prove that

$$\lim_{n\to\infty}\gamma_n = 0. \tag{2.11}$$

In fact, from (2.10), for any given $\varepsilon > 0$, there exists a positive integer n_1 such that

$$\big\langle z - f(z), J(z - x_n)\big\rangle < \varepsilon, \quad \forall n \geqslant n_1,$$

and so $0 \leqslant \gamma_n < \varepsilon$ $\forall n \geqslant n_1$. Since $\varepsilon > 0$ is arbitrary, it implies that

$$\lim_{n\to\infty}\gamma_n = 0.$$

Therefore it follows from (1.9), Lemma 3 and (2.11) that for any $z \in \bigcap_{i=1}^N F(T_i)$ we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \big\|(1-\alpha_{n+1})(T_{n+1}x_n - z) + \alpha_{n+1}\big(f(x_n) - z\big)\big\|^2 \\
&\leqslant (1-\alpha_{n+1})^2\|T_{n+1}x_n - z\|^2 + 2\alpha_{n+1}\big\langle f(x_n) - z, J(x_{n+1} - z)\big\rangle \\
&\leqslant (1-\alpha_{n+1})^2\|x_n - z\|^2 \\
&\quad + 2\alpha_{n+1}\big\langle f(x_n) - f(z) + f(z) - z, J(x_{n+1} - z)\big\rangle \\
&\leqslant (1-\alpha_{n+1})^2\|x_n - z\|^2 + 2\alpha_{n+1}\alpha\|x_n - z\| \cdot \|x_{n+1} - z\| \\
&\quad + 2\alpha_{n+1}\big\langle f(z) - z, J(x_{n+1} - z)\big\rangle \\
&\leqslant (1-\alpha_{n+1})^2\|x_n - z\|^2 + \alpha_{n+1}\alpha\big\{\|x_n - z\|^2 + \|x_{n+1} - z\|^2\big\} \\
&\quad + 2\alpha_{n+1}\big\langle f(z) - z, J(x_{n+1} - z)\big\rangle.
\end{aligned} \tag{2.12}$$

Since the normalized duality mapping J defined by (1.1) is odd, i.e., $J(-x) = -J(x), x \in E$, therefore we have

$$\big\langle f(z) - z, J(x_{n+1} - z)\big\rangle = \big\langle z - f(z), J(z - x_{n+1})\big\rangle \leqslant \gamma_{n+1}.$$

Substituting it into (2.12) and simplifying, we have

$$\begin{aligned}
&(1-\alpha\alpha_{n+1})\|x_{n+1} - z\|^2 \\
&\quad \leqslant \big(1-\alpha_{n+1}(2-\alpha)\big)\|x_n - z\|^2 + \alpha_{n+1}^2\|x_n - z\|^2 + 2\alpha_{n+1}\gamma_{n+1} \\
&\quad \leqslant \big(1-\alpha_{n+1}(2-\alpha)\big)\|x_n - z\|^2 + \alpha_{n+1}^2 M + 2\alpha_{n+1}\gamma_{n+1}.
\end{aligned} \tag{2.13}$$

Since $\alpha_n \to 0$, therefore there exists a positive integer n_2 such that

$$1 - \alpha\alpha_{n+1} > \frac{1}{2}, \quad \forall n \geqslant n_2.$$

It follows from (2.13) that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leqslant \frac{1-\alpha_{n+1}(2-\alpha)}{1-\alpha\alpha_{n+1}}\|x_n - z\|^2 \\
&\quad + 2\alpha_{n+1}\{\alpha_{n+1}M + 2\gamma_{n+1}\}, \quad \forall n \geqslant n_2.
\end{aligned} \tag{2.14}$$

Again since

$$\frac{1-\alpha_{n+1}(2-\alpha)}{1-\alpha\alpha_{n+1}} = 1 - \frac{2\alpha_{n+1}(1-\alpha)}{1-\alpha\alpha_{n+1}} \leqslant 1 - 2\alpha_{n+1}(1-\alpha),$$

it follows from (2.14) that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leqslant \big\{1 - 2\alpha_{n+1}(1-\alpha)\big\}\|x_n - z\|^2 \\
&\quad + 2\alpha_{n+1}\{\alpha_{n+1}M + 2\gamma_{n+1}\}, \quad \forall n \geqslant n_2.
\end{aligned} \tag{2.15}$$

Take $a_n = \|x_n - z\|^2$, $\lambda_n = 2\alpha_{n+1}(1-\alpha)$, $b_n = 2\alpha_{n+1}\{\alpha_{n+1}M + 2\gamma_{n+1}\}$ and $c_n = 0$, $\forall n \geqslant n_2$, in Lemma 2. By the assumptions, it is know that $\sum_{n=0}^\infty \lambda_n = \infty$, $b_n = o(\lambda_n)$ and $\sum_{n=0}^\infty c_n = 0$, hence the conditions in Lemma 2 are satisfied, and so we have

$$\lim_{n\to\infty}\|x_n - z\| = 0, \quad \text{i.e.,} \quad x_n \to z \in \bigcap_{i=}^N F(T_i). \tag{2.16}$$

The sufficiency of conclusion (1) of Theorem 1 is proved.

Necessity

Suppose that the sequence $\{x_n\}$ defined by (1.9) converges strongly to a fixed point $p \in \bigcap_{i=1}^N F(T_i)$. In view of (1.12), we know that

$$\|Sx_n - x_n\| \leqslant \|Sx_n - p\| + \|x_n - p\| \leqslant 2\|x_n - p\| \to 0 \quad (n \to \infty).$$

The necessity of condition (c) is proved.

Since each $T_i : C \to C$, $i = 1, 2, \dots, N$, is nonexpansive, we get

$$\|T_{n+1}x_n - p\| \leqslant \|x_n - p\| \to 0, \quad \text{i.e.,} \quad T_{n+1}x_n \to p \quad (\text{as } n \to \infty).$$

Again from (1.9) we have that

$$\begin{aligned} \alpha_{n+1}\big\|f(x_n) - T_{n+1}x_n\big\| &= \|x_{n+1} - T_{n+1}x_n\| \\ &\leqslant \|x_{n+1} - p\| + \|T_{n+1}x_n - p\| \\ &\leqslant \|x_{n+1} - p\| + \|x_n - p\| \to 0 \quad (\text{as } n \to \infty). \end{aligned}$$

Therefore we have

$$\limsup_{n\to\infty} \alpha_{n+1}\big\|f(x_n) - T_{n+1}x_n\big\| = \limsup_{n\to\infty} \alpha_{n+1}\big\|f(p) - p\big\| = 0.$$

By the assumption that $p \neq f(p)$, $\forall p \in \bigcap_{i=1}^N \mathrm{Fix}(T_i)$, this implies that

$$\limsup_{n\to\infty} \alpha_{n+1} = 0,$$

i.e.,

$$\lim_{n\to\infty} \alpha_n = 0.$$

The necessity of condition (a) is proved.

Take $f = 0$, $C = \{x \in E\colon \|x\| \leqslant 1\}$ (closed unit ball in E) and $T_i = (-I) : C \to C$, $\forall i = 1, 2, \dots, N$, in (1.9), where I is the identity mapping. Since each T_i, $i = 1, 2, \dots, N$, is nonexpansive and 0 is the unique common fixed point of $T_1, T_2, \dots, T_N$ in C, hence we have

$$\begin{aligned} x_{n+1} &= (-1)(1 - \alpha_{n+1})x_n = (-1)^2(1 - \alpha_{n+1})(1 - \alpha_n)x_{n-1} \\ &= \cdots = (-1)^{n+1}\prod_{i=1}^{n+1}(1 - \alpha_i)x_0. \end{aligned}$$

If $x_n \to 0 \in \bigcap_{i=1}^N \mathrm{Fix}(T_i)$, we have

$$0 = \lim_{n\to\infty} \|x_{n+1} - 0\| = \lim_{n\to\infty} \prod_{i=1}^{n+1}(1 - \alpha_i)\|x_0 - 0\|.$$

This implies that

$$\prod_{i=1}^{\infty}(1 - \alpha_i) = 0, \quad \text{i.e.,} \quad \sum_{i=1}^{\infty} \alpha_i = \infty.$$

The necessity of condition (b) is proved.

Summing up the about argument, the conclusion (1) of Theorem 1 is proved.

Proof of conclusion (2) *of Theorem* 1

Indeed, if $\lim_{n\to\infty} x_n = z = P(f)$ for each $f \in \Pi_C$, then by Theorem 4.1 in Xu [14], we have

$$\lim_{n\to\infty} x_n = \lim_{t\to 0} z_t = P(f) = z \in \bigcap_{i=1}^{N} F(T_i) \quad \text{for each } f \in \Pi_C,$$

and

$$\big\langle (f - I)P(f), J\big(P(f) - y\big)\big\rangle \geqslant 0, \quad \forall y \in \bigcap_{i=1}^{N} F(T_i),$$

where $\{z_t\}$ is the sequence defined by (2.1).

The proof of Theorem 1 is completed.

3. Applications to some recent theorems

The following theorem can be obtain from Theorem 1 with $N = 1$ immediately.

Theorem 2. *Let E be a uniformly smooth Banach space, C be a nonempty closed convex subset of E, $f \in \Pi_C$, $T : C \to C$ be a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$. *Suppose further that $x_0 \in C$ is a given point, $\{\alpha_n\}$ is a sequence in* $(0, 1)$ *and $\{x_n\}$ is the iterative sequence defined by*

$$x_{n+1} = \alpha_{n+1} f(x_n) + (1 - \alpha_{n+1})T x_n. \tag{3.1}$$

Then the following conclusions hold:

(1) *$\{x_n\}$ converges strongly to a common fixed point of T if and only if*
- (a) $\lim_{n\to\infty} \alpha_n = 0$;
- (b) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- (c) $\|x_n - T x_n\| \to 0$ $(n \to \infty)$;

(2) *if the sequence $\{x_n\}$ defined by* (3.1) *converges strongly to some common fixed point $z \in$* $\mathrm{Fix}(T)$ *and if $P(f) = z = \lim_{n\to\infty} x_n$ for each $f \in \Pi_C$, then $P(f)$ solves the following variational inequality*:

$$\big\langle (f - I)P(f), J\big(P(f) - p\big)\big\rangle \geqslant 0, \quad \forall p \in \mathrm{Fix}(T).$$

Now we are in a position to apply Theorem 2 to generalize and improve some recent new results.

Theorem 3. (Xu [10, Theorem 4.2]) *Let E be a uniformly smooth Banach space, C be a nonempty closed convex subset of E, $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $f \in \Pi_C$ with a contractive constant α, $x_0 \in C$ be any given point, $\{\alpha_n\}$ be a real sequence in* $(0, 1)$ *and $\{x_n\}$ be the iterative sequence defined by* (3.1). *If the following conditions are satisfied*:

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$,

then the sequence $\{x_n\}$ *converges strongly to* $P(f)$ *which solves the following variational inequality*:

$$\big\langle (f-I)P(f), J\big(P(f)-p\big)\big\rangle \geqslant 0, \quad \forall p \in \mathrm{Fix}(T).$$

Proof. In order to prove the conclusion of Theorem 3, it suffices to show that under the conditions (i)–(iii) of Theorem 3, we have

$$\|x_n - Tx_n\| \to 0 \quad (\text{as } n \to \infty).$$

In fact, for given $f \in \Pi_C$ with a contractive constant $\alpha \in (0,1)$ and for any given $p \in \mathrm{Fix}(T)$, by the same method as given in proof of (2.3), we can prove that

$$\|x_n - p\| \leqslant \max\left\{\|x_0 - p\|, \frac{1}{1-\alpha}\big\|f(p)-p\big\|\right\}, \quad \forall n \geqslant 0.$$

This implies that the sequence $\{x_n\}$ is bounded and so the sequence $\{Tx_n\}$ and $\{f(x_n)\}$ both are bounded. Therefore there exists a constant $M > 0$ such that

$$M = \sup_{n \geqslant 0}\big\|f(x_n) - Tx_n\big\| < \infty.$$

In view of (3.1) we have that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \big\|\alpha_{n+1} f(x_n) + (1-\alpha_{n+1})Tx_n - \big[\alpha_n f(x_{n-1}) + (1-\alpha_n)Tx_{n-1}\big]\big\| \\
&= \big\|(1-\alpha_{n+1})(Tx_n - Tx_{n-1}) \\
&\quad + (\alpha_{n+1}-\alpha_n)\big(f(x_{n-1}) - Tx_{n-1}\big) + \alpha_{n+1}\big(f(x_n) - f(x_{n-1})\big)\big\| \\
&\leqslant (1-\alpha_{n+1})\|x_n - x_{n-1}\| + |\alpha_{n+1}-\alpha_n|\big\|f(x_{n-1}) - Tx_{n-1}\big\| \\
&\quad + \alpha_{n+1}\alpha\|x_n - x_{n-1}\| \\
&\leqslant \big(1-\alpha_{n+1}(1-\alpha)\big)\|x_n - x_{n-1}\| + |\alpha_{n+1}-\alpha_n| M.
\end{aligned} \tag{3.2}$$

If the condition $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ is satisfied, then taking $a_n = \|x_n - x_{n-1}\|$, $\lambda_n = \alpha_{n+1}(1-\alpha)$, $b_n = 0$ and $c_n = |\alpha_{n+1} - \alpha_n| M$, $\forall n \geqslant 0$, in Lemma 2, we know that all conditions in Lemma 2 are satisfied. Hence we have that

$$\|x_{n+1} - x_n\| \to 0 \quad (\text{as } n \to \infty). \tag{3.3}$$

If the condition $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$ is satisfied, then take

$$\begin{aligned}
a_n &= \|x_n - x_{n-1}\|, \\
b_n &= \alpha_n \frac{|\alpha_{n+1} - \alpha_n| M}{\alpha_n} = \left|1 - \frac{\alpha_{n+1}}{\alpha_n}\right| \alpha_n M, \\
c_n &= 0, \quad \forall n \geqslant 0,
\end{aligned}$$

in Lemma 2, we know that all conditions in Lemma 2 are satisfied. Hence (3.3) also holds.

Again it follows from (3.1) that

$$x_{n+1} - x_n = \alpha_{n+1}\big(f(x_n) - Tx_n\big) + Tx_n - x_n.$$

This implies that

$$\begin{aligned}\|Tx_n - x_n\| &\leqslant \|x_{n+1} - x_n\| + \alpha_{n+1}\big\|f(x_n) - Tx_n\big\| \\ &\leqslant \|x_{n+1} - x_n\| + \alpha_{n+1}M \to 0 \quad (\text{as } n \to \infty).\end{aligned}$$

This completes the proof of Theorem 3. □

The following result can be obtained from Theorem 3 immediately.

Theorem 4. (Wittmann [9, Theorem 2]) *Let H be a real Hilbert space, C be a nonempty closed convex subset of H, $T : C \to C$ be a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$. *Let $u, x_0 \in C$ be any given points, $\{x_n\}$ be the iterative sequence defined by*

$$x_{n+1} = \alpha_{n+1}u + (1 - \alpha_{n+1})Tx_n, \quad \forall n \geqslant 0. \tag{3.4}$$

If $\{\alpha_n\}$ be a sequence in $(0, 1)$ *satisfying the conditions* (i)–(iii) *in Theorem* 3, *then the sequence $\{x_n\}$ converges strongly to a fixed point $z = Pu \in F(T)$ which is a solution of the following variational inequality*:

$$\langle u - Pu, Pu - y\rangle \geqslant 0, \quad \forall y \in F(T).$$

Proof. In fact, in Theorem 3 take $f = u$, the conclusion of Theorem 4 is obtained from Theorem 3 immediately. □

Remark 1. Theorem 3 not only generalizes and improves the main result in Wittmann [13] and Xu [15] but also generalizes and improves the main results in Halpern [7] and Lions [9].

Theorem 5. *Let E be a uniformly smooth Banach space, C be a nonempty closed convex subset of E, $f \in \Pi_C$, T_i, $i = 1, 2, \ldots, N$, be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N \mathrm{Fix}(T_i) \neq \emptyset$ satisfying the following conditions*:

(i) $T_N T_{N-1} \cdots T_1 = T_1 T_N \cdots T_3 T_2 = \cdots = T_{N-1} T_{N-2} \cdots T_1 T_N$;
(ii)

$$\begin{aligned}\bigcap_{i=1}^N \mathrm{Fix}(T_i) &= \mathrm{Fix}(T_1 T_N \cdots T_3 T_2) \\ &= \cdots \\ &= \mathrm{Fix}(T_{N-1} T_{N-2} \cdots T_1 T_N) \\ &= \mathrm{Fix}(T_N T_{N-1} \cdots T_1) := F(S),\end{aligned}$$

where

$$S = T_N T_{N-1} \cdots T_1. \tag{3.5}$$

Suppose further that $x_0 \in C$ is a given point, $\{\alpha_n\}$ is a sequence in $[0, 1]$ *and $\{x_n\}$ is the iterative sequence defined by* (1.9). *If the following conditions are satisfied*:

(a) $\lim_{n\to\infty} \alpha_n = 0$;
(b) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(c) $\sum_{n=0}^{\infty} |\alpha_{n+N} - \alpha_n| < \infty$,

then $\{x_n\}$ *converges strongly to a common fixed point* $P(f) \in \bigcap_{i=1}^N \mathrm{Fix}(T_i)$ *which solves the following variational inequality*:

$$\left\langle (f - I)P(f), J\big(P(f) - p\big)\right\rangle \geqslant 0, \quad \forall p \in \bigcap_{i=1}^{N} F(T_i).$$

Proof. It follows from (2.3) that for any $n \geqslant 0$ and for any $p \in \bigcap_{i=1}^N F(T_i)$,

$$\|x_n - p\| \leqslant \max\left\{\|x_0 - p\|, \frac{\|f(p) - p\|}{1-\alpha}\right\}. \tag{3.6}$$

This implies that $\{x_n\}$ is bounded, and so $\{f(x_n)\}$, $T_{n+1}x_n$ both are bounded. Let

$$M = \sup_{n\geqslant 0}\big\{\big\|f(x_n)\big\| + \|T_{n+1}x_n\|\big\} < \infty. \tag{3.7}$$

Next we prove that

$$x_{n+1} - T_{n+1}x_n \to 0 \quad (\text{as } n \to \infty). \tag{3.8}$$

Indeed, from (1.9) and (3.7) we have

$$\|x_{n+1} - T_{n+1}x_n\| = \alpha_{n+1}\big\|f(x_n) - T_{n+1}x_n\big\| \leqslant \alpha_{n+1}M \to 0.$$

This shows that (3.8) is true.

Now we prove that

$$x_{n+N} - x_n \to 0 \quad (\text{as } n \to \infty). \tag{3.9}$$

Indeed, from (1.9) we have

$$\begin{aligned}
\|x_{n+N} - x_n\| &= \big\|\alpha_{n+N}f(x_{n+N-1}) + (1-\alpha_{n+N})T_{n+N}x_{n+N-1} \\
&\quad - \big[\alpha_n f(x_{n-1}) + (1 - \alpha_n T_n x_{n-1})\big]\big\| \\
&= \big\|(1-\alpha_{n+N})[T_{n+N}x_{n+N-1} - T_n x_{n-1}] \\
&\quad + (\alpha_{n+N} - \alpha_n)\big[f(x_{n-1}) - T_n x_{n-1}\big] \\
&\quad + \alpha_{n+N}\big[f(x_{n+N-1} - f(x_{n-1})\big]\big\| \\
&\leqslant (1-\alpha_{n+N})\|x_{n+N-1} - x_{n-1}\| \quad (\text{since } T_{n+N} = T_n) \\
&\quad + |\alpha_{n+N} - \alpha_n|M + \alpha_{n+N}\alpha\|x_{n+N-1} - x_{n-1}\| \\
&= \big(1 - \alpha_{n+N}(1-\alpha)\big)\|x_{n+N-1} - x_{n-1}\| + |\alpha_{n+N} - \alpha_n|M.
\end{aligned}$$

Taking $a_n = \|x_{n+N-1} - x_{n-1}\|$, $\lambda_n = \alpha_{n+N}(1-\alpha)$, $b_n = 0$ and $c_n = |\alpha_{n+N} - \alpha_n|M$, we know that all conditions in Lemma 2 are satisfied. By Lemma 2, $\|x_{n+N} - x_n\| \to 0$ (as $n \to \infty$). The desired result is obtained.

Next we prove that

$$x_n - T_{n+N}T_{n+N-1}\cdots T_{n+1} \to 0 \quad (\text{as } n \to \infty). \tag{3.10}$$

In view of (3.9), it suffices to show that $x_{n+N} - T_{n+N}T_{n+N-1}\cdots T_{n+1} \to 0$ (as $n \to \infty$). In fact, from (3.8) we have

$$x_{n+N} - T_{n+N}x_{n+N-1} \to 0. \tag{1*}$$

Again by (3.8), $x_{n+N-1} - T_{n+N-1}x_{n+N-2} \to 0$. Thus the nonexpansiveness of T_{n+N} implies that

$$T_{n+N}x_{n+N-1} - T_{n+N}T_{n+N-1}x_{n+N-2} \to 0. \tag{2*}$$

Similarly

$$T_{n+N}T_{n+N-1}x_{n+N-2} - T_{n+N}T_{n+N-1}T_{n+N-2}x_{n+N-2} \to 0. \tag{3*}$$

$$\vdots$$

$$T_{n+N}T_{n+N-1}\cdots T_{n+2}x_{n+1} - T_{n+N}T_{n+N-1}T_{n+N-2}\cdots T_{n+2}T_{n+1}x_n \to 0. \tag{N*}$$

Adding these N sequences yields

$$x_{n+N} - T_{n+N}T_{n+N-1}\cdots T_{n+1}x_n \to 0 \quad (\text{as } n \to \infty).$$

The desired is proved.

Finally, we prove that

$$x_n - T_N T_{N-1}\cdots T_1 x_n \to 0 \quad (\text{as } n \to \infty). \tag{3.11}$$

Indeed, it is easy to see that

$$\begin{aligned}
&\text{If } n \pmod N = 1, \text{ then } T_{n+N}T_{n+N-1}\cdots T_{n+1} = T_1T_N\cdots T_2;\\
&\text{If } n \pmod N = 2, \text{ then } T_{n+N}T_{n+N-1}\cdots T_{n+1} = T_2T_1T_N\cdots T_3;\\
&\quad\vdots\\
&\text{If } n \pmod N = N, \text{ then } T_{n+N}T_{n+N-1}\cdots T_{n+1} = T_NT_{N-1}\cdots T_1.
\end{aligned}$$

In view of condition (i),

$$T_NT_{N-1}\cdots T_1 = T_1T_N\cdots T_3T_2 = \cdots = T_{N-1}T_{N-2}\cdots T_1T_N$$

therefore we have

$$T_NT_{N-1}\cdots T_1 = T_{n+N}T_{n+N-1}\cdots T_{n+1}, \quad \forall n \geqslant 1.$$

This implies that

$$x_n - T_NT_{N-1}\cdots T_1x_n = x_n - T_{n+N}T_{n+N-1}\cdots T_{n+1}x_n \to 0 \quad (\text{as } n \to \infty),$$

i.e.,

$$\|x_n - T_NT_{N-1}\cdots T_1x_n\| \to 0 \quad (\text{as } n \to \infty).$$

Therefore all conditions in Theorem 1 are satisfied. The conclusion of Theorem 5 can be obtained from Theorem 1 immediately. □

Remark 2. Theorem 5 is an improvement and generalization of Theorem 3.2 in Bauschke [1], Theorem 3.1 in O'Hara, Pillay and Xu [16] and Theorem 10 in Jung [8].

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